A Decidable Quantified Fragment of Set Theory Involving Ordered Pairs with Applications to Description Logics

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Abstract

We present a decision procedure for a quantified fragment of set theory, called $\forall_0^\pi$, involving ordered pairs and some operators to manipulate them. When our decision procedure is applied to $\forall_0^\pi$-formulae whose quantifier prefixes have length bounded by a fixed constant, it runs in nondeterministic polynomial-time.

Related to the fragment $\forall_0^\pi$, we also introduce a description logic, $\mathcal{DL}\langle\forall_0^\pi\rangle$, which provides an unusually large set of constructs, such as, for instance, Boolean constructs among roles. The set-theoretic nature of the description logics semantics yields a straightforward reduction of the knowledge base consistency problem for $\mathcal{DL}\langle\forall_0^\pi\rangle$ to the satisfiability problem for $\forall_0^\pi$-formulae with quantifier prefixes of length at most 2, from which the NP-completeness of reasoning in $\mathcal{DL}\langle\forall_0^\pi\rangle$ follows. Finally, we extend this reduction to cope with SWRL rules.

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1 Introduction

Computable Set Theory is a research field, started around thirty years ago, devoted to the study of the decision problem for fragments of set theory (see [4, 7] for a thorough account of the state-of-the-art until 2001). The most efficient decision procedures devised in this context have been implemented in the inferential core of the system ÆtnaNova/Referee, described in [8, 17, 19].

The first unquantified sublanguage of set theory that has been proved decidable is *Multi-Level Syllogistic* (in short MLS). MLS involves the set predicates $\in$, $\subseteq$, $=$, the Boolean set operators $\cup$, $\cap$, $\setminus$, and the connectives of propositional logic (cf. [10]). Subsequently, several extensions of MLS with various combinations of operators (such as singleton, powerset, unionset, etc.) and predicates (on finiteness, transitivity, etc.) have been proved to have a solvable satisfiability problem. Also, some extensions of MLS with various map[1] constructs have been shown to be decidable (cf. [9, 5]).

Concerning *quantified* fragments, of particular interest to us is the restricted quantified fragment of set theory $\forall_0$, which has been proved to have a decidable satisfiability problem

[1] According to [20], we use the term 'maps' to denote sets of ordered pairs.

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in [2]. We recall that $\forall_0$-formulae are propositional combinations of restricted quantified prenex formulae $(\forall y_1 \in z_1) \cdots (\forall y_n \in z_n)p$, where p is a Boolean combination of atoms of the forms $x \in y$, $x = y$, and no z_j is a y_i (i.e., nesting among quantified variables is not allowed). The same paper considered also the extension with another sort of variables representing *single-valued* maps, the map domain operator, and terms of the form $f(t)$ (representing the value of the map f on a function-free term t). However, neither one-to-many, nor many-to-one, nor many-to-many relationships can be represented in this language. We observe that the $\forall_0$-fragment is very close to the undecidability boundary, as shown in [18]. In fact, if nesting among quantified variables in prenex formulae of type $(\forall y_1 \in z_1) \cdots (\forall y_n \in z_n)p$ are allowed and a predicate stating that a set is an unordered pair is also admitted, then it turns out that the satisfiability for the resulting collection of formulae is undecidable.

In this paper we present a decision procedure for the novel fragment of set theory $\forall_{\mathbf{0}}^{\pi}$, which extends the fragment $\forall_0$ with ordered pairs and various constructs related to them, thus further thinning the gap between the decidable and the undecidable. A considerable amount of set-theoretic constructs can be expressed by $\forall_{\mathbf{0}}^{\pi}$-formulae, in particular constructs on *multi-valued* maps like map inverse, Boolean operator among maps, map transitivity, and so on. Furthermore, when restricted to formulae with quantifier nesting bounded by a constant, our decision procedure runs in nondeterministic polynomial-time. This fragment has also interesting applications in the field of *knowledge representation*.

Applications of Computable Set Theory to knowledge representation have been recently proposed in [6], where the correspondence between (decidable) fragments of set theory and *Description Logics* (a well established framework for knowledge representation systems; see [1] for a quite complete overview) is exploited by introducing the very expressive description logic $\mathcal{DL}\langle \mathsf{MLSS}_{2,m}^{\times} \rangle$.

Description logics are a family of logic based formalisms widely used in knowledge representation. In particular, several results and decision procedures devised in this context have been profitably employed in the area of the Semantic Web (cf. [11]). The key problem in description logic is to determine whether a knowledge base $\mathcal{K}$ is *consistent* (knowledge base consistency is formally described in Section 4), and many other reasoning tasks can be reduced to it. Unfortunately, this problem is ExpTime-hard (cf. [1, Theorem 3.27, page 132]) also for $\mathcal{AL}$, a basic description logic with a very limited expressive power. However, [6] shows how a better computational complexity can be achieved by imposing some limitations on the usage of existential quantification and number restrictions (definitions of these two constructs are reported in Table 1).

The quantified nature of the language $\forall_{\mathbf{0}}^{\pi}$ and the pair-related constructs it provides allow a straightforward mapping of numerous description logic constructs to $\forall_{\mathbf{0}}^{\pi}$-formulae. The resulting description logic, called $\mathcal{DL}\langle \forall_{\mathbf{0}}^{\pi} \rangle$, extends those presented in [6] with several constructs like, for instance, role transitivity, self restrictions, and role identity. It also allows *finite* existential restrictions of the form $\exists R.\{a_1, \ldots, a_n\}$ to be used without limitations. Furthermore, it turns out that the consistency problem for $\mathcal{DL}\langle \forall_{\mathbf{0}}^{\pi} \rangle$-knowledge bases is NP-complete. This is a quite significant result since in most of the cases in which Boolean operators among roles are present the consistency problem turns out to be NExpTime-hard (cf. [14]).

Finally, we observe that SWRL rules (cf. [12]) can be easily embedded in $\mathcal{DL}\langle \forall_{\mathbf{0}}^{\pi} \rangle$ without disrupting decidability. SWRL rules are a simple form of Horn-style rules, which were proposed with the aim of increasing the expressive power of description logics. Here we consider only a restricted set of SWRL rules, namely those which do not contain *data literals*. Extending description logics with SWRL rules in general leads to undecidability. In [16] this issue has been overcome by restricting the applicability of rules to a finite set of named individuals. Another approach, studied in [13], consists in restricting to rules which can be *internalized*, i.e. rules which can be converted into knowledge base statements.

The paper is organized as follows. Section 2 presents the precise syntax and semantics of the language $\forall_{\mathbf{0}}^{\pi}$. A decision procedure for $\forall_{\mathbf{0}}^{\pi}$ is then developed in Section 3. In Section 4 the correspondences of $\forall_{\mathbf{0}}^{\pi}$ with description logics are exploited by introducing the novel description logic $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$, whose extension with SWRL rules is studied in Section 5. Finally, concluding remarks and some hints to future work are given in Section 6.

2 The language $\forall_{\mathbf{0}}^{\pi}$

The language $\forall_{\mathbf{0}}^{\pi}$ is a quantified fragment of set theory which contains a denumerable infinity of *variables*, $\mathit{Vars} = \{x, y, z, \ldots\}$, the binary *pairing* operator $[\cdot,\cdot]$, the monadic function $\bar{\pi}(\cdot)$, which represents the non-pair members of a set, the relators $\in, =$, the Boolean connectives of propositional logic $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, parentheses, and the universal quantifier $\forall$.

A *quantifier-free* $\forall_{\mathbf{0}}^{\pi}$*-formula* is any propositional combination of *atomic* $\forall_{\mathbf{0}}^{\pi}$*-formulae*. These are expressions of the following types:

$$x \in \bar{\pi}(z), \quad [x, y] \in z, \quad x = y, \tag{1}$$

with $x, y, z \in \mathit{Vars}$. Intuitively, terms of the form $[x, y]$ represent ordered pairs of sets.

A *simple prenex* $\forall_{\mathbf{0}}^{\pi}$*-formula* is a formula $Q_1 \cdots Q_n \varphi$, with $n \geq 0$, where φ is a quantifier-free $\forall_{\mathbf{0}}^{\pi}$-formula, each Q_i is a restricted universal quantifier of form $(\forall x \in \bar{\pi}(y))$ or of the form $(\forall [x, x'] \in y)$ (we will refer to x and x' as *quantified variables* and to y as *domain variable*), and no variable can occur both as a quantified and a domain variable, i.e., roughly speaking, no x can be a y.

Finally, a $\forall_{\mathbf{0}}^{\pi}$*-formula* is any finite conjunction of simple prenex $\forall_{\mathbf{0}}^{\pi}$-formulae.

Semantics of the $\forall_{\mathbf{0}}^{\pi}$-language is based upon the von Neumann standard cumulative hierarchy $\mathcal{V}$ of sets, which is defined as follows:

$$\begin{array}{rcll} \mathcal{V}_0 & = & \emptyset & \\ \mathcal{V}_{\gamma+1} & = & \mathcal{P}(\mathcal{V}_\gamma)\,, & \text{for each ordinal } \gamma \\ \mathcal{V}_\lambda & = & \bigcup_{\mu<\lambda} \mathcal{V}_\mu\,, & \text{for each limit ordinal } \lambda \\ \mathcal{V} & = & \bigcup_{\gamma \in On} \mathcal{V}_\gamma\,, & \end{array}$$

where $\mathcal{P}(\cdot)$ is the powerset operator and On denotes the class of all ordinals.

A $\forall_{\mathbf{0}}^{\pi}$*-interpretation* is a pair $\mathbf{I} = (M_{\mathbf{I}}, \pi_{\mathbf{I}})$, where $M_{\mathbf{I}}$ is a total function which maps each variable into a set of $\mathcal{V}$, and $\pi_{\mathbf{I}}$ is a *pairing function* over sets. We recall that a *pairing function* π is a binary operation over sets such that $\pi(u, v) = \pi(u', v') \iff u = u' \wedge v = v'$ and the class $u \times_\pi v =_{\text{Def}} \{\pi(s, t) : s \in u \wedge t \in v\}$ is a set of $\mathcal{V}$, for all $u, v, u', v' \in \mathcal{V}$.

Let W be a finite subset of Vars, we say that $\mathbf{I}' = (M_{\mathbf{I}'}, \pi_{\mathbf{I}})$ is a W*-variant* of $\mathbf{I}$ if $M_{\mathbf{I}'} y = M_{\mathbf{I}} y$, for $y \in \mathit{Vars} \backslash W$. To any term of the form x, $[x, y]$, and $\bar{\pi}(x)$, a $\forall_{\mathbf{0}}^{\pi}$-interpretation $\mathbf{I}$ associates a set in $\mathcal{V}$ as follows:

$$\begin{array}{rcl} \mathbf{I}x & =_{\text{Def}} & M_{\mathbf{I}} x \\ \mathbf{I}[x, y] & =_{\text{Def}} & \pi_{\mathbf{I}}(\mathbf{I}x, \mathbf{I}y) \\ \mathbf{I}\bar{\pi}(x) & =_{\text{Def}} & \mathbf{I}x \setminus \{\pi_{\mathbf{I}}(u, v) : u, v \in \mathcal{V}\}, \end{array}$$

for all $x, y \in \mathit{Vars}$.

A $\forall_{\mathbf{0}}^{\pi}$-interpretation evaluates atomic $\forall_{\mathbf{0}}^{\pi}$-formulae to the truth values $\mathbf{t}$ (true) and $\mathbf{f}$ (false) in the usual way, by interpreting '$\in$' and '$=$' as the membership and the equality relations between sets, respectively. Evaluation of quantifier-free $\forall_{\mathbf{0}}^{\pi}$-formulae is carried out according to the standard rules of propositional logic, and simple prenex $\forall_{\mathbf{0}}^{\pi}$-formulae are evaluated as follows:

- $\mathbf{I}(\forall x \in \bar{\pi}(y))\varphi = \mathbf{t}$ iff $\mathbf{I}'\varphi = \mathbf{t}$ for every $\{x\}$-variant $\mathbf{I}'$ of $\mathbf{I}$ such that $\mathbf{I}'x \in \mathbf{I}'\bar{\pi}(y)$,
- $\mathbf{I}(\forall [x, y] \in z)\varphi = \mathbf{t}$ iff $\mathbf{I}'\varphi = \mathbf{t}$ for every $\{x, y\}$-variant $\mathbf{I}'$ of $\mathbf{I}$ such that $\mathbf{I}'[x, y] \in \mathbf{I}'z$.

A $\forall_{\mathbf{0}}^{\pi}$-interpretation $\mathbf{I}$ which evaluates a $\forall_{\mathbf{0}}^{\pi}$-formula φ to true is said to be a *model* for φ (and we write $\mathbf{I} \models \varphi$). A $\forall_{\mathbf{0}}^{\pi}$-formula is said to be *satisfiable* if it admits a model. Thus the satisfiability problem (in short, s.p.) for $\forall_{\mathbf{0}}^{\pi}$-formulae consists in determining whether a $\forall_{\mathbf{0}}^{\pi}$-formula is satisfiable or not. Observe that in the context of satisfiability, all free variables in a $\forall_{\mathbf{0}}^{\pi}$-formula may be regarded as existentially quantified.

In the following section we present a decision procedure for the s.p. for $\forall_{\mathbf{0}}^{\pi}$-formulae.

3 A decision procedure for $\forall_{\mathbf{0}}^{\pi}$

In this section we solve the s.p. for $\forall_{\mathbf{0}}^{\pi}$-formulae. In particular, we will prove that a $\forall_{\mathbf{0}}^{\pi}$-formula is satisfiable if and only if there exists a finite collection of atomic $\forall_{\mathbf{0}}^{\pi}$-formulae which *represents* a model for the formula. We begin by introducing the notions of skeletal representations and of their realizations: these are, respectively, collections of atomic $\forall_{\mathbf{0}}^{\pi}$-formulae with an acyclic membership relation among their variables, and suitably defined $\forall_{\mathbf{0}}^{\pi}$-interpretations. In particular, we will focus on skeletal representations "completed" w.r.t. the predicate "$=$" over a set of variables V, which we call V-extensional. It turns out, as will be shown in Lemma 2, that each V-extensional skeletal representation is modeled correctly by any realization associated with it. Finally, we prove the main result of this section, namely that a $\forall_{\mathbf{0}}^{\pi}$-formula φ with free variables V is satisfiable if and only if it is satisfied by the realization associated with a suitable V-extensional skeletal representation whose size is bounded by the cardinality of V (cf. Theorem 3). The latter result entails immediately the decidability of the fragment $\forall_{\mathbf{0}}^{\pi}$ of our interest.

Given a $\forall_{\mathbf{0}}^{\pi}$-formula φ, we denote with φ_y^x the formula obtained by replacing each free occurrence of x in φ with y and with $Vars(\varphi)$ the collection of the free variables of φ. Likewise, given a finite collection $\mathcal{S}$ of atomic $\forall_{\mathbf{0}}^{\pi}$-formulae, we denote with $Vars(\mathcal{S})$ the collection of the variables occurring in the formulae of $\mathcal{S}$. In addition, we indicate with $\in_{\mathcal{S}}^{+}$ (the *membership closure* of $\mathcal{S}$) the minimal transitive relation on $Vars(\mathcal{S})$ such that the following conditions hold:

- if "$x \in \bar{\pi}(z)$" $\in \mathcal{S}$, then $x \in_{\mathcal{S}}^{+} z$;
- if "$[x, y] \in z$" $\in \mathcal{S}$, then $x \in_{\mathcal{S}}^{+} z \wedge y \in_{\mathcal{S}}^{+} z$.

A collection $\mathcal{S}$ of atomic $\forall_{\mathbf{0}}^{\pi}$-formulae is a *skeletal representation* if $x \notin_{\mathcal{S}}^{+} x$, for all $x \in Vars(\mathcal{S})$.

Let $\mathcal{S}$ be a skeletal representation. We define the *height* of a variable $x \in Vars(\mathcal{S})$ with respect to $\mathcal{S}$ (which we write $\mathsf{height}_{\mathcal{S}}(x)$) as the length n of the longest $\in_{\mathcal{S}}^{+}$-chain of the form $x_1 \in_{\mathcal{S}}^{+} \ldots \in_{\mathcal{S}}^{+} x_n \in_{\mathcal{S}}^{+} x$ ending at x, with $x_1, \ldots, x_n \in Vars(\mathcal{S})$. Thus, $\mathsf{height}_{\mathcal{S}}(x) = 0$ if $y \notin_{\mathcal{S}}^{+} x$, for all $y \in Vars(\mathcal{S})$.

A skeletal representation $\mathcal{S}$ is said to be V*-extensional*, for a given set of variables V, if the following conditions hold:

- if "$x = y$" $\in \mathcal{S}$, then $x, y \in V$ and α_y^x and α_x^y belong to $\mathcal{S}$, for each atomic formula α in $\mathcal{S}$;
- if "$x = y$" $\notin \mathcal{S}$, for some $x, y \in V$, then the variables x and y must be explicitly *distinguished* in $\mathcal{S}$ either by some variable z, in the sense that "$z \in \bar{\pi}(x)$" $\in \mathcal{S}$ iff "$z \in \bar{\pi}(y)$" $\notin \mathcal{S}$, or by some pair $[z, z']$, in the sense that "$[z, z'] \in x$" $\in \mathcal{S}$ iff "$[z, z'] \in y$" $\notin \mathcal{S}$.

Skeletal representations allow one to define special $\forall_0^\pi$-interpretations, called *realizations*, which were first introduced in [3], though with a slightly different meaning. To this purpose we introduce the following family $\{\pi_n\}_{n\in\mathbb{N}}$ of pairing functions, recursively defined by

$$\begin{array}{rcl} \pi_0(u,v) & =_{\mathrm{Def}} & \{u,\{u,v\}\} \\ \pi_{n+1}(u,v) & =_{\mathrm{Def}} & \{\pi_n(u,v)\}\,, \end{array}$$

for every $u, v \in \mathcal{V}$.

▶ **Definition 1** (Realization). Let $\mathcal{S}$ be a skeletal representation, let V and T be two finite, nonempty, and disjoint sets of variables such that $\mathit{Vars}(\mathcal{S}) \subseteq V \cup T$, and let σ be a bijection from T onto $\{1, 2, \ldots, |T|\}$. We extend the function $\mathsf{height}_{\mathcal{S}}(\cdot)$ also to variables $x \in (V \cup T) \setminus \mathit{Vars}(\mathcal{S})$ by putting for such variables $\mathsf{height}_{\mathcal{S}}(x) =_{\mathrm{Def}} 0$.

Then the *realization* of $\mathcal{S}$ relative to (V, T) is the $\forall_0^\pi$-interpretation $\mathbf{R} = (M_{\mathbf{R}}, \pi_{\mathbf{R}})$ such that $\pi_{\mathbf{R}} =_{\mathrm{Def}} \pi_{|V|+|T|}$ and, recursively on $\mathsf{height}_{\mathcal{S}}(x)$ for $x \in V \cup T$,

$$M_{\mathbf{R}}x =_{\mathrm{Def}} \{\mathbf{R}y : \text{“}y \in \bar{\pi}(x)\text{”} \in \mathcal{S}\} \cup \{\mathbf{R}[y,z] : \text{“}[y,z] \in x\text{”} \in \mathcal{S}\} \cup s(x)\,,$$

where

$$s(x) =_{\mathrm{Def}} \begin{cases} \{\{k+1, k, \sigma(x)\}\} & \text{if } x \in T \\ \emptyset & \text{otherwise,} \end{cases}$$

with $k = |V| \cdot (|V| + |T| + 3)$.[2] ◀

Realizations have useful properties, stated by the following lemma.

▶ **Lemma 2.** *Let $\mathcal{S}$, V, T, σ, and k be as in Definition 1 and let $\mathbf{R}$ be the realization of $\mathcal{S}$ relative to (V, T). If $\mathcal{S}$ is V-extensional, then for every $x, y, z \in V \cup T$ the following conditions hold:*

(R1) $\mathbf{R}x \neq \pi_{\mathbf{R}}(u, v)$ *for* $u, v \in \mathcal{V}$*;*
(R2) $\mathbf{R}x \neq \{k+1, k, i\}$ *for* $1 \leq i \leq |T|$*;*
(R3) $\mathbf{R}x = \mathbf{R}y$ *iff either* “$x = y$”$\in \mathcal{S}$ *or* x *and* y *coincide;*
(R4) $\mathbf{R}x \in \mathbf{R}\bar{\pi}(y)$ *iff* “$x \in \bar{\pi}(y)$”$\in \mathcal{S}$*;*
(R5) $\mathbf{R}[x, y] \in \mathbf{R}z$ *iff* “$[x, y] \in z$”$\in \mathcal{S}$*.*

Proof. To prove **(R1)**, we establish the more general property

$$\text{if } \mathsf{height}_{\mathcal{S}}(x) \leq n \leq |V| + |T|, \quad \text{then} \quad \mathbf{R}x \neq \pi_n(u, v), \text{ for } x \in V \cup T \text{ and } u, v \in \mathcal{V}. \tag{2}$$

Let $n \leq |V| + |T|$ and let us assume by way of contradiction that $\mathbf{R}x = \pi_n(u, v)$ for some $u, v \in \mathcal{V}$ and some $x \in V \cup T$ of minimal height such that $0 \leq \mathsf{height}_{\mathcal{S}}(x) \leq n$.

We can rule out at once the case in which $n = 0$, as in this case $\mathsf{height}_{\mathcal{S}}(x) = 0$, so that $|\mathbf{R}x| \leq 1$, and therefore $\mathbf{R}x \neq \pi_0(u, v)$, since $|\pi_0(u, v)| = 2$.

Thus, we can assume that $n > 0$. Let us consider first the case in which $\mathsf{height}_{\mathcal{S}}(x) = 0$. If $x \in V$ then, by the very definition of realization, we have $\mathbf{R}x = \emptyset \neq \pi_n(u, v)$. On the other hand, if $x \in T$, then $\mathbf{R}x = \{\{k+1, k, \sigma(x)\}\}$ and since $|\{k+1, k, \sigma(x)\}| > |\pi_{n-1}(u, v)|$ and $\pi_n(u, v) = \{\pi_{n-1}(u, v)\}$, it follows that $\mathbf{R}x \neq \pi_n(u, v)$. In both cases we found a contradiction, so that we must have $\mathsf{height}_{\mathcal{S}}(x) > 0$.

On the other hand, if $\mathsf{height}_{\mathcal{S}}(x) > 0$, our absurd hypothesis $\mathbf{R}x = \pi_n(u, v) = \{\pi_{n-1}(u, v)\}$ and the definition of realization imply that either

[2] We are assuming that integers are represented *à la* von Neumann, namely $0 =_{\mathrm{Def}} \emptyset$ and, recursively, $n + 1 =_{\mathrm{Def}} n \cup \{n\}$.

(i) $\pi_{n-1}(u,v) = \{k+1, k, \sigma(x)\}$, but provided that $x \in T$, or
(ii) $\pi_{n-1}(u,v) = \mathbf{R}y$, for some y such that "$y \in \bar{\pi}(x)$" $\in \mathcal{S}$, or
(iii) $\pi_{n-1}(u,v) = \mathbf{R}[y,z] = \pi_{|V|+|T|}(\mathbf{R}y, \mathbf{R}z)$, for some y, z such that "$[y,z] \in x$" $\in \mathcal{S}$.

We can exclude at once case **(i)**, since $|\pi_{n-1}(u,v)| \leq 2 < |\{k+1, k, \sigma(x)\}|$. Case **(ii)** can be excluded as well, since it would contradict the minimality of $\mathsf{height}_{\mathcal{S}}(x)$, as $\mathsf{height}_{\mathcal{S}}(y) < \mathsf{height}_{\mathcal{S}}(x)$. In case **(iii)**, from elementary properties of our pairing functions π_i it would follow that $|V| + |T| = n - 1$, contradicting our initial assumption that $n \leq |V| + |T|$. Thus (2) holds.

In view of (2), to establish **(R1)** it is now enough to observe that $\mathsf{height}_{\mathcal{S}}(x) < |V| + |T|$.

Next, since $\mathsf{rank}(\{k+1, k, i\}) = k+2$, for $1 \leq i \leq |T|$ (as $k > |T|$),[3] to establish **(R2)** it will be enough to show that $\mathsf{rank}(\mathbf{R}x) \neq k+2$, for $x \in V \cup T$. Thus, let $x \in V \cup T$. If $y \in^{+}_{\mathcal{S}} x$, for some $y \in T$, then $\mathsf{rank}(\mathbf{R}x) \geq \mathsf{rank}(\mathbf{R}y) \geq k+3$. The same conclusion can be reached also in the case in which $x \in T$. On the other hand, if $y \notin^{+}_{\mathcal{S}} x$, for any $y \in T$, and $x \in V$, it can easily be proved by induction on $\mathsf{height}_{\mathcal{S}}(x)$ that $\mathsf{rank}(\mathbf{R}x) \leq (|V| + |T| + 3) \cdot \mathsf{height}_{\mathcal{S}}(x) \leq |V| \cdot (|V| + |T| + 3) = k$. Hence, in any case $\mathsf{rank}(\mathbf{R}x) \neq k+2$ holds, proving **(R2)**.

Concerning **(R3)**, we observe preliminarily that if "$x = y$" $\in \mathcal{S}$, then $\mathbf{R}x = \mathbf{R}y$ is a direct consequence of the V-extensionality of $\mathcal{S}$. Thus, to complete the proof of **(R3)** it is enough to show that if $\mathbf{R}x = \mathbf{R}y$, for distinct variables $x, y \in V \cup T$, then "$x = y$" $\in \mathcal{S}$. So, assume that "$x = y$" $\notin \mathcal{S}$, for two distinct variables $x, y \in V \cup T$ and consider first the case in which either x or y, say y, is a variable in T. From the definition of realization it follows that $\{k+1, k, \sigma(y)\} \in \mathbf{R}y$, while from **(R2)** and the fact that $\{k+1, k, \sigma(y)\}$ is not a pair with respect to $\pi_{|V|+|T|}$, it follows that $\{k+1, k, \sigma(y)\} \notin \mathbf{R}x$, unless $x \in T$ and $\{k+1, k, \sigma(y)\} = \{k+1, k, \sigma(x)\}$. But in such a case, we would have $\sigma(x) = \sigma(y)$ and therefore x and y must coincide, contradicting our initial assumption that x and y are distinct variables. Therefore we have $\mathbf{R}x \neq \mathbf{R}y$.

Next, let us assume that $x, y \in V$. We will induction on $\max(\mathsf{height}_{\mathcal{S}}(x), \mathsf{height}_{\mathcal{S}}(y))$. From the V-extensionality of $\mathcal{S}$ it follows that x, y are distinguished in $\mathcal{S}$ by a variable z or by a pair $[z', z'']$. Let us first assume that x, y are distinguished in $\mathcal{S}$ by a variable z. If "$z \in \bar{\pi}(x)$" $\in \mathcal{S}$ and "$z \in \bar{\pi}(y)$" $\notin \mathcal{S}$, then for all w such that "$w \in \bar{\pi}(y)$" $\in \mathcal{S}$ we have $\mathbf{R}z \neq \mathbf{R}w$ by the inductive hypothesis, since $\mathsf{height}_{\mathcal{S}}(z) < \mathsf{height}_{\mathcal{S}}(x)$ and $\mathsf{height}_{\mathcal{S}}(w) < \mathsf{height}_{\mathcal{S}}(y)$. Furthermore, from **(R1)** it follows also that $\mathbf{R}z \neq \mathbf{R}[w, w']$, for all w, w' such that "$[w, w'] \in y$" $\in \mathcal{S}$. Thus $\mathbf{R}z \in \mathbf{R}x \setminus \mathbf{R}y$. If "$z \in \bar{\pi}(y)$" $\in \mathcal{S}$ and "$z \in \bar{\pi}(x)$" $\notin \mathcal{S}$ we can prove that $\mathbf{R}z \in \mathbf{R}y \setminus \mathbf{R}x$ in an analogous way. In both case we have $\mathbf{R}x \neq \mathbf{R}y$. On the other hand, if x, y are distinguished by a pair $[z', z'']$, we can argue as follows. Assume first that "$[z', z''] \in x$" $\in \mathcal{S}$ and "$[z', z''] \in y$" $\notin \mathcal{S}$. Plainly, $\mathbf{R}[z', z''] \in \mathbf{R}x$. If $\mathbf{R}[z', z''] \in \mathbf{R}y$, then by **(R1)** $\mathbf{R}[z', z''] = \mathbf{R}[w', w'']$, for a pair $[w', w'']$ such that "$[w', w''] \in y$" $\in \mathcal{S}$. Since $\pi_{|V|+|T|}$ is a pairing function, we have $\mathbf{R}z' = \mathbf{R}w'$ and $\mathbf{R}z'' = \mathbf{R}w''$. Considering that $\mathsf{height}_{\mathcal{S}}(z'), \mathsf{height}_{\mathcal{S}}(z'') < \mathsf{height}_{\mathcal{S}}(x)$ and that $\mathsf{height}_{\mathcal{S}}(w'), \mathsf{height}_{\mathcal{S}}(w'') < \mathsf{height}_{\mathcal{S}}(y)$, the inductive hypothesis yields that

- z' and w' coincide or "$z' = w'$" is in $\mathcal{S}$, and
- z'' and w'' coincide or "$z'' = w''$" is in $\mathcal{S}$.

But then, by the V-extensionality of $\mathcal{S}$, "$[z', z''] \in y$" would be in $\mathcal{S}$, a contradiction. Hence, $\mathbf{R}[z', z''] \in \mathbf{R}x \setminus \mathbf{R}y$. Analogously, if "$[z', z''] \in x$" $\notin \mathcal{S}$ and "$[z', z''] \in y$" $\in \mathcal{S}$, we have $\mathbf{R}[z', z''] \in \mathbf{R}y \setminus \mathbf{R}x$. Therefore, in both cases we have $\mathbf{R}x \neq \mathbf{R}y$, proving **(R3)**.

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[3] We recall that the *rank* of a set $u \in \mathcal{V}$ denotes the least ordinal γ such that $u \subseteq \mathcal{V}_\gamma$ (i.e., $u \in \mathcal{V}_{\gamma+1}$).

The cases **(R4)** and **(R5)** are easy consequences of **(R1)**, **(R2)**, and **(R3)**. Details are left to the reader. This completes the proof of the lemma. ◀

Realizations act as *minimal models* for skeletal representations, in the sense that if V, T are two disjoint sets of variables, $\mathcal{S}$ is a V-extensional skeletal representation such that $\mathit{Vars}(\mathcal{S}) \subseteq V \cup T$, and $\mathbf{R}$ is the realization of $\mathcal{S}$ relative to (V, T) (and to a bijection σ), then $\mathbf{R} \models \alpha$ if and only if $\alpha \in \mathcal{S}$.

In the next theorem we show how skeletal representations can be used to witness the satisfiability of $\forall_{\mathbf{0}}^{\pi}$-formulae.

▶ **Theorem 3.** *Let φ be a $\forall_{\mathbf{0}}^{\pi}$-formula, and let $V = \mathit{Vars}(\varphi)$. Then φ is satisfiable iff there exists a V-extensional skeletal representation $\mathcal{S}$ such that:*

(i) $\mathit{Vars}(\mathcal{S}) \subseteq V \cup T$, for some T such that $1 \leq |T| < 2|V|$;
(ii) $\mathbf{R} \models \varphi$, where $\mathbf{R}$ is the realization of $\mathcal{S}$ relative to (V, T).

Proof. To prove the theorem, it is enough to exhibit a skeletal representation $\mathcal{S}$ that satisfies conditions (i) and (ii) above, given a model $\mathbf{I}$ for φ.

Thus, let $\mathbf{I}$ be a model for φ and let $\Sigma = \{\mathbf{I}x : x \in V\}$. As shown in [3], there exists a collection Σ_0 of size strictly less than $|\Sigma|$ which witnesses all the inequalities among the members of Σ, in the sense that $s \cap \Sigma_0 \neq s' \cap \Sigma_0$ for any two distinct $s, s' \in \Sigma$. Let us *split* the pairs present in Σ_0 (relative to the pairing function $\pi_{\mathbf{I}}$ of $\mathbf{I}$) forming the collection

$$\Sigma_1 =_{\text{Def}} \{s : s \in \Sigma_0 \wedge (\forall u, v \in \mathcal{V})(s \neq \pi_{\mathbf{I}}(u, v))\} \cup \bigcup \{\{u, v\} : \pi_{\mathbf{I}}(u, v) \in \Sigma_0\}.$$

Then we put

$$\Sigma_2 =_{\text{Def}} \begin{cases} \Sigma_1 \setminus \Sigma & \text{if } \Sigma_1 \setminus \Sigma \neq \emptyset \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

and let T be any collection of variables in Vars, not already occurring in φ, such that $|T| = |\Sigma_2|$ (so that $|T| \geq 1$). Notice that $|T| \leq 2|\Sigma_0| + 1 < 2|V|$.

Finally, we define our skeletal representation as the collection $\mathcal{S}$ of atomic $\forall_{\mathbf{0}}^{\pi}$-formulae such that:

$$\begin{aligned} \text{“}x \in \bar{\pi}(y)\text{”} \in \mathcal{S} &\iff \mathbf{I}x \in \mathbf{I}\bar{\pi}(y) \\ \text{“}[x, y] \in z\text{”} \in \mathcal{S} &\iff \mathbf{I}[x, y] \in \mathbf{I}z \\ \text{“}x = y\text{”} \in \mathcal{S} &\iff \mathbf{I}x = \mathbf{I}y \ \text{ and } \ x, y \in V \end{aligned}$$

for all $x, y, z \in V \cup T$.

As can be easily verified, the above construction process yields a V-extensional skeletal representation $\mathcal{S}$ satisfying condition (i) of the theorem.

We prove next that also condition (ii) is satisfied, i.e. $\mathbf{R} \models \varphi$ holds, where $\mathbf{R}$ is the realization of $\mathcal{S}$ relative to (V, T). This amounts to showing that $\mathbf{R}$ models correctly all conjuncts of φ. These are simple prenex $\forall_{\mathbf{0}}^{\pi}$-formulae whose free variables belong to $V \cup T$ and whose domain variables belong to V, which are correctly modeled by $\mathbf{I}$. It will therefore be enough to prove the following general property stating that

$$\mathbf{I} \models \psi \Longrightarrow \mathbf{R} \models \psi, \tag{3}$$

for every simple prenex $\forall_{\mathbf{0}}^{\pi}$-formula ψ such that $\mathit{Vars}(\psi) \subseteq V \cup T$ and whose domain variables, if any, belong to V.

We prove (3) by induction on the length of the quantifier prefix of ψ.

When ψ is quantifier-free, (3) follows from propositional logic, by observing that the definition of $\mathcal{S}$ together with conditions **(R3)**, **(R4)**, and **(R5)** of Lemma 2 yield that $\mathbf{I}\alpha = \mathbf{R}\alpha$, for each atomic $\forall_{\mathbf{0}}^{\pi}$-formula α such that $Vars(\alpha) \subseteq V \cup T$.

For the inductive step, let ψ have either the form $(\forall x \in \bar{\pi}(y))\chi$ or the form $(\forall [x,y] \in z)\chi$, with χ a simple prenex $\forall_{\mathbf{0}}^{\pi}$-formula having one less quantifier than ψ. For the sake of simplicity, we consider here only the case in which ψ has the form $(\forall x \in \bar{\pi}(y))\chi$, as the other case can be dealt with much in the same manner. We remark that, by hypothesis, the domain variable y in $(\forall x \in \bar{\pi}(y))\chi$ belongs to V.

Let us assume that $\mathbf{I} \models \psi$. To complete the inductive proof of (3) we need to show that $\mathbf{R} \models \psi$. From $\mathbf{I} \models \psi$ it follows that $\mathbf{I} \models (w \in \bar{\pi}(y)) \rightarrow \chi_w^x$, for every variable w, and in particular for every variable $w \in W$, where $W =_{\text{Def}} \{w \in V \cup T : \text{“}w \in \bar{\pi}(y)\text{”} \in \mathcal{S}\}$. Let $w \in W$. We clearly have $\mathbf{I} \models w \in \bar{\pi}(y)$, and therefore $\mathbf{I} \models \chi_w^x$. Plainly, $Vars(\chi_w^x) \subseteq V \cup T$. In addition, all domain variables in χ_w^x belong to V, since this is the case for all domain variables in χ and w can not appear in χ_w^x as a domain variable, since x is a quantified variable of ψ and as such can not appear also as a domain variable in ψ, and therefore in χ. Hence, by inductive hypothesis, we have $\mathbf{R} \models \chi_w^x$ and, *a fortiori*, $\mathbf{R} \models (w \in \bar{\pi}(y)) \rightarrow \chi_w^x$.

Notice that the latter relation holds also for $w \in (V \cup T) \setminus W$, since in this case $\mathbf{I} \not\models (w \in \bar{\pi}(y))$ and therefore, as observed above, $\mathbf{R} \not\models (w \in \bar{\pi}(y))$. Thus we have

$$\mathbf{R} \models (w \in \bar{\pi}(y)) \rightarrow \chi_w^x, \tag{4}$$

for every $w \in V \cup T$. We show that (4) implies $\mathbf{R} \models (\forall x \in \bar{\pi}(y))\chi$, which is what we want to prove.

Indeed, if by contradiction $\mathbf{R} \not\models (\forall x \in \bar{\pi}(y))\chi$, then $\mathbf{R}' \not\models (x \in \bar{\pi}(y)) \rightarrow \chi$, for some $\{x\}$-variant $\mathbf{R}'$ of $\mathbf{R}$, so that $\mathbf{R}' \models (x \in \bar{\pi}(y))$ and $\mathbf{R}' \not\models \chi$. But then

$$\mathbf{R}'x \in \mathbf{R}'\bar{\pi}(y) = \mathbf{R}\bar{\pi}(y) \subseteq \{\mathbf{R}z : \text{“}z \in \bar{\pi}(y)\text{”} \in \mathcal{S}\}.$$

Therefore $\mathbf{R}'x = \mathbf{R}z_0$, for some variable z_0 (in $V \cup T$) such that the literal “$z_0 \in \bar{\pi}(y)$” belongs to $\mathcal{S}$. Thus we have $\mathbf{R} \models z_0 \in \bar{\pi}(y)$ and $\mathbf{R} \not\models (z_0 \in \bar{\pi}(y)) \rightarrow \chi_{z_0}^x$, contradicting (4). Hence, $\mathbf{R} \models (\forall x \in \bar{\pi}(y))\chi$ holds, completing the inductive proof of (3) and, in turn, the proof of condition (ii) of the theorem. ◀

Theorem 3 yields a decision test for the s.p. for $\forall_{\mathbf{0}}^{\pi}$-formulae, as the number of possible V-extensional skeletal representations satisfying condition (i) of the theorem is finite, for any given $\forall_{\mathbf{0}}^{\pi}$-formula, and condition (ii) is effectively verifiable. In the following section, we analyze the s.p. for $\forall_{\mathbf{0}}^{\pi}$-formulae from a complexity point of view.

3.1 Complexity issues

The s.p. for propositional logic can be easily reduced to the one for $\forall_{\mathbf{0}}^{\pi}$-formulae as follows. Given a propositional formula Q, we construct in linear time a quantifier-free $\forall_{\mathbf{0}}^{\pi}$-formula φ_Q, by replacing each propositional variable p in Q with a corresponding atomic $\forall_{\mathbf{0}}^{\pi}$-formula $x_p \in \bar{\pi}(U)$, where U is a set variable distinct from all set variables x_p so introduced. It is then immediate to check that Q is propositionally satisfiable if and only if the resulting $\forall_{\mathbf{0}}^{\pi}$-formula φ_Q is satisfiable. Thus the NP-hardness of the satisfiability of $\forall_{\mathbf{0}}^{\pi}$-formulae follows immediately.

Having shown a lower bound for the s.p. for $\forall_{\mathbf{0}}^{\pi}$-formulae, we next give an upper bound for it, proving that it is in the NExpTime class and, furthermore, when restricted to a certain useful collection of $\forall_{\mathbf{0}}^{\pi}$-formulae, it is NP-complete.

As proved in Theorem 3, satisfiability of a $\forall_0^\pi$-formula φ can be tested by first guessing a skeletal representation $\mathcal{S}$, whose size is polynomial in the size $|\varphi|$ of φ (since $|\,\mathit{Vars}(\mathcal{S})| < 3 \cdot |\,\mathit{Vars}(\varphi)|$), and then verify that the formula φ is modeled correctly by the realization $\mathbf{R}$ of S relative to (V,T), where $V = \mathit{Vars}(\varphi)$ and $T = \mathit{Vars}(\mathcal{S}) \setminus \mathit{Vars}(\varphi)$. Construction of the realization $\mathbf{R}$ takes polynomial time, however to verify that $\mathbf{R} \models \varphi$ can take exponential time. Indeed, it is easy to check that $\mathbf{R}$ models correctly φ if and only if it satisfies the *expansion* $\mathsf{Exp}_{\mathcal{S}}(\varphi)$ of φ relative to $\mathcal{S}$, which we define shortly. For a simple prenex $\forall_0^\pi$-formula ψ, we put

$$\mathsf{exp}_{\mathcal{S}}(\psi) \;=_{\mathrm{Def}} \begin{cases} \psi & \text{if } \psi \text{ is quantifier-free,} \\ \displaystyle\bigwedge_{\text{“}x' \in \bar{\pi}(y)\text{”} \in \mathcal{S}} \mathsf{exp}_{\mathcal{S}}(\chi^{x}_{x'}) & \text{if } \psi = (\forall x \in \bar{\pi}(y))\chi, \\ \displaystyle\bigwedge_{\text{“}[x',y'] \in z\text{”} \in \mathcal{S}} \mathsf{exp}_{\mathcal{S}}(\chi^{x,\,y}_{x',y'}) & \text{if } \psi = (\forall [x,y] \in z)\chi. \end{cases}$$

Then we put

$$\mathsf{Exp}_{\mathcal{S}}(\varphi) \;=_{\mathrm{Def}} \mathsf{exp}_{\mathcal{S}}(\varphi_1) \wedge \ldots \wedge \mathsf{exp}_{\mathcal{S}}(\varphi_n),$$

where $\varphi_1, \ldots, \varphi_n$ are the (simple prenex) conjuncts of φ. If ℓ is the longest quantifier prefix of the formulae $\varphi_1, \ldots, \varphi_n$, then it turns out that $|\mathsf{Exp}_{\mathcal{S}}(\varphi)| = \mathcal{O}(|\varphi|^{2\ell}) = \mathcal{O}(|\varphi|^{2\cdot|\varphi|})$, and therefore to test whether $\mathbf{R} \models \mathsf{Exp}_{\mathcal{S}}(\varphi)$ takes at most exponential time, showing that the s.p. for $\forall_0^\pi$-formula is in NExpTime.

However, the same proof shows that if we restrict to the collection of $\forall_0^\pi$-formulae whose quantifier prefixes are bounded by a constant $h \geq 0$, which we call $(\forall_0^\pi)^{\leq h}$, then $|\mathsf{Exp}_{\mathcal{S}}(\varphi)|$ is only polynomial in $|\varphi|$, for any $(\forall_0^\pi)^{\leq h}$-formula φ, and therefore to test whether $\mathbf{R}$ models correctly $\mathsf{Exp}_{\mathcal{S}}(\varphi)$, and in turn to test whether $\mathbf{R} \models \varphi$, takes polynomial time in $|\varphi|$, proving the following result:

▶ **Corollary 4.** *The s.p. for $(\forall_0^\pi)^{\leq h}$-formulae is* NP*-complete, for any $h \geq 0$.* ◀

In the rest of the paper we describe some applications of $\forall_0^\pi$-formulae in the field of knowledge representation. More specifically, in the next section we introduce a novel description logic whose consistency problem can be reduced to the s.p. for $(\forall_0^\pi)^{\leq 2}$-formulae. Such description logic will then be extended with Horn-style rules in Section 5.

4 The description logic $\mathcal{DL}\langle\forall_0^\pi\rangle$

Description logics are a family of logic-based formalisms which allow to represent knowledge about a domain of interest in terms of *concepts* (which denote sets of elements), *roles* (which represent relations between elements), and *individuals* (which denote domain elements). Each language in this family is mainly characterized by its set of *constructors*, which allow to form complex terms starting from *concept names*, *role names*, and *individual names* (see Table 1 for the syntax and semantics of the most widely used description logic constructs). A description logic *knowledge base* is a finite set of statements which define constraints on the domain structure.

Description logic semantics[4] is given in terms of *interpretations*. An interpretation $\mathcal{I}$ consists of a nonempty *domain* $\Delta^{\mathcal{I}}$ and an interpretation function assigning to each concept

[4] Here we are recalling the *descriptive* semantics. There are several other semantics that are out of the scope of this paper.

name a subset of $\Delta^{\mathcal{I}}$, to every role name a relation over $\Delta^{\mathcal{I}}$, and to every individual name a domain item in $\Delta^{\mathcal{I}}$. An interpretation $\mathcal{I}$ extends recursively to complex terms. An interpretation $\mathcal{I}$ that satisfies all the constraints of a knowledge base $\mathcal{K}$ is said to be a *model* for $\mathcal{K}$. A knowledge base is said to be *consistent* if it admits a model. Thus the *consistency problem* for description logic knowledge bases is to determine whether a knowledge base is consistent or not.

It turns out that the semantical definitions of several description logic statements Σ may be expressed as formulae of the form

$$\mathcal{I} \models \Sigma \text{ iff } \big(\forall x_1 \in \Delta^{\mathcal{I}}\big) \ldots \big(\forall x_n \in \Delta^{\mathcal{I}}\big)\, \Gamma_\Sigma,$$

where Γ_Σ is a Boolean combination of expressions of the types

$$u \in C^{\mathcal{I}},\ [u,u'] \in R^{\mathcal{I}},\ u = a^{\mathcal{I}},\ u = u',$$

with C, R, a respectively a concept term, a role term, and an individual name, and with u, u' ranging over the variables $x_1, \ldots, x_n$ (see Table 1).

This holds in particular for all the knowledge base statements allowed in the novel description logic $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ defined next.

▶ **Definition 5.** Let $\mathcal{N}^c, \mathcal{N}^r, \mathcal{N}^i$ be the three denumerable, infinite and mutually disjoint collections of, respectively, concept, role, and individual names. $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-concept terms and $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-role terms are formed according to the following syntax rules:

$$\begin{array}{lcl} C, D & \longrightarrow & A \,|\, \top \,|\, \bot \,|\, \neg C \,|\, C \sqcup D \,|\, C \sqcap D \,|\, \{a\} \,|\, \exists R.\mathsf{Self} \,|\, \exists R.\{a\} \\ R, S & \longrightarrow & P \,|\, \mathbb{U} \,|\, R^- \,|\, \neg R \,|\, R \sqcup S \,|\, R \sqcap S \,|\, R_{C|} \,|\, R_{|D} \,|\, R_{C|D} \,|\, \mathsf{id}(C) \,|\, \mathsf{sym}(R) \end{array}$$

where C, D denote $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-concept terms, R, S denote $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-role terms, A, P denote a concept and a role name, respectively, and a denotes an individual name. A $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-knowledge base is then a finite collection of statements of the following types:

$$\begin{array}{lllll} C \equiv D\,, & C \sqsubseteq D\,, & R \equiv S\,, & R \sqsubseteq S\,, & C \sqsubseteq \forall R.D\,, \\ \exists R.C \sqsubseteq D\,, & R \circ R' \sqsubseteq S\,, & \mathsf{Trans}(R)\,, & \mathsf{Ref}(R)\,, & \mathsf{ASym}(R) \end{array}$$

where C, D are $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-concept terms and R, S, R' are $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-role terms.

Notice that the above definition of $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ is not minimal, as we intended to give a clear and immediate overview of its expressive power.

The major limitation of $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ (with respect to other description logics) is that value restriction and existential quantification are restricted to the left-hand side and right-hand side of inclusions, respectively. Moreover, number restrictions are not allowed. On the other hand, the set of allowed constructs is extremely large. In particular, complex role constructors can be used freely, in contrast with most expressive description logics. Additionally, reasoning in $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ is NP-complete, as will be proved in the following theorem.

▶ **Theorem 6.** *The consistency problem for $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-knowledge bases is* NP*-complete.*

Proof. We will show that the consistency problem for $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-knowledge bases reduces to the satisfiability problem for $(\forall_0^{\pi})^{\leq 2}$-formulae.

We begin with observing that we can restrict our attention to $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-knowledge bases containing only statements of the following types:

$$\begin{array}{lllllll} A \equiv \top\,, & A \equiv \neg B\,, & A \equiv B \sqcup B'\,, & A \equiv \{a\}\,, & A \sqsubseteq \forall P.B\,, & \exists P.A \sqsubseteq B\,, & A \equiv \exists P.\{a\}, \\ P \equiv \mathbb{U}\,, & P \equiv \neg Q\,, & P \equiv Q \sqcup Q'\,, & P \equiv Q^-\,, & P \equiv \mathsf{id}(A)\,, & P \equiv Q_{A|}\,, & P \circ P' \sqsubseteq Q, \\ \mathsf{Ref}(P) \end{array}$$

$A^{\mathcal{I}}$	$\subseteq$	$\Delta^{\mathcal{I}}$	(concept name)
$P^{\mathcal{I}}$	$\subseteq$	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$	(role name)
$a^{\mathcal{I}}$	$\in$	$\Delta^{\mathcal{I}}$	(individual name)
$\top^{\mathcal{I}}$	$=$	$\Delta^{\mathcal{I}}$	(universal concept)
$\bot^{\mathcal{I}}$	$=$	$\emptyset$	(bottom concept)
$(\neg C)^{\mathcal{I}}$	$=$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$	(concept negation)
$(C \sqcup D)^{\mathcal{I}}$	$=$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$	(concept union)
$(C \sqcap D)^{\mathcal{I}}$	$=$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$	(concept intersection)
$\{a\}^{\mathcal{I}}$	$=$	$\{a^{\mathcal{I}}\}$	(nominal)
$(\exists R.\mathsf{Self})^{\mathcal{I}}$	$=$	$\{x \in \Delta^{\mathcal{I}} : [x,x] \in R^{\mathcal{I}}\}$	(self restriction)
$(\forall R.C)^{\mathcal{I}}$	$=$	$\{x \in \Delta^{\mathcal{I}} : (\forall [x,y] \in R^{\mathcal{I}})(y \in C^{\mathcal{I}})\}$	(value restriction)
$(\exists R.C)^{\mathcal{I}}$	$=$	$\{x \in \Delta^{\mathcal{I}} : (\exists y \in C^{\mathcal{I}})([x,y] \in R^{\mathcal{I}})\}$	(existential quantifier)
$(\leq nR.C)^{\mathcal{I}}$	$=$	$\{x \in \Delta^{\mathcal{I}} : \lvert\{y \in C^{\mathcal{I}} : [x,y] \in R^{\mathcal{I}}\}\rvert \leq n\}$	(number restrictions)
$(\geq nR.C)^{\mathcal{I}}$	$=$	$\{x \in \Delta^{\mathcal{I}} : \lvert\{y \in C^{\mathcal{I}} : [x,y] \in R^{\mathcal{I}}\}\rvert \geq n\}$	
$(R \subseteq S)^{\mathcal{I}}$	$=$	$\{x \in \Delta^{\mathcal{I}} : (\forall y \in \Delta^{\mathcal{I}})([x,y] \in R^{\mathcal{I}} \rightarrow [x,y] \in S^{\mathcal{I}})\}$	(role-value-map)
$\mathbb{U}^{\mathcal{I}}$	$=$	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$	(universal role)
$(\neg R)^{\mathcal{I}}$	$=$	$(\Delta \times \Delta) \setminus R^{\mathcal{I}}$	(role negation)
$(R \sqcup S)^{\mathcal{I}}$	$=$	$R^{\mathcal{I}} \cup S^{\mathcal{I}}$	(role union)
$(R \sqcap S)^{\mathcal{I}}$	$=$	$R^{\mathcal{I}} \cap S^{\mathcal{I}}$	(role intersection)
$(R^{-})^{\mathcal{I}}$	$=$	$\{[x,y] \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} : [y,x] \in R^{\mathcal{I}}\}$	(role inverse)
$(R_{C\vert})^{\mathcal{I}}$	$=$	$\{[x,y] \in R^{\mathcal{I}} : x \in C^{\mathcal{I}}\}$	(role restrictions)
$(R_{\vert D})^{\mathcal{I}}$	$=$	$\{[x,y] \in R^{\mathcal{I}} : y \in D^{\mathcal{I}}\}$	
$(R_{C\vert D})^{\mathcal{I}}$	$=$	$(R_{C\vert})^{\mathcal{I}} \cap (R_{\vert D})^{\mathcal{I}}$	
$id(C)^{\mathcal{I}}$	$=$	$\{[x,x] : x \in C^{\mathcal{I}}\}$	(role identity)
$(R \circ S)^{\mathcal{I}}$	$=$	$R^{\mathcal{I}} \circ S^{\mathcal{I}}$	(role composition)
$(R^{*})^{\mathcal{I}}$	$=$	$(R^{\mathcal{I}})^{*}$	(transitive closure)
$(\mathsf{sym}(R))^{\mathcal{I}}$	$=$	$R^{\mathcal{I}} \cup (R^{-})^{\mathcal{I}}$	(symmetric closure)
$\mathcal{I} \models C \sqsubseteq D$	$\iff$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$	(inclusion axioms)
$\mathcal{I} \models R \sqsubseteq S$	$\iff$	$R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$	
$\mathcal{I} \models C \equiv D$	$\iff$	$C^{\mathcal{I}} = D^{\mathcal{I}}$	(equivalence axioms)
$\mathcal{I} \models R \equiv S$	$\iff$	$R^{\mathcal{I}} = S^{\mathcal{I}}$	
$\mathcal{I} \models \mathsf{Trans}(R)$	$\iff$	$R^{\mathcal{I}} \circ R^{\mathcal{I}} \subseteq R^{\mathcal{I}}$	(role transitivity)
$\mathcal{I} \models \mathsf{Ref}(R)$	$\iff$	$(\mathsf{id}(\exists R.\top))^{\mathcal{I}} \subseteq R^{\mathcal{I}}$	(role reflexivity)
$\mathcal{I} \models \mathsf{ASym}(R)$	$\iff$	$R^{\mathcal{I}} \cap (R^{-})^{\mathcal{I}} = \emptyset$	(role asymmetry)
$\mathcal{I} \models C(a)$	$\iff$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$	(concept assertion)
$\mathcal{I} \models R(a,b)$	$\iff$	$[a^{\mathcal{I}}, b^{\mathcal{I}}] \in R^{\mathcal{I}}$	(role assertion)

Table 1 Description logic constructs

where A, B, B' are concept names, P, P', Q, Q' are role names, and a is an individual name, since any $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-knowledge base $\mathcal{K}$ can be easily transformed into a knowledge base $\mathcal{K}'$ which contains only statements of these types, and such that $\mathcal{K}$ is consistent if and only if $\mathcal{K}'$ is.

Next, we define a mapping τ from $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-statements to simple prenex $\forall_{\mathbf{0}}^{\pi}$-formulae as follows:

$$\begin{array}{rcl}
\tau(A \equiv \top) & =_{\mathrm{Def}} & (\forall x \in \bar{\pi}(\Delta))\,(x \in \bar{\pi}(A)) \\
\tau(A \equiv \neg B) & =_{\mathrm{Def}} & (\forall x \in \bar{\pi}(\Delta))\,(x \in \bar{\pi}(A) \leftrightarrow x \notin \bar{\pi}(B)) \\
\tau(A \equiv B \sqcup B') & =_{\mathrm{Def}} & (\forall x \in \bar{\pi}(\Delta))\,(x \in \bar{\pi}(A) \leftrightarrow x \in \bar{\pi}(B) \vee x \in \bar{\pi}(B')) \\
\tau(A \equiv \{a\}) & =_{\mathrm{Def}} & (\forall x \in \bar{\pi}(\Delta))\,(x \in \bar{\pi}(A) \leftrightarrow x = a) \wedge a \in \bar{\pi}(A) \\
\tau(A \sqsubseteq \forall P.B) & =_{\mathrm{Def}} & (\forall [x,y] \in P)\,(x \in \bar{\pi}(A) \rightarrow y \in \bar{\pi}(B)) \\
\tau(\exists P.A \sqsubseteq B) & =_{\mathrm{Def}} & (\forall [x,y] \in P)\,(y \in \bar{\pi}(A) \rightarrow x \in \bar{\pi}(B)) \\
\tau(A \equiv \exists P.\{a\}) & =_{\mathrm{Def}} & (\forall x \in \bar{\pi}(\Delta))\,(x \in \bar{\pi}(A) \leftrightarrow [x,a] \in P) \\
\tau(P \equiv \mathbb{U}) & =_{\mathrm{Def}} & (\forall [x,y] \in \Delta)\,([x,y] \in P) \\
\tau(P \equiv \neg Q) & =_{\mathrm{Def}} & (\forall x,y \in \bar{\pi}(\Delta))\,([x,y] \in P \leftrightarrow [x,y] \notin Q) \\
\tau(P \equiv Q \sqcup Q') & =_{\mathrm{Def}} & (\forall x,y \in \bar{\pi}(\Delta))\,([x,y] \in P \leftrightarrow [x,y] \in Q \vee [x,y] \in Q') \\
\tau(P \equiv Q^{-}) & =_{\mathrm{Def}} & (\forall x,y \in \bar{\pi}(\Delta))\,([x,y] \in P \leftrightarrow [y,x] \in Q) \\
\tau(P \equiv Q_{A|}) & =_{\mathrm{Def}} & (\forall x,y \in \bar{\pi}(\Delta))\,([x,y] \in P \leftrightarrow [x,y] \in Q \wedge x \in \bar{\pi}(A)) \\
\tau(P \equiv \mathsf{id}(A)) & =_{\mathrm{Def}} & (\forall x,y \in \bar{\pi}(\Delta))\,([x,y] \in P \leftrightarrow x = y \wedge x \in \bar{\pi}(A)) \\
\tau(P \circ P' \sqsubseteq Q) & =_{\mathrm{Def}} & (\forall [x,y] \in P)\,(\forall [y',z] \in P')\,(y = y' \rightarrow [x,z] \in Q) \\
\tau(\mathsf{Ref}(P)) & =_{\mathrm{Def}} & (\forall [x,y] \in P)\,([x,x] \in P)
\end{array}$$

We remark that in the above definition of the mapping τ we are assuming that the collection *Vars* of the variables of the language $\forall_{\mathbf{0}}^{\pi}$ contains all the concept, role, and individual names. Moreover, we used the same symbol Δ which is also used to denote the domain of a description logic interpretation, under the assumption that $\Delta \notin \mathcal{N}^c \cup \mathcal{N}^r \cup \mathcal{N}^i$. These are just technical assumptions (not strictly necessary for the proof) which have been just introduced to enhance readability of the formulae $\tau(\cdot)$ and to emphasize the strong correlation between the semantical definitions of $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-statements and their corresponding $\forall_{\mathbf{0}}^{\pi}$-formulae.

Now let $\mathcal{K}$ be a $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-knowledge base. We define the $\forall_{\mathbf{0}}^{\pi}$-formula φ, expressing the consistency of $\mathcal{K}$, as follows

$$\begin{array}{lcl}
\varphi & =_{\mathrm{Def}} & \varphi_\Delta \wedge \varphi_C \wedge \varphi_R \wedge \varphi_I \wedge \varphi_{\mathcal{K}} \\
\varphi_\Delta & =_{\mathrm{Def}} & (\forall [x,y] \in \Delta)\,([x,y] \notin \Delta) \\
\varphi_C & =_{\mathrm{Def}} & \bigwedge_{A \in \mathsf{Cpts}} ((\forall x \in \bar{\pi}(A))\,(x \in \bar{\pi}(\Delta)) \wedge (\forall [x,y] \in A)\,([x,y] \notin A)) \\
\varphi_R & =_{\mathrm{Def}} & \bigwedge_{P \in \mathsf{Rls}} ((\forall x \in \bar{\pi}(P))\,(x \notin \bar{\pi}(P)) \wedge (\forall [x,y] \in P)\,(x \in \bar{\pi}(\Delta) \wedge y \in \bar{\pi}(\Delta))) \\
\varphi_I & =_{\mathrm{Def}} & \bigwedge_{a \in \mathsf{Inds}} a \in \bar{\pi}(\Delta) \\
\varphi_{\mathcal{K}} & =_{\mathrm{Def}} & \bigwedge_{\Sigma \in \mathcal{K}} \tau(\Sigma)
\end{array}$$

where $\mathsf{Cpts}, \mathsf{Rls}$, and Inds are respectively the sets of concept, role and individual names occurring in $\mathcal{K}$.

The consistency problem for $\mathcal{K}$ is equivalent to the satisfiability of φ, as we prove next.

Plainly, $\varphi_\Delta, \varphi_C, \varphi_R$, and φ_I guarantee that each model of φ can be easily turned into a $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-interpretation. Additionally, $\varphi_{\mathcal{K}}$ ensures that the $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$-interpretation obtained in this way satisfies all the statements in $\mathcal{K}$.

Conversely, let $\mathcal{I}$ be a model for $\mathcal{K}$. Without loss of generality, we may assume that $\Delta^{\mathcal{I}}$ is a set belonging to the von Neumann hierarchy $\mathcal{V}$ (otherwise, we embed $\Delta^{\mathcal{I}}$ in $\mathcal{V}$). Let

$\mathbf{I} = (M_\mathbf{I}, \pi_\mathbf{I})$ be the $\forall_\mathbf{0}^\pi$-interpretation, induced by $\mathcal{I}$, defined by

$$\begin{array}{rlll} \pi_\mathbf{I}(u,v) & =_{\text{Def}} & \{u, \{u,v\}, \Delta^\mathcal{I}\} & \text{for all } u,v \in \mathcal{V} \\ M_\mathbf{I}\Delta & =_{\text{Def}} & \Delta^\mathcal{I} & \\ M_\mathbf{I}A & =_{\text{Def}} & A^\mathcal{I} & \text{for all } A \in \mathcal{N}^c \\ M_\mathbf{I}P & =_{\text{Def}} & \{\pi_\mathbf{I}(u,v) : [u,v] \in P^\mathcal{I}\} & \text{for all } P \in \mathcal{N}^r \\ M_\mathbf{I}a & =_{\text{Def}} & a^\mathcal{I} & \text{for all } a \in \mathcal{N}^i. \end{array}$$

Since $\mathbf{I}\Delta \in \pi_\mathbf{I}(u,v)$ for all $u,v \in \mathcal{V}$, from the well-foundedness of the membership relation it follows that $\mathbf{I}\Delta$ does not contain any pair (with respect to $\pi_\mathbf{I}$). Thus $x \in \mathbf{I}\bar{\pi}(A) \iff x \in A^\mathcal{I}$ and $\pi_\mathbf{I}(x,y) \in \mathbf{I}P \iff [x,y] \in P^\mathcal{I}$ follow from the definition of $\mathbf{I}$, and then $\mathbf{I}\tau(\Sigma) = \mathbf{true}$ if and only if $\mathcal{I}$ satisfies Σ, for all the statements $\Sigma \in \mathcal{K}$.

We conclude the proof by observing that each conjunct in φ contains at most two quantifiers (i.e., φ is a formula of $(\forall_0^\pi)^{\leq 2}$), thus in view of Corollary 4 the satisfiability of φ can be checked in nondeterministic polynomial time, while the NP-hardness of this problem follows directly from the NP-completeness of the satisfiability problem for propositional formulae. ◀

5 Extending $\mathcal{DL}\langle\forall_\mathbf{0}^\pi\rangle$ with SWRL rules

In order to increase the expressive power of description logics, in [12] it was proposed to extend this framework with a simple form of Horn-style rules called SWRL rules. SWRL rules have the form

$$H \to B_1 \wedge \ldots \wedge B_n$$

where $H, B_1, \ldots, B_n$ are *atoms* of the forms $A(x), P(x,y), x = y, x \neq y$, with A a concept name, P a role name, and x,y either SWRL-variables or individual names.

A *binding* $\mathcal{B}(\mathcal{I})$ is any extension of the interpretation $\mathcal{I}$ which assigns a domain item to each SWRL-variable. An interpretation $\mathcal{I}$ satisfies a rule $H \to B_1 \wedge \ldots \wedge B_n$ if each binding $\mathcal{B}(\mathcal{I})$ which satisfies all the atoms $B_1, \ldots, B_n$ satisfies H also.

A $\mathcal{DL}\langle\forall_\mathbf{0}^\pi\rangle$-knowledge base $\mathcal{K}$ extended with a finite set of SWRL rules $\mathcal{R}$ is said to be *satisfiable* if and only if it has a model which satisfies all the rules in $\mathcal{R}$.

The reduction provided in Section 4 can be easily extended to cope with $\mathcal{DL}\langle\forall_\mathbf{0}^\pi\rangle$-knowledge bases extended with finite sets of SWRL rules, as shown in the following theorem.

▶ **Theorem 7.** *The consistency problem for $\mathcal{DL}\langle\forall_\mathbf{0}^\pi\rangle$-knowledge bases extended with finite sets of SWRL rules is decidable.*

Proof. Let $\mathcal{K}$ be a $\mathcal{DL}\langle\forall_\mathbf{0}^\pi\rangle$-knowledge base, and let $\mathcal{R}$ be a finite set of SWRL rules. Let us extend the mapping τ, defined in Theorem 6, to SWRL rules and atoms as follows:

$$\begin{array}{rll} \tau(H \to B_1 \wedge \ldots \wedge B_n) & =_{\text{Def}} & (\forall x_1, \ldots, x_m \in \bar{\pi}(\Delta))\,(\tau(H) \to \tau(B_1) \wedge \ldots \wedge \tau(B_n)) \\ \tau(A(x)) & =_{\text{Def}} & x \in \bar{\pi}(A) \\ \tau(P(x,y)) & =_{\text{Def}} & [x,y] \in P \\ \tau(x = y) & =_{\text{Def}} & x = y \\ \tau(x \neq y) & =_{\text{Def}} & x \neq y \end{array}$$

where $H, B_1, \ldots B_n$ are SWRL atoms, $x_1, \ldots, x_m$ are the SWRL variables occurring in $H \to B_1 \wedge \ldots \wedge B_n$, x,y can be either SWRL variables or individual names, and A, P are respectively a concept and a role name.

We conclude the proof by observing that the following $\forall_{\mathbf{0}}^{\pi}$-formula φ' is satisfiable if and only if the knowledge base $\mathcal{K}$ extended with $\mathcal{R}$ is consistent:

$$\varphi' =_{\mathrm{Def}} \bigwedge_{\rho \in \mathcal{R}} \tau(\rho) \wedge \varphi,$$

where φ is built from $\mathcal{K}$ as described in Theorem 6, extending Cpts, Rls and Inds with the concept, role and individual names occurring in $\mathcal{R}$, respectively. ◀

6 Conclusions and future works

We have introduced the collection of quantified $\forall_{\mathbf{0}}^{\pi}$-formulae of set theory, which allow the explicit manipulation of ordered pairs, and proved that they have a decidable satisfiability problem. In fact, when restricted to $\forall_{\mathbf{0}}^{\pi}$-formulae whose conjuncts have quantifier prefixes of length bounded by a constant, the satisfiability problem is NP-complete.

In addition, we have introduced the novel description logic $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ and shown that its consistency check is NP-complete, since it can be reduced to the satisfiability test for a $\forall_{\mathbf{0}}^{\pi}$-formula whose conjuncts involve at most two quantifiers. Finally we have extended the description logic $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ with SWRL rules without disrupting the decidability of the knowledge base consistency problem.

In contrast with description logics, the semantics of set theory is *multi-level*, so that sets (and consequently relations) can be nested arbitrarily. In the light of this observation, we intend to investigate whether the description logic $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ can be extended with *meta-modeling* features (cf. [15]), which would allow to state relationships among elements of the conceptual model.

Finally, we intend to investigate if $\forall_{\mathbf{0}}^{\pi}$ (and consequently $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$) can be extended with concrete domains, in order to promote definitively $\forall_{\mathbf{0}}^{\pi}$ as a language for knowledge representation, and, consequently, for the semantic web.

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