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A new family of high-order compact upwind difference schemes with good spectral resolution

Qiang Zhou, Zhaohui Yao, Feng He *, M.Y. Shen

Department of Engineering Mechanics, Tsinghua University, Beijing 100084, People's Republic of China

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Abstract

This paper presents a new family of high-order compact upwind difference schemes. Unknowns included in the proposed schemes are not only the values of the function but also those of its first and higher derivatives. Derivative terms in the schemes appear only on the upwind side of the stencil. One can calculate all the first derivatives exactly as one solves explicit schemes when the boundary conditions of the problem are non-periodic. When the proposed schemes are applied to periodic problems, only periodic bi-diagonal matrix inversions or periodic block-bi-diagonal matrix inversions are required. Resolution optimization is used to enhance the spectral representation of the first derivative, and this produces a scheme with the highest spectral accuracy among all known compact schemes. For non-periodic boundary conditions, boundary schemes constructed in virtue of the assistant scheme make the schemes not only possess stability for any selective length scale on every point in the computational domain but also satisfy the principle of optimal resolution. Also, an improved shock-capturing method is developed. Finally, both the effectiveness of the new hybrid method and the accuracy of the proposed schemes are verified by executing four benchmark test cases.

Keywords: Compact upwind difference scheme; Optimized resolution; Hybrid scheme; Boundary closure; Shock-capturing; High-order scheme; Spectral analysis; Full spatial stability

1. Introduction

Compact high-order finite difference schemes have been extensively studied and widely used to compute problems involving incompressible, compressible and hypersonic flows [13–22,39], computational aeroacoustics [23–27,36,48], computational electromagnetics [28–30] and several other practical applications [31–35]. Using compact stencils that relate various derivatives with function values at discrete nodes, compact schemes not only offer higher order approximations to differential operators but provide higher resolution characteristics for the same number of scheme points in comparison to explicit finite difference schemes. Due to their high formal order, good spectral resolution and their flexibility, compact high-order finite

* Corresponding author. Tel.: +86 10 62787470; fax: +86 10 62782639.
E-mail address: hefeng@tsinghua.edu.cn (F. He).

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difference schemes are the most attractive schemes for flows with multiscales, e.g., turbulence. The smallest scales, which are dictated by the physical viscosity of the flow, become unresolvable with the mesh size, and they suffer from instabilities. There are three remedies [1] for this unwanted behavior: explicit filtering, artificial viscosity and schemes with inherent dissipation. Of these three remedies, the satisfactory way of physically dissipating unresolvable wavenumbers is to use a high-order compact upwind scheme. Upwinding in compact schemes has earlier been incorporated by Tolstykh [2,3] through a Murman-type switch [4]. The CUD-3 and CUD-5 [5] were their most famous schemes. In subsequent years, Tolstykh constructed compact upwind schemes with arbitrary order via linear combinations of ‘‘elementary’’ CUD operators of fixed order (say, third order). Some details of the technique (multioperator) can be found, for example, in [6,7]. However, in their work, no effort was made to optimize the resolution characteristics. An optimized version using a free parameter was introduced by Adams and Shariff [40]. Not adopting the idea of ‘‘multioperator’’, which was interesting but a bit complicated, Adams and Shariff [40] constructed their schemes basically following Lele's technique [12], namely, based on Hermite interpolation, but dropped the requirement of symmetric/antisymmetric coefficients. In this paper, the method of developing compact upwind schemes is almost the same as Adams and Shariff's [40] except that the second derivative appears. We also note that some particular forms of compact upwind schemes have been proposed in [8–10]. Lele [12] emphasized the distinction between the order of approximation and the spectral resolution and applied Pade schemes for the solution of compressible and incompressible flow problems. Mahesh [37] presented and analyzed combined compact uniform grid finite difference schemes (C-D schemes) which evaluate the first and the second derivative simultaneously. The generalized compact (GC) schemes and some of their important properties were discussed by Shen et al. [11]. The GC schemes can be considered as a more general version of the standard Pade schemes discussed by Lele [12]. The appearance of higher derivatives in both C-D schemes and GC schemes gives rise to the spectral accuracy of the first derivative. The family of compact upwind schemes developed in this study can be categorized as GC schemes. They have a speed advantage as well as higher resolution in comparison to other compact schemes. Often, the tri-diagonal matrix inversion is required in applications of the commonly used compact schemes. However, using the proposed schemes, one can sequentially calculate all the derivatives by explicit means if the boundary value problem is solved. When our schemes are applied to periodic problems, only periodic bi-diagonal matrix inversions or periodic block-bi-diagonal matrix inversions are required.

Sengupta et al. [36] pointed out that the definition of G–K–S stability [38] (also known as Lax stability) might be too weak and hence it would not be a practical option to use those compact schemes for DNS, CAA and CEM, which are only G–K–S stable. They also developed a Fourier–Laplace transform based spectral method for the purpose of evaluating the spectral resolution, numerical stability and dispersion relation preservation (DRP) property of any discrete computing technique. Using this analysis, some well-known compact schemes that were found to be G–K–S and time stable are shown to be unstable for selective length scales [36]. Subsequently, further improvements of this analysis method were obtained in [49–51] to provide results for different spatial and temporal discretization methods. In the following sections, we refer to these schemes as full spatial stable (referred to later on in this paper as FS-stable for short) if they are stable for all interior and boundary points analyzed in isolation by this method. We consider the linear wave equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{1.1}$$

as a model equation, working in the computational plane with a uniform grid of size h and total point number N. It is assumed that computations with nonuniform grids can define analytical mappings between the nonuniform grid and a corresponding uniform grid. The metrics of the mapping may then be used to relate the derivatives on the uniform grid to those on the nonuniform grid. If the Dirichlet boundary condition is adopted on the first point, the FS-stability means that spatial discretization schemes are all stable for points $j = 2, 3, \ldots, N$. In the present paper, our consideration is confined to this situation. In order to achieve FS-stability, a new method for constructing stable boundary schemes is developed. Numerical examples show that schemes that possess the property of FS-stability are asymptotically stable both in the scalar and system case.

If the flow fields involve shock waves, schemes should also be essentially oscillation-free near the discontinuities. In recent years, many efforts have been devoted to the development of high resolution shock-capturing