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On portfolio analysis using oriented fuzzy numbers for the trade-related sector of the Warsaw Stock Exchange

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Abstract

Oriented fuzzy numbers are useful in portfolio management since they convey information regarding uncertainty and imprecision when considering financial markets. One may apply a fuzzy discount factor and an imprecise present value in the form of a trapezoidal oriented fuzzy number. An investor can obtain recommendations on individual stocks (buy, sell, accumulate, reduce). Analogous recommendations are also issued by experts. In such cases, recommendations are mostly based on available data and expert's knowledge and experience. The purpose of the paper is to present a procedure for comparing the accuracy of both types of recommendations. Also, the real impact the recommendations have on potential changes in portfolio composition in trading-related industries is considered. The research uses quotations from companies from the trading sector of the Warsaw Stock Exchange (WSE). Theoretical considerations are presented in the form of an empirical case study.

Keywords: *oriented fuzzy numbers, recommendations, portfolio analysis, imprecision, fuzzy discount factor*

1. Introduction

The objective of the research is to show that the use of a strategy based on oriented fuzzy numbers (OFNs) when establishing the investment recommendation, can generate better results than the decision based on the recommendations of brokers or a passive strategy. The authors also wanted to answer a question about which results would be the least imprecise. The main purpose of the paper is to implement an innovative technique of portfolio composition. In decision-making, investors might use fundamental or technical analysis if they have the appropriate educational background and are capable of identifying all the factors that contribute to the right decision. The range of factors that can influence the investment decision-making process varies depending on the person [11]. However, individual investors do not always possess the necessary knowledge and experience to run a quantitative analysis before they make a decision. Therefore, they frequently seek a reliable source of information regarding a particular company

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or sector. Also, very often they prefer the information to be expressed in a natural language as it is easier for them to recognize and interpret it.

Linguistic information can take all sorts of forms, one of which is the recommendation of stockbrokers. It is their knowledge and experience that the investors seek [32, 34]. In [14] the research into what factors influence investors' short-term decisions to hold or sell company stocks based on expert recommendations was carried out. Also in [2] the impact of psychology in decision-making was discussed focusing on factors, psychological and emotional, that influenced final decisions. A possible implication of understanding the impact of behavioural finance on decision-making, including the opportunity to learn (based on the past investment decision mistakes) was described in [3, 10]. An analysis of the long-term stock performance of recommendations and the properties of accompanied earnings forecasts for initiations and non-initiations to evaluate the reporting, selection, and processing explanations for analyst optimism were presented in [5]. Significant discrepancies between the recommendations given by experts and the actual price were discussed in [30]. It proved that "buy" recommendations were more beneficial for investors if they acted promptly but also that the initial price reactions were incomplete. A similar analysis was shown in [6] where the actual investment value of short-term recommendations was considered.

Most of the research mentioned above shows and discusses stockbrokers' recommendations from the point of view of possible earnings and investors' income. The analysis in this paper, proposed by the authors, also takes the earnings into account, however, it departs from most of the work in the area by focusing on imprecision because in times of a bearish market or in unfavourable economic times, the gains can be non-existence and then the investors must limit their losses. Both experience and knowledge and subsequently the recommendations are usually imprecise and inaccurate as imprecision is one of the characteristics of any financial market information [36]. When dealing with imprecision and inaccuracy of recommendations and following [9], it can be stated that an application of imprecise linguistic assessments for decision analysis is beneficial as it implements a flexible framework for representing the given information more directly and adequately. Linguistic labels always have an individual semantic value belonging to a given linguistic term set. Each label may be transformed into a quantitative form using a given methodology as reviewed in [8, 33].

An ex-ante analysis can be carried out to evaluate the efficiency of a recommendation, taking into account that any unknown future state of affairs is uncertain [20, 21]. It must be assumed that the recommendation may or may not be realised in the future and, consequently, it can be assessed if the information was correct or incorrect. We can model with a certain probability the uncertainty derived from the ignorance of future potential states of affairs. To do so we have to point out a specific time when the given state of affairs is known to the observer. Such an approach enables the use of trapezoidal oriented fuzzy numbers (TrOFNs) while making a portfolio investment decision [18–20, 28, 31, 35]. In the paper, individual companies listed on the Warsaw Stock Exchange (WSE) from trade-related sectors are included in a portfolio. Present Values (PVs) are also described by trapezoidal oriented fuzzy numbers. It should be noted that the analysis using a fuzzy discount factor is less complex than the one based on a return rate [22, 28].

By definition, any security gives the owner a right to obtain future financial profit payable at a specified time (maturity). The profit value may be represented by an anticipated future value of capital. If the present value of future cash flows is regarded as an approximated value, then TrOFNs are applicable.

The main advantage of TrOFNs is that they are simpler in the case of basic operations. A multi- or a single-asset financial portfolio is regarded as an arbitrary, finite set of assets, elements of which can be positively or negatively oriented. A considered asset is understood as a fixed security in a long position. The procedure presented in the paper starts with a test of the fuzzy discount factor. Next, a weighted sum of positively/negatively oriented discount factors is calculated separately because adding oriented fuzzy numbers is not associative. Then, a portfolio discount factor is calculated as a weighted addition of both sums. In our procedure, we apply energy and entropy measures to estimate the imprecision risk of analysed investment portfolios.

The paper is organized as follows. Introduction 1 is followed by Section 2 which presents a brief review of trapezoidal fuzzy numbers and their main characteristics. Moreover, the imprecision of oriented fuzzy numbers (energy and entropy measures) is described. In Section 3 we present the implemented procedure in a sequence of consecutive steps which allows us to compare the accurateness of different kinds of recommendations. The notion of imprecisely estimated PVs and expected discount factor is shown. In Section 4 case studies presenting all strategies are compared: (1) a fuzzy portfolio approach based on TrOFNs S1, (2) a stockbrokers' recommendation strategy S2 and (3) a passive portfolio strategy S3. Section 5 concludes the findings of the research.

2. The origin of trapezoidal fuzzy numbers

Fuzzy numbers (FNs), initially introduced by Dubois and Prade [4], constitute the largest set of numbers in fuzzy theory. We can divide them into smaller groups depending on their characteristics, e.g., ordered fuzzy numbers and oriented fuzzy numbers [27].

2.1. Oriented fuzzy numbers

Kosiński [12, 13] presented ordered fuzzy numbers. The numbers were introduced in an intuitive manner as a fuzzy number model of an imprecise number. Ordered fuzzy number S was determined for any sequence of numbers $(a,\, b,\, c,\, d)$ as an ordered pair of continuous real functions $(f_S,\, g_S)$ defined on the interval $[0,\, 1]$, i.e., $f_S : [0,\, 1] \to \mathrm{UP_s}$ and $g_S : [0,\, 1] \to \mathrm{DOWN_s}$, where $\mathrm{UP_s} = [a,\, b]$ and $\mathrm{DOWN_s} = [c,\, d]$. Functions f_S and g_S, are respectively called rising (UP) and falling (DOWN), which satisfy the conditions: $f_S\,(0) = a$, $f_S\,(1) = b$, $g_S\,(1) = c$, $g_S\,(0) = d$. Currently, the above numbers are regarded as Kosński's numbers.

However, further research exposed a significant drawback of Kosiński's theory as it was proved that there exist such ordered fuzzy numbers which are not fuzzy numbers. Therefore, the notion of ordered fuzzy numbers was revised by Piasecki [23]. Eventually, the shortcomings of Kosiński's theory resulted in distinguishing a group of oriented fuzzy numbers (OFNs) according to Definition 1.

Definition 1. For any monotonic sequence $(a,\, b,\, c,\, d) \subset \mathbb{R}$, oriented fuzzy number – OFN $\overleftrightarrow{\mathcal{L}} = \overleftrightarrow{\mathcal{L}}\,(a,\, b,\, c,\, d, S_L, E_L)$ is a pair composed of an orientation $\overrightarrow{a,\, d} = (a,\, d)$ and a fuzzy number defined by its membership function $\mu_L(\cdot|a,\ b,\ c,\ d, S_L, E_L) \in [0,\ 1]^{\mathbb{R}}$

$$\mu_L(x) = \mu_L(x|a,\ b,\ c,\ d) = \begin{cases} 0, & x \notin [a,\ d] & \equiv & [d,\ a] \\ S_L, & x \in [a,\ b[& \equiv &]a,\ b] \\ 1, & x \in [b,\ c] & \equiv & [c,\ b] \\ E_L, & x \in]c,\ d] & \equiv & [c,\ d[\end{cases} \tag{1}$$

where the starting function $S_L \in [0,\ 1]^{[a,\,b]}$ and the ending function $E_L \in [0,\ 1]^{[c,\,d]}$ are semi-continuous from above, monotone function satisfying the condition $[\mathcal{L}]_{0+} = [a,\ d]$ [23].

2.2. Trapezoidal oriented fuzzy numbers

From the notion of oriented fuzzy numbers, trapezoidal oriented fuzzy numbers (TrOFNs) can be distinguished as a special presentation of OFNs. TrOFNs, defined by (*a, b, c, d*), can be divided into positively and negatively oriented. When the orientation is positive, a rise in the value of an estimated TrOFN is expected and vice versa, when the orientation is negative, a drop in the value of an estimated TrOFN is expected which is presented as follows:

$$a < d \to (\overrightarrow{a,\ d}) \tag{2}$$

$$a > d \to (\overleftarrow{a,\ d}) \tag{3}$$

Definition of Tr OFNs proposed by Piasecki in [23] is as follows:

Definition 2. For any monotonic sequence $(a,\ b,\ c,\ d) \subset \mathbb{R}$, TrOFN $\overleftrightarrow{Tr}(a,\ b,\ c,\ d) = \overleftrightarrow{\mathcal{T}}$ is OFN $\overleftrightarrow{\mathcal{T}} \in \mathbb{K}_{Tr}$ is determined explicitly by its membership functions $\mu_T \in [0,\ 1]^{\mathbb{R}}$

$$\mu_T(x) = \mu_{T_r}(x|a,\ b,\ c,\ d) = \begin{cases} 0, & x \notin [a,\ d] \equiv [d,\ a] \\ \dfrac{x-a}{b-a}, & x \in [a,\ b[\equiv]a,\ b] \\ 1, & x \in [b,\ c] \equiv [c,\ b] \\ \dfrac{x-d}{c-d}, & x \in]c,\ d] \equiv [c,\ d[\end{cases} \tag{4}$$

For further research, we must also define arithmetic operations on fuzzy numbers (extended sum $\boxplus$ and dot product $\boxdot$):

$$\overleftrightarrow{Tr}(a,\ b,\ c,\ d) \boxplus \overleftrightarrow{Tr}(p-a, q-b, r-c, s-d)$$

$$\begin{cases} = \overleftrightarrow{Tr}(\min\{p,q\}, q, r, \max\{r,s\}), (q<r) \vee (q=r \wedge p \le s) \\ \overleftrightarrow{Tr}(\max\{p,q\}, q, r, \min\{r,s\}), (q>r) \vee (q=r \wedge p > s) \end{cases} \tag{5}$$

$$\beta \boxdot \overleftrightarrow{Tr}(a,\ b,\ c,\ d) = \overleftrightarrow{Tr}(\beta \cdot a, \beta \cdot b, \beta \cdot c, \beta \cdot d) \tag{6}$$

2.3. Energy and entropy measures

In fuzzy theory, we encounter such phenomena as ambiguity and indistinctness. Ambiguity risk results from unequivocal recommendations, while indistinctness comes from the inability to distinguish between recommended and non-recommended decisions. The higher the level of ambiguity, the higher the risk of choosing the wrong alternative decision among the available ones. To evaluate the levels of ambiguity and indistinctness we use energy and entropy measures, which are significant indicators in the fuzzy approach, as they allow us to assess the imprecision risk - estimated as a sum of ambiguity and indistinctness levels [22, 26].

In the case of TrOFN $(a,\ b,\ c,\ d)$, the entropy d and energy e measures are determined in a following manner:

$$d\left(\mathrm{TrOFN}(a,\ b,\ c,\ d)\right) = \frac{1}{2} \cdot |d + c - b - a| \tag{7}$$

$$e\left(\mathrm{TrOFN}(a,\ b,\ c,\ d)\right) = \frac{1}{4} \cdot |d - c + b - a| \tag{8}$$

In the research, the findings are determined by a relative profit (also referred to as imprecise assets benefit) defined in a portfolio approach by a function $\varpi : \left(\mathbb{K}_{Tr}\right)^2 \times [0,\ 1] \to \mathbb{K}_{Tr}$ given as follows:

$$(K, L, \lambda) = (\lambda \boxdot K) \boxplus ((1 - \lambda) \boxdot L) \tag{9}$$

Theorem 1. For any real number $\lambda \in [0,\ 1]$, we have:

- for any pair $\left(\overleftrightarrow{K}, \overleftrightarrow{L}\right) \in \left(\mathbb{K}_{Tr}^{-} \times \mathbb{K}_{Tr}^{-}\right) \cup \left(\left(\mathbb{K}_{Tr}^{+} \cup \mathbb{R}\right) \times \left(\mathbb{K}_{Tr}^{+} \cup \mathbb{R}\right)\right)$

$$d\left(\varpi\left(\overleftrightarrow{K}, \overleftrightarrow{L}, \lambda\right)\right) = \lambda \cdot d\left(\overleftrightarrow{K}\right) + (1 - \lambda) \cdot d\left(\overleftrightarrow{L}\right) \tag{10}$$

$$e\left(\varpi\left(\overleftrightarrow{K}, \overleftrightarrow{L}, \lambda\right)\right) = \lambda \cdot e\left(\overleftrightarrow{K}\right) + (1 - \lambda) \cdot e\left(\overleftrightarrow{L}\right) \tag{11}$$

- for any pair $\left(\overleftrightarrow{K}, \overleftrightarrow{L}\right) \in \left(\left(\mathbb{K}_{Tr}^{+} \cup \mathbb{R}\right) \times \mathbb{K}_{Tr}^{-}\right)$

$$d\left(\varpi\left(\overleftrightarrow{K}, \overleftrightarrow{L}, \lambda\right)\right) \leq \begin{cases} \lambda \cdot d\left(\overleftrightarrow{K}\right) - (1 - \lambda) \cdot d\left(\mathrm{Core}\left(\overleftrightarrow{L}\right)\right), & \varpi\left(\overleftrightarrow{K}, \overleftrightarrow{L}, \lambda\right) \in \mathbb{K}_{Tr}^{+} \cup \mathbb{R} \\ (1 - \lambda) \cdot d\left(\overleftrightarrow{L}\right) - \lambda \cdot d\left(\mathrm{Core}\left(\overleftrightarrow{K}\right)\right), & \varpi\left(\overleftrightarrow{K}, \overleftrightarrow{L}, \lambda\right) \in \mathbb{K}_{Tr}^{-} \cup \mathbb{R} \end{cases} \tag{12}$$

where $\mathrm{Core}\,(A) = \{x : \mu_A(x) = 1\}, A \in F(\mathbb{R})$

- for any pair $\left(\overleftrightarrow{K}, \overleftrightarrow{L}\right) \in \left(\left(\mathbb{K}_{Tr}^{+} \cup \mathbb{R}\right) \times \mathbb{K}_{Tr}^{-}\right) \cup \left(\mathbb{K}_{Tr}^{-} \times \left(\mathbb{K}_{Tr}^{+} \cup \mathbb{R}\right)\right)$

$$e\left(\varpi\left(\overleftrightarrow{K}, \overleftrightarrow{L}, \lambda\right)\right) \leqslant \min\left\{\lambda e\left(\overleftrightarrow{K}\right), (1 - \lambda)\, e\left(\overleftrightarrow{L}\right)\right\} \tag{13}$$

For more information about TrOFNs see [18, 26, 27].

3. Portfolio approach. Implemented procedure

The algorithm used in the proposed portfolio analysis (introduced in [18]) originates from an imprecise oriented present value, derived from stock quotes presented in the form of Japanese candles, which we describe with the use of trapezoidal oriented fuzzy numbers (the method is presented in [25]) and also the expected return rate. In [21] the present values were defined as a current equivalent of payments available at a given time (now or in the future). Due to that fact, it is commonly assumed that the future cash flow can be imprecise. The natural consequence of such an assumption is the evaluation of a present value as a fuzzy number. Such a present value is to be called a fuzzy present value. In [1, 7, 15, 16] the validity of the use of trapezoidal oriented fuzzy numbers as a tool of imprecise portfolio arithmetic was proved. Moreover, the evaluation of PVs should be supplemented by the forecast of changes in the closest prices of PV. Those changes can be anticipated subjectively. In our analysis, the assessment of fuzzy PVs is supplemented by the forecast of the closest changes. Such fuzzy PV is called fuzzy oriented present value. Using trapezoidal oriented fuzzy numbers in portfolio analysis is more useful than non-oriented fuzzy numbers, which was extensively described in [27]. For this reason, the fuzzy oriented present value can be estimated with the use of oriented fuzzy numbers [17–20, 25, 28]. Certainly, there are many different ways of estimating present values in the portfolio analysis. However, in the fuzzy portfolio approach, neither the traditional present value nor the traditional return rates formulas are applicable, therefore, they were not presented.

It must also be stressed that any operations of oriented fuzzy numbers are more complex than the operations of trapezoidal oriented fuzzy numbers [23, 27]. An approximation of oriented fuzzy numbers by trapezoidal oriented fuzzy numbers has been discussed in detail in [24]. In the fuzzy portfolio approach, the most convenient tools are a fuzzy expected present value, a fuzzy expected return rate and/or a fuzzy expected discount factor. In [29] it was indicated that a fuzzy oriented return rate is a fuzzy number, though unfortunately, it is not a trapezoidal oriented fuzzy number. Yet, a fuzzy oriented expected discount factor (OEDF) is a trapezoidal oriented fuzzy number and, additionally, it has the same orientation as a fuzzy oriented present value (OPV). Therefore, the fuzzy oriented expected discount factor is a better tool in assessing securities than a fuzzy oriented expected return rate [28].

We consider a multi-asset financial portfolio which is an arbitrary, finite set of assets of trade-related fixed securities in a long position (also referred to as stocks, shares or assets). Generally, both short and long positions can traditionally be considered in portfolio analysis. However, in the case of investing in stocks on capital markets, especially when individual (retail) investors are involved, it is more of a conventional practice to assume a long position. In our case "long" can be regarded not only from the point of view of the bullish intent, which is the expectation that the price will increase in time, but also with the reference to a measurement of time. Elements of analysed such portfolio have positive or negative orientation and each of them is determined by:

- its price $\check{P}_i \in \mathbb{R}^+$,
- its imprecise present value

$$\overleftrightarrow{PV}_i = \overleftrightarrow{Tr}\left(V_s^{(i)}, V_f^{(i)}, V_l^{(i)}, V_e^{(i)}\right) \tag{14}$$

where the monotonic sequence $\left(V_s, V_f, \check{P}, V_l, V_e\right)$ is determined by: $\check{P}$ – quoted price, $[V_s, V_e] \subset \mathbb{R}^+$ – interval of all PV possible values, $[V_f, V_l] \subset [V_s, V_e]$ – interval of all prices that do not significantly differ from the quoted price $\check{P}$ [20, 28],

- its EDF $\bar{v}_i$ defined by $\bar{v} = (1 + \bar{r})^{-1}$ (we assume that the expected value $\bar{r}$ of the distribution and the expected discount factor $\bar{v}$ exist).

The expected increase in price will be reflected in the present value determined by a positively oriented TrOFN, a decrease – a negatively oriented TrOFN. Japanese candles are the best representation of trapezoidal oriented PVs [25].

The main objective of the analysis is the comparison of energy and entropy measures of the imprecise oriented discount factors (OEDF) of the portfolios determined by each of these measures. To achieve that goal, we use the method described in detail in [18]. The input data include the stock quotes on a given date described by Japanese candlesticks and we present them in the form of trapezoidal oriented fuzzy numbers (TrOFNs). The procedure of the above-mentioned transformation was presented in [25]. Present values (PVs) described in such a manner can have a positive or negative orientation. For this reason in Step 2 of the procedure, we must determine $\overleftrightarrow{PV}^+$ and $\overleftrightarrow{PV}^-$, respectively of those securities of positive orientation (from the economic point of view those are the ones that prices are expected to increase) and those of negative orientation (prices are expected to drop). Following the fuzzy portfolio approach in Step 3, we calculate an oriented fuzzy present value of an entire portfolio as a generalised sum of imprecise present values of increasing and decreasing securities. A detailed motivation and idea of a generalised sum is presented in [23, 27]. In the following step, we determine a fuzzy oriented discount factor [28] of raising $\overleftrightarrow{V}^+$ and falling $\overleftrightarrow{V}^-$ securities and consequently their generalised sum. In Steps 6 and 7, we estimate entropy and energy measures which finally allow us to assess the imprecision of obtained results. We can then run the comparative analysis of the presented strategies.

When analysing the strategies (S1, S2, S3), we implemented the procedure that consists of 7 steps that are as follows:

Step 1. For each security, on a fixed date we determine the PV equal to TrOFN $\overleftrightarrow{PV}_i$, describing its Japanese candles.

Step 2. We distinguish the sums of rising securities π^+ and falling securities π^- based on:

$$\overleftrightarrow{PV}^+ = \overleftrightarrow{Tr}\left(V_s^{(+)}, V_f^{(+)}, V_l^{(+)}, V_e^{(+)}\right) \tag{15}$$

$$\overleftrightarrow{PV}^- = \overleftrightarrow{Tr}\left(V_s^{(-)}, V_f^{(-)}, V_l^{(-)}, V_e^{(-)}\right) \tag{16}$$

Step 3. We calculate the imprecise PV of a whole portfolio π^* as

$$\overleftrightarrow{PV}^* = \overleftrightarrow{PV}^+ \boxplus \overleftrightarrow{PV}^- = \overleftrightarrow{Tr}\left(V_s^{(+)}, V_f^{(+)}, V_l^{(+)}, V_e^{(+)}\right) \boxplus \overleftrightarrow{Tr}\left(V_s^{(-)}, V_f^{(-)}, V_l^{(-)}, V_e^{(-)}\right). \tag{17}$$

Step 4. We determine imprecise OEDF of respective portfolios π^+ and π^- using:

$$\overleftrightarrow{V}^+ = \bar{Tr}\left(\sum_{Y_i \in \pi^+} \frac{\bar{v}^+ \cdot p_i^{(+)}}{v_i} \cdot D_s^{(i)}, \sum_{Y_i \in \pi^+} \frac{\bar{v}^+ \cdot p_i^{(+)}}{v_i} \cdot D_f^{(i)}, \right.$$
$$\left. \sum_{Y_i \in \pi^+} \frac{\bar{v}^+ \cdot p_i^{(+)}}{v_i} \cdot D_l^{(i)}, \sum_{Y_i \in \pi^+} \frac{\bar{v}^+ \cdot p_i^{(+)}}{v_i} \cdot D_e^{(i)}\right) \tag{18}$$

$$\overleftrightarrow{\mathcal{V}}^- = \bar{T}r\left(\sum_{Y_i\in\pi^-}\frac{\bar{v}^-\cdot p_i^{(-)}}{v_i}\cdot D_s^{(i)},\sum_{Y_i\in\pi^-}\frac{\bar{v}^-\cdot p_i^{(-)}}{v_i}\cdot D_f^{(i)},\right.$$
$$\left.\sum_{Y_i\in\pi^-}\frac{\bar{v}^-\cdot p_i^{(-)}}{v_i}\cdot D_l^{(i)},\sum_{Y_i\in\pi^-}\frac{\bar{v}^-\cdot p_i^{(-)}}{v_i}\cdot D_e^{(i)}\right) \tag{19}$$

Step 5. We calculate an imprecise OEDF of a portfolio π^*

$$\overleftrightarrow{\mathcal{V}}^* = \overleftrightarrow{Tr}\left(D_s^*,\,D_f^*,\,D_l^*,\,D_e^*\right) = \left(\frac{\bar{v}^*\cdot q^+}{\bar{v}^+}\boxdot\overleftrightarrow{\mathcal{V}}^+\right)\boxplus\left(\frac{\bar{v}^*\cdot q^-}{\bar{v}^-}\boxdot\overleftrightarrow{\mathcal{V}}^-\right), \tag{20}$$

where: $M^+ = \sum_{Y_{i\in\pi^+}}\breve{P}_i$,$M^- = \sum_{Y_{i\in\pi^-}}\breve{P}_i$, $M^* = M^+ + M^-$ are the values of the portfolios π^+, π^-, π^*,

$p_i^{+/-} = \dfrac{\breve{P}_i}{M^{+/-}}$ represent the shares of a given asset $Y_i \in \pi^{+/-}$ in an individual portfolio ($\pi^{+/-}$),

$q_i^{+/-} = \dfrac{M^{(+/-)}}{M^*}$ represent the share $q^{+/-}$ of an individual portfolio $\pi^{+/-}$ in a portfolio π^*,

$\bar{v}^{+/-} = \left(\sum_{Y_{i\in\pi^{+/-}}}\frac{p_i^{+/-}}{\bar{v}_i}\right)^{-1}$, $\bar{v}^* = \left(\frac{q^+}{\bar{v}_+}+\frac{q^-}{\bar{v}_-}\right)^{-1}$ are the EDFs $\bar{v}^+$/$\bar{v}^-$/$\bar{v}^*$ of the portfolios π^+,$\pi-$ and π^*.

Step 6. We determine the energy measure

$$d\left(\overleftrightarrow{\mathcal{V}}^{+/-}\right) = \sum_{Y_i\in\pi^{(+/-)}}\frac{\bar{v}^{+/-}\cdot q_i^{(+/-)}}{\bar{v}_i}\cdot d\left(\overleftrightarrow{\mathcal{V}}\,(Y_i)\right) \tag{21}$$

that meets the condition

$$d\left(\overleftrightarrow{\mathcal{V}}^*\right) \le \begin{cases}\dfrac{\bar{v}^*\cdot q^+}{\bar{v}^+}\cdot d\left(\overleftrightarrow{\mathcal{V}}^+\right) - \dfrac{\bar{v}^*\cdot q^-}{\bar{v}^-}\cdot d\left(\mathrm{Core}\left(\overleftrightarrow{\mathcal{V}^-}\right)\right), & \overleftrightarrow{\mathcal{V}}^* \in \mathbb{K}_{Tr}^+\cup\mathbb{R}\\[2ex] \dfrac{\bar{v}^*\cdot q^-}{\bar{v}^-}\cdot d\left(\overleftrightarrow{\mathcal{V}}^-\right) - \dfrac{\bar{v}^*\cdot q^+}{\bar{v}^+}\cdot d\left(\mathrm{Core}\left(\overleftrightarrow{\mathcal{V}^+}\right)\right), & \overleftrightarrow{\mathcal{V}}^* \in \mathbb{K}_{Tr}^-\cup\mathbb{R}\end{cases} \tag{22}$$

Step 7. We determine the entropy measure

$$e\left(\overleftrightarrow{\mathcal{V}}^{+/-}\right) = \sum_{Y_i\in\pi^{(+/-)}}\frac{\bar{v}^{+/-}\cdot q_i^{(+/-)}}{\bar{v}_i}\cdot e\left(\overleftrightarrow{\mathcal{V}}\,(Y_i)\right) \tag{23}$$

that meets the condition

$$e\left(\overleftrightarrow{\mathcal{V}}^*\right) \leqslant \min\left\{\frac{\bar{v}^*\cdot q^+}{\bar{v}^+}\cdot e\left(\overleftrightarrow{\mathcal{V}}^+\right),\ \frac{\bar{v}^*\cdot q^-}{\bar{v}^-}\cdot e\left(\overleftrightarrow{\mathcal{V}}^-\right)\right\} \tag{24}$$

4. Findings

The research focuses on a portfolio consisting of the assets of companies listed on the Warsaw Stock Exchange (WSE). All of the analysed companies are included in WIG (the main index on WSE) and come from the sector related to trade (trade-related industries) and for each of them, brokers' recommendations were issued. Seven such companies were satisfying the above-mentioned conditions. On the chosen day the present value of each company was described by a Japanese candlestick, which was then transformed into a trapezoidal oriented fuzzy number. The white candlestick is represented by a positively oriented TrOFN, and the black one by a negatively oriented TrOFN respectively [25].

The portfolio is analysed based on three different approaches therefore we will distinguish Portfolio π_1 (TrOFNs) – Strategy S1, Portfolio π_2 (brokers' recommendations) – Strategy S2, and Portfolio π_3 (passive) – Strategy S3. The composition of the initial analysed Portfolio π^*, indicating the number of individual securities, is shown in Table 1.

Table 1. Composition of initial Portfolio π^*

Variable	Name	Tick	No. of stocks
Y_{ABE}	ABSA (ABPL)	ABE	700
Y_{ALE}	ALLEGRO	ALE	900
Y_{ANR}	ANSWEAR	ANS	1100
Y_{ASB}	ASBIS	ASB	1900
Y_{DAD}	DADELO	DAD	2100
Y_{OPN}	OPONEO.PL	OPN	800
Y_{TOA}	TOYA	TOA	1400

Step 1. Based on the closing of WSE session on April 20, 2022, we run the procedure for each analysed asset (PV equal to TrOFN $\overleftrightarrow{PV_i}$). A quoted price $\breve{P}_Y$ of each single element becomes the initial price on the following day (April 21, 2022). The obtained results for the initial Portfolio π^* – oriented present value, its energy and entropy measures, and quoted prices are presented in Table 2.

Table 2. Results of elements of portfolio π^*

Variable	oriented present value	Quoted price	Energy measure	Entropy measure
Y_{ABE}	$\overleftrightarrow{Tr}(46.15, 46.15, 47.90, 47.95)$	47.90	1.7750	0.0125
Y_{ALE}	$\overleftrightarrow{Tr}(27.05, 27.49, 27.91, 28.52)$	27.85	0.9450	0.2625
Y_{ANR}	$\overleftrightarrow{Tr}(26.20, 26.00, 25.85, 25.80)$	26.00	0.2750	0.0625
Y_{ASB}	$\overleftrightarrow{Tr}(11.60, 11.60, 12.00, 12.12)$	12.22	0.4600	0.0300
Y_{DAD}	$\overleftrightarrow{Tr}(13.85, 13.85, 13.20, 12.95)$	13.00	0.7750	0.0625
Y_{OPN}	$\overleftrightarrow{Tr}(48.00, 48.30, 48.80, 49.30)$	48.80	0.9000	0.2000
Y_{TOA}	$\overleftrightarrow{Tr}(6.22, 6.22, 6.25, 6.35)$	6.34	0.0800	0.0250

We establish that five analysed companies (variables Y_{ABE}, Y_{ALE}, Y_{ASB}, Y_{OPN} and Y_{TOA}) are characterised by positively oriented TrOFNs. In this case, the investor will anticipate an increase of their value in the future. This means that the investor might have an opportunity in the future to sell the securities with a profit. The other two variables (Y_{ANR} and Y_{DAD}) are determined by negatively oriented TrOFNs, therefore, the investor expects a drop in their value, which might present an opportunity to buy the stocks of those companies in the future at a lower price. For each variable, the above-mentioned initial price for the following day is given. Also, energy and entropy measures are calculated for each security. The linear portfolio analysis is not possible for considered portfolio π^*.

Step 2. Based on those mentioned above positively and negatively oriented variables, the respective sums of TrOFNs were calculated with the use of (15) and (16).

$$\overleftrightarrow{PV}^{+} = \overleftrightarrow{Tr}(125798.00, 126434.00, 129239.00, 130591.00) \tag{25}$$

$$\overleftrightarrow{PV}^{-} = \overleftrightarrow{Tr}(57905.00, 57685.00, 56155.00, 55575.00) \tag{26}$$

Step 3. Using (17), we compute portfolio π^* determined by TrOFN ($\overleftrightarrow{PV}$)

$$\overleftrightarrow{PV}^{*} = \overleftrightarrow{Tr}(183703.00, 184119.00, 185394.00, 186166.00). \tag{27}$$

The $\overleftrightarrow{PV}^{*}$ (27) is a positively oriented TrOFNs, therefore, we assume that an increase in the value of the portfolio is expected in the future.

Step 4. Imprecise OEDFs of portfolios π^+ and π^- were calculated using (18) and (19).

Step 5. Imprecise OEDF of π^* was calculated using (20).

Step 6. Energy measure was calculated using (21).

Step 7. The entropy measure was calculated using (23).

All the results obtained from Steps 4–7, as well as the values of imprecise OEDF, are shown in Table 3

Table 3. Characteristics of Portfolio π^+, π^- and π^*.

Portfolio	EDF $\bar{v}$	Imprecise OEDF $\overleftrightarrow{V}$
π^+	0.9858	$\overleftrightarrow{Tr}(0.9836, 0.9885, 1.0103, 1.0209)$
π^-	1.0624	$\overleftrightarrow{Tr}(1.0436, 1.0394, 1.0130, 1.0030)$
π^*	1.0077	$\overleftrightarrow{Tr}(1.0007, 1.0030, 1.0111, 1.0158)$

The data presented in Table 3 enabled the computation of the energy and entropy measures of imprecise OEDF $\overleftrightarrow{V}^*$ of Portfolio π^* which reached the levels of $d(\overleftrightarrow{V}^*) = 0.0116$ and $e(\overleftrightarrow{V}^*) = 0.0018$, respectively.

Based on the above premises of TrOFNs analysis we would advise the investor to:

- Buy additional shares of Y_{ABE}, Y_{ALE}, Y_{ASB}, Y_{OPN} and Y_{TOA} to sell them in the future at a higher price.
- Sell some shares of Y_{ANR} and Y_{DAD} to buy them cheaper in some time.

4.1. Case study. Strategy S1

Portfolio π_1 is the example in which investors implement strategy S1 following the recommendations of TrOFNs (the recommendations of TrOFNs are dominant). Therefore, the investors decide to:

1. sell 100 shares of ANR.

2. Sell 100 shares of DAD.
3. Buy 100 shares of ABE.
4. Buy 100 shares of ALE.
5. Buy 100 shares of ASB.
6. Buy 100 shares of OPN.
7. Buy 100 shares of TOA.

After implementing the above changes, the investor now has a portfolio π_1^* consisting of several assets presented in Table 4 and the proposed procedure (Steps 1–7) is implemented again.

Table 4. Composition of Portfolio π_1^*

Variable	Name	Tick	No. of stocks
Y_{ABE}	ABSA (ABPL)	ABE	800
Y_{ALE}	ALLEGRO	ALE	1000
Y_{ANR}	ANSWEAR	ANS	1000
Y_{ASB}	ASBIS	ASB	2000
Y_{DAD}	DADELO	DAD	2000
Y_{OPN}	OPONEO.PL	OPN	900
Y_{TOA}	TOYA	TOA	1500

Step 1. Based on the closing session (WSE) on June 17, 2022 (Friday) for each analysed asset, its PV equal to TrOFN $\overleftrightarrow{PV_i}$ (describing its Japanese candles) is determined. A quoted price $\breve{P}_Y$ of each single element of the observed portfolio becomes the initial price on the subsequent following session, June 20, 2022 (Monday).

Step 2. Based on the above-mentioned positively and negatively oriented variables, the respective sums of TrOFNs were calculated with the use of (15) and (16).

$$\overleftrightarrow{PV}_1^+ = \overleftrightarrow{Tr}(93570.00, 94360.00, 95730.00, 99050.00), \tag{28}$$

$$\overleftrightarrow{PV}_1^- = \overleftrightarrow{Tr}(69865.00, 69050.00, 67600.00, 66725.00). \tag{29}$$

Step 3. Using (17) we compute portfolio π^* determined by TrOFN ($\overleftrightarrow{PV}$).

$$\overleftrightarrow{PV}_1^* = \overleftrightarrow{Tr}(163435.00, 163410.00, 163330.00, 163330.00). \tag{30}$$

The $\overleftrightarrow{PV}^*$ (30) is a negatively oriented TrOFNs, therefore, we assume that a decrease in the value of the portfolio is expected in the future.

All the results obtained from Step 4–7, as well as the values of imprecise OEDFs, are shown in Table 5.

Table 5. Characteristics of Portfolio π_1^+, π_1^- and π_1^*

Portfolio	EDF $\bar{v}_1$	Imprecise OEDF $\overleftrightarrow{V_1}$
π_1^+	0.9886	$\overleftrightarrow{Tr}(0.9621, 0.9703, 0.9842, 1.0187)$
π_1^-	1.0025	$\overleftrightarrow{Tr}(1.0291, 1.0165, 0.9954, 0.9825)$
π_1^*	0.9942	$\overleftrightarrow{Tr}(0.9894, 0.9891, 0.9888, 0.9888)$

The obtained data from Table 5 enabled the computation of the energy and entropy measures of imprecise OEDF $\overleftrightarrow{\mathcal{V}_1^*}$ of Portfolio π_1^* which reached the levels of $d(\overleftrightarrow{\mathcal{V}_1^*}) = 0.0005$ and $e(\overleftrightarrow{\mathcal{V}_1^*}) = 0.0001$, respectively.

4.2. Case study. Strategy S2

Portfolio π_2 is the case in which the investor implements strategy S2 and decides to seek additional support of brokers' recommendations. It is commonly believed that brokers employed in brokerage houses are highly qualified in stock investments. Therefore, in this strategy the recommendations of brokers are dominant. The official recommendations of brokers issued for analysed companies in the given time are presented in Table 6.

Table 6. Recommendations of brokers for analysed companies

Company	Recommendation	Date of recommendation
ABSA	BUY	2022-04-28
ALLEGRO	HOLD	2022-04-13
ANSWEAR	BUY	2022-04-25
ASBIS	BUY	2022-04-26
DADELO	HOLD	2022-04-25
OPONEO.PL	HOLD	2022-04-28
TOYA	HOLD	2022-04-25

Comparing the recommendations in strategy S1 (Portfolio π_1) and in strategy S2 (Portfolio π_2) we notice that recommendations for analysed entities in the assumed time horizon do not differ significantly among themselves. When recommendations in both strategies are identical, the investor makes the same decision regarding an individual company. However, in strategy S2 if the recommendations of TrOFNs and experts are different, the investor accepts brokers' recommendations.

Therefore, the investors decide to:

1. Buy 100 shares of ABE.
2. Buy 100 shares of ANR.
3. Buy 100 shares of ASB.

The number of shares of the remaining four companies stays the same. Finally, we obtain the following Portfolio π_2^* presented in Table 7.

Table 7. Composition of Portfolio π_2^*

Variable	Name	Tick	No. of stocks
Y_{ABE}	ABSA (ABPL)	ABE	800
Y_{ALE}	ALLEGRO	ALE	900
Y_{ANR}	ANSWEAR	ANS	1200
Y_{ASB}	ASBIS	ASB	2000
Y_{DAD}	DADELO	DAD	2100
Y_{OPN}	OPONEO.PL	OPN	800
Y_{TOA}	TOYA	TOA	1400

Similarly to the previous strategy, all steps are repeated analogously.

Step 1. PV equal to TrOFN PV_i (describing its Japanese candles) is determined for each analysed asset based on the closing of the WSE session on June 17, 2022. Next, the quoted price $\breve{P}_Y$ of each single element of the observed portfolio becomes the initial price on the following day (June 20, 2022).

Step 2. Portfolios π_2^+ and π_2^- determined by TrOFN $(\overleftrightarrow{PV})$ are calculated

$$\overleftrightarrow{PV}_2^+ = \overleftrightarrow{Tr}(87362.00, 88071.00, 89358.00, 92350.00) \quad (31)$$

$$\overleftrightarrow{PV}_2^- = \overleftrightarrow{Tr}(73875.00, 73029.00, 71378.00, 70492.00) \quad (32)$$

Step 3. We compute portfolios π^* determined by TrOFN ($\overleftrightarrow{PV}$)

$$\overleftrightarrow{PV}_2^* = \overleftrightarrow{Tr}(161237.00, 161100.00, 160736.00, 160736.00). \quad (33)$$

The $\overleftrightarrow{PV}_2^*$ (33) is defined by a negatively oriented TrOFNs, therefore, we expect a further decrease in the value of the portfolio in the future.

Results from Steps 4–7 (EDFs of the given portfolios π_2^+, π_2^- and π_2^* and the values of imprecise OEDFs) are given in Table 8.

Table 8. Characteristics of Portfolio π_2^+, π_2^- and π_2^*

Portfolio	EDF $\bar{v}_2$	Imprecise OEDF $\overleftrightarrow{\mathcal{V}}_2$
π_2^+	0.9891	$\overleftrightarrow{Tr}(0.9636, 0.9714, 0.9855, 1.0189)$
π_2^-	1.0275	$\overleftrightarrow{Tr}(1.0293, 1.0169, 0.9939, 0.9817)$
π_2^*	1.0057	$\overleftrightarrow{Tr}(0.9920, 0.9911, 0.9891, 0.9891)$

The entropy and energy measures of imprecise OEDF $\overleftrightarrow{\mathcal{V}}_2^*$ of Portfolio π_2^* reached the levels of $d(\overleftrightarrow{\mathcal{V}}_2^*) = 0.0024$ and $e(\overleftrightarrow{\mathcal{V}}_2^*) = 0.0002$.

4.3. Comparative analysis of strategies S1, S2 and S3

We will compare the results of another possible strategy S3 ("passive" strategy in which the investors did not use any of the available recommendations) with the results obtained from strategies S1 and S2. In Table 9 we can see the results of the analysed portfolios, energy and entropy measures of imprecise OEDF $\overleftrightarrow{\mathcal{V}}_2^*$ in the initial point (S0) and after implementing analysed strategies (S1, S2, S3).

Table 9. The value of given Portfolios, energy, and entropy measures for strategies S1, S2, S3 and the initial point S0

Strategies	M_i^*	$d(\overleftrightarrow{\mathcal{V}}_i^*)$	$e(\overleftrightarrow{\mathcal{V}}_i^*)$
S0	185629.00	0.0116	0.0018
S1	16664.00	0.0005	0.0001
S2	163914.00	0.0024	0.0002
S3	156614.00	0.0022	0.0002

By comparing the obtained results for the initial point S0 and strategies S1, S2, S3 we can see that:

$$M_0^* > M_1^* > M_2^* > M_3^* \quad (34)$$

$$d(\overleftrightarrow{\mathcal{V}}_1^*) < d(\overleftrightarrow{\mathcal{V}}_3^*) < d(\overleftrightarrow{\mathcal{V}}_2^*) < d(\overleftrightarrow{\mathcal{V}}_0^*) \quad (35)$$

$$e(\overleftrightarrow{\mathcal{V}_1^*}) < e(\overleftrightarrow{\mathcal{V}_2^*}) = e(\overleftrightarrow{\mathcal{V}_3^*}) < e(\overleftrightarrow{\mathcal{V}_0^*}) \tag{36}$$

It means that if the investor used the recommendations of OFNs (S1) then, despite the unfavourable economic situation and massive drops on WSE, the investor lost the least (the smallest loss of all strategies). Additionally, the obtained information was the least imprecise. That allows us to conclude that in some situations recommendations of OFNs are better than the recommendations of an expert (broker). Also, OFNs recommendations let the investor gain better results than the "passive" strategy.

5. Conclusions

Oriented fuzzy numbers include information uncertainty and imprecision related to the financial market and therefore they are frequently used in portfolio analysis and management. This, in turn, allows us to use an expected fuzzy discount factor. Furthermore, that enables the use of imprecise present value. As a final result, we get various recommendations, e.g. buy, sell, accumulate or reduce, regarding individual assets in an investor's portfolio. Similar recommendations are regularly issued by experts relying on their knowledge and experience which also are imprecise.

The main purpose of the paper was to analyse the published recommendations of experts with recommendations obtained with the use of OFNs and also with a "passive" strategy. We considered the accuracy and actual impact of those recommendations on the portfolio composition and operations. The analysed companies came from the trade-related sector of WIG index on WSE.

Trapezoidal oriented fuzzy numbers (TrOFNs) are used to determine the present value of securities included in the created portfolio on the day of issuance of the recommendation. The elements of the portfolio have either positive or negative orientations. Theoretical considerations are illustrated by empirical case studies.

A comparison of the usefulness of recommendations made by experts (stockbrokers) and by the fuzzy portfolio analysis was conducted with the use of the fuzzy discount factor.

The results show that the fuzzy portfolio analysis (S1) was more accurate for the companies in the trade-related sector assessed by a forecast of the closest change in prices than the recommendations of the brokers.

Using OFNs an initial decrease in the value of a portfolio was anticipated. The decrease occurred in both of analysed strategies (S1 and S2), however, in the case of S1 the decrease was of a smaller value.

The case study shows that recommendations obtained with the use of OFNs are better if we consider energy and entropy measures (their values are smaller). It can also result from the fact that OFN recommendations, unlike most brokers' recommendations, are short-term recommendations.

The results based on OFNs are better because they are less imprecise.

The future direction of the research is to run the analysis concerning mixed strategies regarding the recommendation. Also, the research will be extended to securities from foreign stock exchanges. The proposed approach is limited by the number of brokers' recommendations and their irregularity.

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