On regular polytopes of 2-power order

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Abstract

For each $d \geq 3$, $n \geq 5$, and $k_1, k_2, \ldots, k_{d-1} \geq 2$ with $k_1 + k_2 + \ldots + k_{d-1} \leq n-1$, we show how to construct a regular d-polytope whose automorphism group is of order 2^n and whose Schläfli type is $\{2^{k_1}, 2^{k_2}, \ldots, 2^{k_{d-1}}\}$.

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1 Introduction

In the 1960s, Branko Grünbaum suggested to the geometric community to study generalizations of the concept of regular polytopes that he called polystromata. His work greatly influenced Ludwig Danzer and Egon Schulte who developed, along those lines, the theory of what are now called abstract regular polytopes. The comprehensive book written by Peter McMullen and Egon Schulte [25] is nowadays seen as the reference on the subject.

Abstract polytopes are a generalization of the classical notion of convex geometric polytopes to more general structures. The highly symmetric examples are the most studied. They include not only classical regular polytopes such as the Platonic solids, but also non-degenerate regular maps on surfaces. Another famous example is the 11-cell that Grünbaum discovered in 1977 [18] by gluing together eleven hemi-icosahedra in such a way that the "geometry around each vertex" would look like a hemi-dodecahedron.

An abstract regular polytope is a partially-ordered set endowed with a rank function, satisfying certain conditions that arise naturally from a geometric setting. The order of the automorphism group of a regular polytope is also called the *order* of the regular polytope.

Over the years, an impressive amount of work has been produced in the study of abstract regular polytopes. It is impossible to exhaustively list everything here. One of the paths that has been followed, and in which this work stands, was to produce experimental data and then analyse the data to discover theoretical results. The authors would like to single out reference [23] which

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was accepted for publication by Grünbaum, who was a strong supporter of that approach of collecting experimental data, analyzing them and stating conjectures that could then be proved by developing new mathematical tools. We refer to [3, 4, 5, 9, 12, 13, 14, 15, 16, 20, 21, 22] for examples of such works. The atlas [7] contains information about all regular polytopes with order at most 2000. Most groups appearing in this atlas are soluble groups. There are not that many theoretical constructions of regular polytopes for soluble groups (see for instance [17, 19, 24]). In [26] Schulte and Weiss proposed the following problem.

Problem 1.1 *For any positive integer n, characterize regular polytopes of order 2^n.*

The number of pairwise nonisomorphic groups of order 2^n grows extremely rapidly with n. For instance Besche, Eick and O'Brien announced in [1] that there are 49 487 365 422 groups of order 2^{10}. Hence an analysis of these groups one by one is hopeless. However, Problem 1.1 was solved for groups of order 2^{10} by Conder in [7] using low index normal subgroups of finitely presented groups.

Let $d \geq 3$, $n \geq 10$, and $k_1, k_2, \ldots, k_{d-1} \geq 2$. Let $\mathcal{P}$ be a regular d-polytope of order 2^n and type $\{2^{k_1}, 2^{k_2}, \ldots, 2^{k_{d-1}}\}$. We know, from Conder [6] (see Proposition 2.2 below), that $k_1 + k_2 + \ldots + k_{d-1} \leq n - 1$. Moreover Conder proved that for every odd integer n there exists a regular polytope of order 2^n. For even values of n, existence of such polytopes follows from [10, Theorem 6.3] (see also [8, Theorem 3.4]). Cunningham and Pellicer [11] classified regular 3-polytopes with $k_1 + k_2 = n - 1$, and the authors constructed in [19] a regular 3-polytope for each k_1, k_2, n with $k_1 + k_2 \leq n - 1$. These results contributed to the study of Problem 1.1.

Computer searches for small values of n suggest that Problem 1.1 is a very difficult one. Nevertheless, in this paper, we contribute to this problem in the following way. For each $d \geq 4$, we construct a regular d-polytope for each $k_1, k_2, \ldots, k_{d-1}, n$ with $k_1 + k_2 + \ldots + k_{d-1} \leq n - 1$, showing that all possible Schläfli types can be achieved for regular polytopes of order 2^n. Our main theorem can be stated as follows.

Theorem 1.2 *For any integers $d, n, k_1, k_2, \ldots, k_{d-1}$ such that $d \geq 3$, $n \geq 5$, $k_1, k_2, \ldots, k_{d-1} \geq 2$ and $k_1 + k_2 + \ldots + k_{d-1} \leq n - 1$, there exists a string C-group $(G, \{\rho_0, \rho_1, \ldots, \rho_{d-1}\})$ of order 2^n and type $\{2^{k_1}, 2^{k_2}, \ldots, 2^{k_{d-1}}\}$.*

The paper is organized as follows. In Section 2, we give the necessary definitions to understand this paper and we recall some results that we use in Section 3 to prove Theorem 1.2.

2 Background definitions and preliminaries

Abstract regular polytopes with a chosen base flag are in one-to-one correspondence with string C-groups; see [25, Section 2E]. We take here the viewpoint of string C-groups because it is the easiest and the most efficient one to define abstract regular polytopes. Let G be a group and let $S = \{\rho_0, \ldots, \rho_{d-1}\}$ be a generating set of involutions of G. For $I \subseteq \{0, \ldots, d-1\}$, let G_I denote the group generated by $\{\rho_i : i \in I\}$. Suppose that

* for any $i, j \in \{0, \ldots, d-1\}$ with $|i - j| > 1$, ρ_i and ρ_j commute (the *string property*);
* for any $I, J \subseteq \{0, \ldots, d-1\}$, $G_I \cap G_J = G_{I \cap J}$ (the *intersection property*).

Then the pair (G, S) is called a *string C-group of rank d* and the *order* of (G, S) is simply the order of G. The *type* of (G, S) is the ordered set $\{k_1, \ldots, k_{d-1}\}$, where k_i is the order of $\rho_{i-1}\rho_i$. It is natural to assume that each k_i is at least 3 for otherwise the generated group is a direct product of two smaller groups and the corresponding string C-group is called *degenerate*. By the intersection property, S is a minimal generating set of G.

If (G, S) only satisfies the string property, it is called a *string group generated by involutions*, or an *sggi* for short. The following proposition is called the *quotient criterion* for a string C-group.

Proposition 2.1 [25, Theorem 2E17] *Let* $\Gamma = \langle \rho_0, \rho_1, \ldots, \rho_{d-1}\rangle$ *be an sggi, and let* $(\Delta, \{\sigma_0, \sigma_1, \ldots, \sigma_{d-1}\})$ *be a string C-group. If the mapping* $\rho_j \mapsto \sigma_j$ *for* $j = 0, \ldots, d-1$ *induces a homomorphism* $\pi : \Gamma \to \Delta$*, which is one-to-one on the subgroup* $\Gamma_{d-1} = \langle \rho_0, \rho_1, \ldots, \rho_{d-2}\rangle$ *or on* $\Gamma_0 = \langle \rho_1, \rho_2, \ldots, \rho_{d-1}\rangle$*, then* $(\Gamma, \{\rho_0, \rho_1, \ldots, \rho_{d-1}\})$ *is also a string C-group.*

Let (G, S) be a string C-group and let $\mathcal{P}$ be its corresponding regular polytope. Then the rank, the order, and the type of (G, S) mean the *rank*, the *order*, and the *(Schläfli) type* of $\mathcal{P}$, respectively. A regular polytope $\mathcal{P}$ is called a regular *d-polytope*, if $\mathcal{P}$ has rank d. Conder [6] obtained a lower bound for the order of a regular polytope.

Proposition 2.2 [6, Theorem 3.2] *If* $\mathcal{P}$ *is a regular d-polytope of type* $\{k_1, k_2, \ldots, k_{d-1}\}$*, then* $\mathcal{P}$ *has order at least* $2k_1k_2\ldots k_{d-1}$.

If the lower bound in Proposition 2.2 is attained, $\mathcal{P}$ is called *tight*. A *string Coxeter group* $[k_1, k_2, \ldots, k_{d-1}]$ is defined as the following group:

$$\langle \rho_0, \rho_1, \ldots, \rho_{d-1} \mid \quad \rho_i^2 = 1 \text{ for } 0 \le i \le d-1, (\rho_i\rho_{i+1})^{k_{i+1}} = 1 \text{ for } 0 \le i \le d-2, \\ (\rho_i\rho_j)^2 = 1 \text{ for } 0 \le i < j-1 < d-1\rangle.$$

Proposition 2.3 [6, Theorem 5.3] *For every sequence* $(k_1, k_2, \ldots, k_{d-1})$ *of* $d-1$ *even integers greater than 2, there exists a tight regular d-polytope* $\mathcal{P}$ *of order* $2k_1k_2\ldots k_{d-1}$ *and type* $\{k_1, k_2, \ldots, k_{d-1}\}$*. In particular, one can take the string Coxeter group* $[k_1, k_2, \ldots, k_{d-1}]$ *with standard generators* $\rho_0, \rho_1, \ldots, \rho_{d-1}$*, and let* $\mathrm{Aut}(\mathcal{P})$ *be the quotient obtained by adding all relations of the form* $[\rho_i, (\rho_{i+1}\rho_{i+2})^2] = 1$ *for* $0 \le i \le d-3$ *and* $[(\rho_i\rho_{i+1})^2, \rho_{i+2}]$ *for* $0 \le i \le d-3$.

The following proposition gives some constructions for string C-groups of order 2^n.

Proposition 2.4 [19, Theorem 1.2] *Let* $n \ge 10$*,* $s, t \ge 2$ *and* $n-s-t \ge 1$*. Set* $R = \{\rho_0^2, \rho_1^2, \rho_2^2, (\rho_0\rho_1)^{2^s}, (\rho_1\rho_2)^{2^t}, (\rho_0\rho_2)^2, [(\rho_0\rho_1)^4, \rho_2], [\rho_0, (\rho_1\rho_2)^4]\}$ *and define*

$$H = \begin{cases} \langle \rho_0, \rho_1, \rho_2 \mid R, [(\rho_0\rho_1)^2, \rho_2]^{2^{\frac{n-s-t-1}{2}}}\rangle & \text{when } n-s-t \text{ is odd}, \\ \langle \rho_0, \rho_1, \rho_2 \mid R, [(\rho_0\rho_1)^2, (\rho_1\rho_2)^2]^{2^{\frac{n-s-t-2}{2}}}\rangle & \text{when } n-s-t \text{ is even}. \end{cases}$$

Then $(H, \{\rho_0, \rho_1, \rho_2\})$ *is a string C-group of order* 2^n *and type* $\{2^s, 2^t\}$.

Let G be a group. For $a, b \in G$, we use $[a, b]$ as an abbreviation for the *commutator* $a^{-1}b^{-1}ab$ of a and b. The following proposition is a basic property of commutators and its proof is straightforward.

Proposition 2.5 *Let* G *be a group. Then, for any* $a, b, c \in G$*,* $[ab, c] = [a, c]^b[b, c]$ *and* $[a, bc] = [a, c][a, b]^c$.

Finally, we will also use the following result in the proof of our theorem.

Proposition 2.6 *Let $H = \langle a, b, c\rangle$ be a group such that $a^2 = b^2 = c^2 = (ac)^2 = [(ab)^2, c] = 1$. Then $[a, (bc)^2] = [a, (bc)^4] = [(ab)^4, c] = 1$, $\langle (ab)^2\rangle \trianglelefteq H$ and $\langle (bc)^2\rangle \trianglelefteq H$.*

Proof: Since $(ac)^2 = 1$, we have $[a, c] = 1$. By Proposition 2.5, $1 = [(ab)^2, c] = [a, c]^{bab}[bab, c] = [bab, c] = [a, bcb]^b = [a, bcbcc]^b = [a, c]^b[a, (bc)^2]^{cb} = [a, (bc)^2]^{cb}$, that is, $[a, (bc)^2] = 1$. It follows that $[(ab)^4, c] = 1$ and $[a, (bc)^4] = 1$. Note that $(ab)^a = ba = (ab)^{-1}$ and $(ab)^b = ba = (ab)^{-1}$. Since $H = \langle a, b, c\rangle$, we have $\langle (ab)^2\rangle \trianglelefteq H$, and similarly, $\langle (bc)^2\rangle \trianglelefteq H$, as required. □

3 Proof of Theorem 1.2

The proof of this theorem is constructive. The cases where $5 \leq n \leq 9$ can be checked using Conder's list [7]. Let $d, n, k_1, k_2, \ldots, k_{d-1}$ be integers such that $d \geq 3$, $n \geq 10$, and $k_1, k_2, \ldots, k_{d-1} \geq 2$ with $k_1 + k_2 + \ldots + k_{d-1} \leq n - 1$. Define

$$G = \begin{cases} \langle \rho_0, \rho_1, \ldots, \rho_{d-1} \mid R_1, R_2, R_3, [(\rho_{d-3}\rho_{d-2})^2, \rho_{d-1}]^{2^{\frac{l-1}{2}}}\rangle & \text{for } l \text{ odd}, \\ \langle \rho_0, \rho_1, \ldots, \rho_{d-1} \mid R_1, R_2, R_3, [(\rho_{d-3}\rho_{d-2})^2, (\rho_{d-2}\rho_{d-1})^2]^{2^{\frac{l-2}{2}}}\rangle & \text{for } l \text{ even}, \end{cases}$$

where

$$R_1 = \{\rho_i^2, (\rho_j\rho_k)^2, (\rho_h\rho_{h+1})^{2^{k_{h+1}}} \mid 0 \leq i \leq d-1,\ \ 0 \leq j < k-1 \leq d-2,\ \ 0 \leq h \leq d-2\}, \quad (1)$$
$$R_2 = \{[(\rho_i\rho_{i+1})^2, \rho_{i+2}] \mid 0 \leq i \leq d-4\}\}, \quad (2)$$
$$R_3 = \{[\rho_{d-3}, (\rho_{d-2}\rho_{d-1})^4], [(\rho_{d-3}\rho_{d-2})^4, \rho_{d-1}]\}, \quad (3)$$

and $l = n - (k_1 + k_2 + \ldots + k_{d-1})$. Note that $l \geq 1$.

The case where $d = 3$ has been dealt with in Proposition 2.4. Hence we may assume from now on that $d \geq 4$. For convenience, write $o(g)$ for the order of g in G.

Let

$$K = \langle \rho_0, \rho_1, \ldots, \rho_{d-1} \mid R_1, R_2, [(\rho_{d-3}\rho_{d-2})^2, \rho_{d-1}]\rangle,$$

where R_1 and R_2 are given by equations (1) and (2). Then $[(\rho_i\rho_{i+1})^2, \rho_{i+2}] = 1$ for $0 \leq i \leq d-3$ in K, and by Proposition 2.6, $[\rho_i, (\rho_{i+1}\rho_{i+2})^2] = 1$. Then $(K, \{\rho_0, \rho_1, \ldots, \rho_{d-1}\})$ is a string C-group of order $2^{1+k_1+k_2+\ldots+k_{d-1}}$ by Proposition 2.3, with type $\{2^{k_1}, 2^{k_2}, \ldots, 2^{k_{d-1}}\}$. In particular, the listed exponents are the true orders of the corresponding elements in K, that is, $o(\rho_i) = 2$ for $0 \leq i \leq d-1$, $o(\rho_i\rho_{i+1}) = 2^{k_{i+1}}$ for $0 \leq i \leq d-2$ and $o(\rho_i\rho_j) = 2$ for $0 \leq i < j-1 < d-1$.

Note that $[(\rho_{d-3}\rho_{d-2})^2, \rho_{d-1}] = 1$ in K. Proposition 2.6 implies that

$$[(\rho_{d-3}\rho_{d-2})^4, \rho_{d-1}] = [\rho_{d-3}, (\rho_{d-2}\rho_{d-1})^4] = 1$$

in K. Furthermore,

$$[(\rho_{d-3}\rho_{d-2})^2, (\rho_{d-2}\rho_{d-1})^2] = [(\rho_{d-3}\rho_{d-2})^2, \rho_{d-1}][(\rho_{d-3}\rho_{d-2})^2, \rho_{d-2}\rho_{d-1}\rho_{d-2}]^{\rho_{d-1}}$$
$$= [(\rho_{d-3}\rho_{d-2})^{-2}, \rho_{d-1}]^{\rho_{d-2}\rho_{d-1}} = 1.$$

In particular, in K we have $[(\rho_{d-3}\rho_{d-2})^2, \rho_{d-1}]^{2^{\frac{l-1}{2}}} = 1$ when l is odd, and

$$[(\rho_{d-3}\rho_{d-2})^2, (\rho_{d-2}\rho_{d-1})^2]^{2^{\frac{l-2}{2}}} = 1$$

when l is even. Thus $\rho_0, \rho_1, \ldots, \rho_{d-1}$ in K satisfy the same relations as do $\rho_0, \rho_1, \ldots, \rho_{d-1}$ in G, and hence the map: $\rho_i \mapsto \rho_i$, $0 \leq i \leq d-1$, induces an epimorphism α from G to K. This also implies, in G, that $o(\rho_i) = 2$ for $0 \leq i \leq d-1$, $o(\rho_i\rho_{i+1}) = 2^{k_{i+1}}$ for $0 \leq i \leq d-2$, and $o(\rho_i\rho_j) = 2$ for $0 \leq i < j-1 < d-1$, because these are true in K.

Let $K_1 = \langle \rho_0, \rho_1, \ldots, \rho_{d-2} \rangle \leq K$ and $G_1 = \langle \rho_0, \rho_1, \ldots, \rho_{d-2} \rangle \leq G$. Then the restriction $\alpha|_{G_1}$ of α on G_1 is an epimorphism from G_1 to K_1. Now we prove that $\alpha|_{G_1}$ is actually an isomorphism from G_1 to K_1. To do that, let

$$L = \langle \rho_0, \rho_1, \ldots, \rho_{d-2} \mid \overline{R}_1, R_2 \rangle,$$

where $\overline{R}_1 = \{\rho_i^2, (\rho_j\rho_k)^2, (\rho_h\rho_{h+1})^{2^{k_{h+1}}} \mid 0 \leq i \leq d-2, 0 \leq j < k-1 \leq d-3, 0 \leq h \leq d-3\}$ and R_2 is given by equation (2). Now, an easy argument similar to the argument used for K shows that $|L| = 2^{1+k_1+k_2+\ldots+k_{d-2}}$.

Clearly, $\rho_0, \rho_1, \ldots, \rho_{d-2}, 1$ in L satisfy the same relations as do $\rho_0, \rho_1, \ldots, \rho_{d-2}, \rho_{d-1}$ in K, and therefore, the map $\rho_i \mapsto \rho_i$ for $0 \leq i \leq d-2$, with $\rho_{d-1} \mapsto 1$, induces an epimorphism from K to L, whose restriction on K_1 is an epimorphism from K_1 to L. On the other hand, $\rho_0, \rho_1, \ldots, \rho_{d-2}$ in $K_1 \leq K$ satisfy the same relations as do $\rho_0, \rho_1, \ldots, \rho_{d-2}$ in L, and hence there is an epimorphism from L to K_1. These facts yield that the map $\rho_i \mapsto \rho_i$ for $0 \leq i \leq d-2$ induces an isomorphism from K_1 to L, because $|L|$ is finite.

Similarly, $\rho_0, \rho_1, \ldots, \rho_{d-2}, 1$ in L satisfy the same relations as do $\rho_0, \rho_1, \ldots, \rho_{d-2}, \rho_{d-1}$ in G, and $\rho_0, \rho_1, \ldots, \rho_{d-2}$ in $G_1 \leq G$ satisfy the same relations as do $\rho_0, \rho_1, \ldots, \rho_{d-2}$ in L. A similar argument to the one of the above paragraph permits to conclude that the map $\rho_i \mapsto \rho_i$ for $0 \leq i \leq d-2$ induces an isomorphism from G_1 to L. This, together with the isomorphism from K_1 to L in the above paragraph, implies that $\alpha|_{G_1}$ is an isomorphism from G_1 to K_1.

Since $(K, \{\rho_0, \rho_1, \ldots, \rho_{d-1}\})$ is a string C-group, $(G, \{\rho_0, \rho_1, \ldots, \rho_{d-1}\})$ is a string C-group by Proposition 2.1, which has type $\{2^{k_1}, 2^{k_2}, \ldots, 2^{k_{d-1}}\}$. To finish the proof, we are only left to show that $|G| = 2^n$. We prove this by induction on d. It is true for $d = 3$ by Proposition 2.4, and we may let $d \geq 4$.

Let $N = \langle (\rho_0\rho_1)^2 \rangle \leq G$. Since $o(\rho_0\rho_1) = 2^{k_1}$ in G, we have $|N| = 2^{k_1-1}$. By Proposition 2.6, $N \trianglelefteq G$, because $[\rho_0, \rho_j] = [\rho_1, \rho_j] = 1$ for any $j \geq 3$. Clearly, $G/N \cong M$ with

$$M = \begin{cases} \langle \rho_0, \rho_1, \ldots, \rho_{d-1} \mid R_1, R_2, R_3, [(\rho_{d-3}\rho_{d-2})^2, \rho_{d-1}]^{2^{\frac{l-1}{2}}}, (\rho_0\rho_1)^2 \rangle & \text{for } l \text{ odd,} \\ \langle \rho_0, \rho_1, \ldots, \rho_{d-1} \mid R_1, R_2, R_3, [(\rho_{d-3}\rho_{d-2})^2, (\rho_{d-2}\rho_{d-1})^2]^{2^{\frac{l-2}{2}}}, (\rho_0\rho_1)^2 \rangle & \text{for } l \text{ even,} \end{cases}$$

where R_1, R_2, R_3 are given by equations (1), (2) and (3). Write $M_1 = \langle \rho_1, \rho_2, \ldots, \rho_{d-1} \rangle \leq M$.

Since $o(\rho_0\rho_1) = 2^{k_1} \geq 4$ in G, $\langle \rho_0, \rho_1 \rangle$ is a dihedral group of order $2^{k_1+1} \geq 8$, implying that $\rho_0 \notin N$ and $\rho_1 \notin N$. In particular, $o(\rho_0 N) = 2$ in G/N, and therefore, $o(\rho_0) = 2$ in M. Since $(G, \{\rho_0, \rho_1, \ldots, \rho_{d-1}\})$ is a string C-group, $\langle \rho_1, \rho_2, \ldots, \rho_{d-1} \rangle \cap \langle \rho_0 \rangle N \leq \langle \rho_1, \rho_2, \ldots, \rho_{d-1} \rangle \cap \langle \rho_0, \rho_1 \rangle = \langle \rho_1 \rangle$. If $\rho_1 \in \langle \rho_0 \rangle N$ then either $\rho_1 \in N$ or $\rho_0\rho_1 \in N$, both of which are impossible. It follows that $\langle \rho_1, \rho_2, \ldots, \rho_{d-1} \rangle \cap \langle \rho_0 \rangle N = 1$ in G, and hence $\langle \rho_1, \rho_2, \ldots, \rho_{d-1} \rangle N \cap \langle \rho_0 \rangle N = N(\langle \rho_1, \rho_2, \ldots, \rho_{d-1} \rangle \cap \langle \rho_0 \rangle N) = N$ in G/N. This implies that $\langle \rho_0 \rangle \cap M_1 = 1$ in M. Since $(\rho_0\rho_j)^2 = 1$ in M for any $j \geq 2$, we have $M = \langle \rho_0 \rangle \times M_1$ and hence $|M| = 2|M_1|$.

Let

$$A = \begin{cases} \langle \rho_1, \ldots, \rho_{d-1} \mid R_1^-, R_2^-, R_3, [(\rho_{d-3}\rho_{d-2})^2, \rho_{d-1}]^{2^{\frac{l'-1}{2}}} \rangle & \text{for } l' \text{ odd,} \\ \langle \rho_1, \ldots, \rho_{d-1} \mid R_1^-, R_2^-, R_3, [(\rho_{d-3}\rho_{d-2})^2, (\rho_{d-2}\rho_{d-1})^2]^{2^{\frac{l'-2}{2}}} \rangle & \text{for } l' \text{ even,} \end{cases}$$

where $R_1^- = \{\rho_i^2, (\rho_j\rho_k)^2, (\rho_h\rho_{h+1})^{2^{k_{h+1}}} \mid 1 \leq i \leq d-1, 1 \leq j < k-1 \leq d-2, 1 \leq h \leq d-2\}$, $R_2^- = \{[(\rho_i\rho_{i+1})^2, \rho_{i+2}] \mid 1 \leq i \leq d-4\}\}$ and R_3 is given by equation (3). Note that $l' = (n-k_1) - k_2 - \ldots - k_{d-1}$.

Let $n - k_1 \geq 10$. Since $(G, \{\rho_0, \rho_1, \ldots, \rho_{d-1}\})$ is a string C-group, by taking $n - k_1$ in A as n in G, $(A, \{\rho_1, \rho_2, \ldots, \rho_{d-1}\})$ is a string C-group of rand $d-1$. Then the inductive hypothesis implies that $|A| = 2^{n-k_1}$. Now we claim that this is also true for $n - k_1 \leq 9$.

Note that $d-1 \geq 3$, $l' \geq 1$ and $k_1, k_2, \ldots, k_{d-1} \geq 2$. Since $1+2(d-2) \leq l'+k_2+k_3+\ldots+k_{d-1} = n-k_1 \leq 9$, we have $3 \leq d-1 \leq 5$ and $5 \leq n-k_1 \leq 9$.

First assume $d-1 = 3$. Then $A = \langle \rho_1, \rho_2, \rho_3 \rangle$ and $l' = (n-k_1) - k_2 - k_3$. If $n - k_1 = 5$, then $(l', k_2, k_3) = (1, 2, 2)$, and using Magma [2] we easily check that $|A| = 2^5 = 2^{n-k_1}$. If $n - k_1 = 6$, then $(l', k_2, k_3) \in \{(1,2,3), (1,3,2), (2,2,2)\}$; if $n - k_1 = 7$, then $(l', k_2, k_3) \in \{(1,2,4), (1,4,2), (1,3,3), (2,2,3), (2,3,2), (3,2,2)\}$; if $n - k_1 = 8$, then $(l', k_2, k_3) \in \{(1,2,5), (1,5,2), (1,3,4), (1,4,3), (2,2,4), (2,4,2), (2,3,3), (3,2,3), (3,3,2), (4,2,2)\}$; if $n - k_1 = 9$, then $(l', k_2, k_3) \in \{(1,2,6), (1,6,2), (1,3,5), (1,5,3), (1,4,4), (2,2,5), (2,5,2), (2,3,4), (2,4,3), (3,2,4), (3,4,2), (3,3,3), (4,2,3), (4,3,2), (5,2,2)\}$. For each (l', k_2, k_3), Magma computations show that $|A| = 2^{n-k_1}$.

Assume $d-1 = 4$. Then $A = \langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ and $7 \leq (n-k_1) = l'+k_2+k_3+k_4 \leq 9$. If $n-k_1 = 7$, then $(l', k_2, k_3, k_4) = (1,2,2,2)$; if $n-k_1 = 8$, then $(l', k_2, k_3, k_4) \in \{(1,3,2,2), (1,2,3,2), (1,2,2,3), (2,2,2,2)\}$; if $n-k_1 = 9$, then $(l', k_2, k_3, k_4) \in \{(1,4,2,2), (1,2,4,2), (1,2,2,4), (1,3,3,2), (1,3,2,3), (1,2,3,3), (2,3,2,2), (2,2,3,2), (2,2,2,3), (3,2,2,2)\}$.

Assume $d-1 = 5$. Then $A = \langle \rho_1, \rho_2, \rho_3, \rho_4, \rho_5 \rangle$ and $n-k_1 = 9$; furthermore, $(l', k_2, k_3, k_4, k_5) = (1,2,2,2,2)$. Again using Magma [2], for each case we have $|A| = 2^{n-k_1}$, as claimed.

Clearly, $1, \rho_1, \rho_2, \ldots, \rho_{d-1}$ in A satisfy the same relations as $\rho_0, \rho_1, \rho_2, \ldots, \rho_{d-1}$ in M. Thus the map $\rho_0 \mapsto 1$, $\rho_i \mapsto \rho_i$ for $1 \leq i \leq d-1$, induces an epimorphism β from M to A and hence the restriction $\beta|_{M_1}$ is an epimorphism from M_1 to A. On the other hand, $\rho_1, \rho_2, \ldots, \rho_{d-1}$ in $M_1 \leq M$ satisfy the same relations as $\rho_1, \rho_2, \ldots, \rho_{d-1}$ in A, and therefore, there is an epimorphism from A to M_1. Thus, $\beta|_{M_1}$ is an isomorphism from M_1 to A and in particular, $|M_1| = |A| = 2^{n-k_1}$. Now, we have $|G| = |G/N||N| = |M||N| = 2|M_1||N| = 2 \cdot 2^{n-k_1} \cdot 2^{k_1-1} = 2^n$. This completes the proof. □

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