Towards Plane Spanners of Degree 3

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Abstract

Let S be a finite set of points in the plane that are in convex position. We present an algorithm that constructs a plane $\frac{3+4\pi}{3}$-spanner of S whose vertex degree is at most 3. Let Λ be the vertex set of a finite non-uniform rectangular lattice in the plane. We present an algorithm that constructs a plane $3\sqrt{2}$-spanner for Λ whose vertex degree is at most 3. For points that are in the plane and in general position, we show how to compute plane degree-3 spanners with a linear number of Steiner points.

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1 Introduction

Let S be a finite set of points in the plane. A *geometric graph* is a graph $G = (S, E)$ with vertex set S and edge set E consisting of line segments connecting pairs of vertices. The *length* (or *weight*) of any edge (p, q) in E is defined to be the Euclidean distance $|pq|$ between p and q. The *length* of any path in G is defined to be the sum of the lengths of the edges on this path. For any two vertices p and q of S, their *shortest-path distance* in G, denoted by $|pq|_G$, is a minimum length of any path in G between p and q. For a real number $t \geqslant 1$, the graph G is a *t-spanner* of S if for any two points p and q in S, $|pq|_G \leq t|pq|$. The smallest value of t for which G is a t-spanner is called the *stretch factor* of G. A large number of algorithms have been proposed for constructing t-spanners for any given point set; see [18].

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A *plane* spanner is a spanner whose edges do not cross each other. Chew [7] was the first to prove that plane spanners exist. Chew proved that the L_1-Delaunay triangulation of a finite point set has stretch factor at most $\sqrt{10} \approx 3.16$ (observe that lengths in this graph are measured in the Euclidean metric). In the journal version [8], Chew proves that the Delaunay triangulation based on a convex distance function defined by an equilateral triangle is a 2-spanner. Dobkin *et al.* [11] proved that the L_2-Delaunay triangulation is a t-spanner for $t = \frac{\pi(1+\sqrt{5})}{2} \approx 5.08$. Keil and Gutwin [16] improved the upper bound on the stretch factor to $t = \frac{4\pi}{3\sqrt{3}} \approx 2.42$. This was subsequently improved by Cui *et al.* [9] to $t = 2.33$ for the case when the point set is in convex position. Currently, the best result is due to Xia [19], who proved that t is less than 1.998. For points that are in convex position the current best upper bound on the stretch factor of plane spanners is 1.88 that was obtained by Amani *et al.* [1]. Regarding lower bounds, by considering the four vertices of a square, it is obvious that a plane t-spanner with $t < \sqrt{2}$ does not exist. Mulzer [17] has shown that every plane spanning graph for the vertices of a regular 21-gon has stretch factor at least 1.41611. Dumitrescu and Ghosh [13] improved the lower bound to 1.4308 for the vertices of a regular 23-gon.

The *degree* of a spanner is its maximum vertex degree. Das and Heffernan [10] showed the existence of spanners of maximum degree 3. Moreover, 3 is the lower bound on the maximum degree of a t-spanner, for any constant $t > 1$, because a Hamiltonian path through a set of points arranged in a grid has unbounded stretch factor; see [18] for more details. Even for points that are in convex position, 3 is a lower bound on the degree (see Kanj *et al.* [15]).

The problem of constructing bounded-degree spanners that are plane and have small stretch factor has received considerable attention (e.g., see [5, 6, 15]). Bonichon *et al.* [5] proved the existence of a degree 4 plane spanner with stretch factor 156.82. A simpler algorithm by Kanj *et al.* [15] constructs a degree 4 plane spanner with stretch factor 20; for points that are in convex position, this algorithm gives a plane spanner of degree at most 3 with the same stretch factor. Dumitrescu and Ghosh [12] considered plane spanners for uniform grids. For the infinite uniform square grid, they proved the existence of a plane spanner of degree 3 whose stretch factor is at most 2.607; the lower bound is $1 + \sqrt{2}$.

In this paper we consider bounded-degree plane spanners. In Section 3 we present an algorithm that computes a plane $\frac{3+4\pi}{3} \approx 5.189$-spanner of degree 3 for points in convex position. In Section 4 we consider finite non-uniform rectangular grids; we present an algorithm that computes a degree 3 plane spanner whose stretch factor is at most $3\sqrt{2} \approx 4.25$. In Section 5 we show that any plane t-spanner for points in the plane that are in general position, can be converted to a plane $(t + \epsilon)$-spanner of degree at most 3 that uses a linear number of Steiner points, where $\epsilon > 0$ is an arbitrary small constant.

2 Preliminaries

For any two points p and q in the plane let pq denote the line segment between p and q, $\ell(p, q)$ denote the line passing through p and q, $R(p{\to}q)$ denote the ray emanating from p and passing through q, and let $D(p, q)$ denote the closed disk that has pq as a diameter. Moreover, let $L(p, q)$ denote the lune of p and q, which is the intersection of the two closed disks of radius $|pq|$ that are centered at p and q.

Let S be a finite and non-empty set of points in the plane. We denote by $CH(S)$ the boundary of the convex hull of S. The *diameter* of S is the largest distance among the distances between all pairs of points of S. Any pair of points whose distance is equal to the diameter is called a *diametral pair*. Each point of diametral pair is called a *diametral point*.

Algorithm 1 MATCHING(C_1, C_2)

Input: Linearly separated chains C_1 and C_2 with the vertices of $C_1 \cup C_2$ in convex position.
Output: A matching between the points of C_1 and the points of C_2.

if $C_1 = \emptyset$ or $C_2 = \emptyset$ **then** **return** $\emptyset$
$(a, b) \leftarrow$ a closest pair of vertices between C_1 and C_2 such that $a \in C_1$ and $b \in C_2$
$C_1', C_1'' \leftarrow$ the two chains obtained by removing a from C_1
$C_2', C_2'' \leftarrow$ the two chains obtained by removing b from C_2
return $\{ab\} \cup$ MATCHING$(C_1', C_2') \cup$ MATCHING(C_1'', C_2'')

▶ **Observation 1.** *Let S be a finite set of at least two points in the plane, and let $\{p, q\}$ be any diametral pair of S. Then, the points of S lie in $L(p, q)$.*

▶ **Theorem 2** (See Theorem 7.11 in [3])**.** *If C_1 and C_2 are convex polygonal regions with $C_1 \subseteq C_2$, then the length of the boundary of C_1 is at most the length of the boundary of C_2.*

▶ **Lemma 3** (Amani *et al.* [1])**.** *Let a, b, and c be three points in the plane, and let $\beta = \angle abc$. Then, $\frac{|ab|+|bc|}{|ac|} \leqslant \frac{1}{\sin(\beta/2)}$.*

▶ **Lemma 4** (Proof in the full version of the paper [4])**.** *Let a and b be two points in the plane. Let c be a point that is on the boundary or in the interior of $L(a, b)$. Then, $\angle acb \geqslant \frac{\pi}{3}$.*

3 Plane Spanners for Points in Convex Position

In this section we consider degree-3 plane spanners for points that are in convex position. Let S be a finite set of points in the plane that are in convex position. Consider the two chains that are obtained from $CH(S)$ by removing any two edges. Let τ be the larger stretch factor of these two chains. In Section 3.1 we present an algorithm that computes a plane $(2\tau + 1)$-spanner of maximum degree 3 for S. Based on that, in Section 3.2 we show how to compute a plane $\frac{3+4\pi}{3}$-spanner of maximum degree 3 for S. Moreover, we show that if S is centrally symmetric, then there exists a plane $(\pi + 1)$-spanner of degree 3 for S.

3.1 Spanner for Convex Double Chains

Let C_1 and C_2 be two chains of points in the plane that are separated by a straight line. Let S_1 and S_2 be the sets of vertices of C_1 and C_2, respectively, and assume that $S_1 \cup S_2$ is in convex position. Let τ be a real number. In this section we show that if the stretch factor of each of C_1 and C_2 is at most τ, then there exists a plane $(2\tau + 1)$-spanner for $S_1 \cup S_2$ whose maximum vertex degree is 3.

In order to build such a spanner, we join C_1 and C_2 by a set of edges that form a matching. Thus, the spanner consists of C_1, C_2, and a set E of edges such that each edge has one endpoint in C_1 and one endpoint in C_2. The set E is a matching, i.e., no two edges of E are incident to a same vertex. We show how to compute E recursively. Let (a, b) be the closest pair of vertices between C_1 and C_2; see Figure 1. Add this closest pair (a, b) to E. Then remove (a, b) from C_1 and C_2, and recurse on the two pairs of chains obtained on each side of $\ell(a, b)$. Stop the recursion as soon as one of the chains is empty. Given C_1 and C_2, the algorithm MATCHING computes a set E.

▶ **Theorem 5.** *Let $C_1 = (S_1, E_1)$ and $C_2 = (S_2, E_2)$ be two linearly separated chains of points in the plane, each with stretch factor at most τ, such that $S_1 \cup S_2$ is in convex*

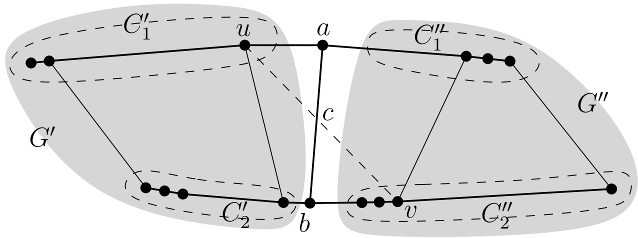


Figure 1 Illustration of the proof of Theorem 5.

position. Let E be the set of edges returned by algorithm MATCHING(C_1, C_2). *Then, the graph $G = (S_1 \cup S_2, E_1 \cup E_2 \cup E)$ is a plane $(2\tau + 1)$-spanner for $S_1 \cup S_2$ in which the degree of each endpoint of C_1 and C_2 is at most 2 and every other vertex has degree at most 3.*

In the rest of this section we prove Theorem 5. The degree and planarity constraints follows from the algorithm. However, in the full version of the paper [4] we prove these constraints by induction. Now, we prove that the stretch factor of G is at most $2\tau + 1$. The proof is by induction on $\min\{|S_1|, |S_2|\}$. As for the base cases, if $|S_1| = 0$, then $G = C_2$ is a τ-spanner. If $|S_2| = 0$, then $G = C_1$ is a τ-spanner. Assume $|S_1| \geqslant 1$ and $|S_2| \geqslant 1$. Let ℓ be a line that separates C_1 and C_2. Without loss of generality assume ℓ is horizontal, C_1 is above ℓ, and C_2 is below ℓ. Let (a, b) be the pair of vertices selected by algorithm MATCHING, where (a, b) is a closest pair of vertices between C_1 and C_2 such that $a \in C_1$ and $b \in C_2$.
Let C_1' and C_1'' be the left and right sub-chains of C_1, respectively, that are obtained by removing a; see Figure 1. We obtain C_2' and C_2'' similarly. Let G' (resp. G'') be the spanner obtained for the vertices of C_1' and C_2' (resp. C_1'' and C_2''). By the induction hypothesis, G' (resp. G'') is a $(2\tau + 1)$-spanner for the vertices of $C_1' \cup C_2'$ (resp. $C_1'' \cup C_2''$).

We are going to prove that for any two points $u, v \in S_1 \cup S_2$ we have $|uv|_G \leqslant (2\tau + 1)|uv|$. If both u and v belong to S_1, or both belong to S_2, then $|uv|_G \leqslant \tau|uv|$; this is valid because each of C_1 and C_2 has stretch factor at most τ. Assume $u \in S_1$ and $v \in S_2$. If $u, v \in G'$ or $u, v \in G''$ then, by the induction hypothesis, $|uv|_G \leqslant (2\tau + 1)|uv|$. Thus, it only remains to prove $|uv|_G \leqslant (2\tau + 1)|uv|$ for the following cases: (a) $u = a$ and $v \in C_2$, (b) $u \in C_1$ and $v = b$, (c) $u \in C_1'$ and $v \in C_2''$, and (d) $u \in C_1''$ and $v \in C_2'$. Because of symmetry we only prove items (a) and (c). The proofs are given in the following two lemmas.

▶ **Lemma 6.** *If $u = a$ and $v \in C_2$, then $|uv|_G \leqslant (2\tau + 1)|uv|$.*

Proof. Note that $|av|_G \leqslant |ab| + |bv|_{C_2} \leqslant |av| + \tau|bv|$, where the second inequality is valid since $|ab| \leqslant |av|$, by our choice of (a, b), and since $|bv|_{C_2} \leqslant \tau|bv|$, given that the stretch factor of C_2 is at most τ. It remains to prove that $|bv| \leqslant 2|av|$. By the triangle inequality we have $|bv| \leqslant |ab| + |av|$. Since $|ab| \leqslant |av|$, we have $|bv| \leqslant 2|av|$. ◀

▶ **Lemma 7.** *If $u \in C_1'$ and $v \in C_2''$, then $|uv|_G \leqslant (2\tau + 1)|uv|$.*

Proof. Since S is in convex position, the polygon Q formed by u, a, v, and b is convex and its vertices appear in the order u, a, v, b. Note that

$$|uv|_G \leqslant |ua|_{C_1} + |ab| + |bv|_{C_2} \leqslant \tau|ua| + |uv| + \tau|bv| = |uv| + \tau(|ua| + |bv|),$$

where the second inequality is valid since $|ab| \leqslant |uv|$, by our choice of (a, b), and since $|ua|_{C_1} \leqslant \tau|ua|$ and $|bv|_{C_2} \leqslant \tau|bv|$, given that the stretch factor of each of C_1 and C_2 is at most τ. It remains to prove that $|ua| + |bv| \leqslant 2|uv|$. Let c be the intersection point of ab and

uv; see Figure 1. By the triangle inequality, we have $|ua| \leqslant |uc| + |ca|$ and $|bv| \leqslant |bc| + |cv|$. It follows that $|ua| + |bv| \leqslant |uv| + |ab|$. Since $|ab| \leqslant |uv|$, we have $|ua| + |bv| \leqslant 2|uv|$. This completes the proof. ◀

3.2 Spanner for Points in Convex Position

In this section we show how to construct plane spanners of degree at most 3 for points that are in convex position.

▶ **Theorem 8.** *Let S be a finite set of points in the plane that is in convex position. Then, there exists a plane spanner for S whose stretch factor is at most $\frac{3+4\pi}{3}$ and whose vertex degree is at most 3.*

Proof. The proof is constructive; we present an algorithm that constructs such a spanner for S. The algorithm performs as follows. Let (p, q) be a diametral pair of S. Consider the convex hull of S. Let C_1 and C_2 be the two chains obtained from $CH(S)$ by removing p and q (and their incident edges). Note that C_1 and C_2 are separated by $\ell(p, q)$. Let G' be the graph on $S \setminus \{p, q\}$ that contains the edges of C_1, the edges of C_2, and the edges obtained by running algorithm $\textsc{Matching}(C_1, C_2)$. By Theorem 5, G' is plane and the endpoints of C_1 and C_2 have degree at most 2. We obtain a desired spanner, G, by connecting p and q, via their incident edges in $CH(S)$, to G'. In other words, $G = (S, E)$, where E is the union of the edges of $CH(S)$ and the edges of $\textsc{Matching}(C_1, C_2)$. A pseudo code for this construction is given in the full version of the paper [4].

Observe that G is plane. Moreover, all vertices of G have degree at most 3; p and q have degree 2. Now we show that the stretch factor of G is at most $\frac{3+4\pi}{3} \approx 5.19$. Note that G consists of $CH(S)$ and a matching which is returned by algorithm $\textsc{Matching}$. Since p and q are diametral points, then by a result of [1], for any point $s \in S \setminus \{p\}$ we have $|ps|_{CH(S)} \leqslant 1.88|ps|$. Since $CH(S) \subseteq G$, we have $|ps|_G \leqslant 1.88|ps|$. By symmetry, the same result holds for q and any point $s \in S \setminus \{q\}$. Since (p, q) is a diametral pair of S, both C_1 and C_2 are in $L(p, q)$. Based on this, in Theorem 11, we will see that both C_1 and C_2 have stretch factor at most $\frac{2\pi}{3}$. Then, by Theorem 5, the stretch factor of G' is at most $\frac{3+4\pi}{3}$. Since $G' \subset G$, for any two points $r, s \in S \setminus \{p, q\}$ we have $|rs|_G \leqslant \frac{3+4\pi}{3}|rs|$. Therefore, the stretch factor of G is at most $\frac{3+4\pi}{3}$. This completes the proof of the theorem. ◀

A point set S is said to be *centrally symmetric* (with respect to the origin), if for every point $p \in S$, point $-p$ also belongs to S.

▶ **Theorem 9.** *Let S be a finite centrally symmetric point set in the plane that is in convex position. Then, there exists a plane spanner for S whose stretch factor is at most $\pi + 1$ and whose vertex degree is at most 3.*

Proof. Let G be the graph obtained by the algorithm presented in the proof of Theorem 8. Recall that G is plane and its maximum vertex degree is at most 3. It remains to show that the stretch factor of G is at most $\pi + 1$. Let (p, q) be the diametral pair of S that is considered by this algorithm. Since S is centrally symmetric, all points of S are in $D(p, q)$. Based on this, in Theorem 10, we will see that both C_1 and C_2 have stretch factor at most $\frac{\pi}{2}$. Then Theorem 5 implies that the stretch factor of G is at most $\pi + 1$. ◀

3.3 Convex Chains with Diametral Endpoints

In this section we analyze the stretch factor of convex chains of points where their endpoints are a diametral pair. Let C be a chain of points. For any two points u and v on C let $\delta_C(u, v)$ denote the path between u and v on C, and let $|uv|_C$ denote the length of $\delta_C(u, v)$.

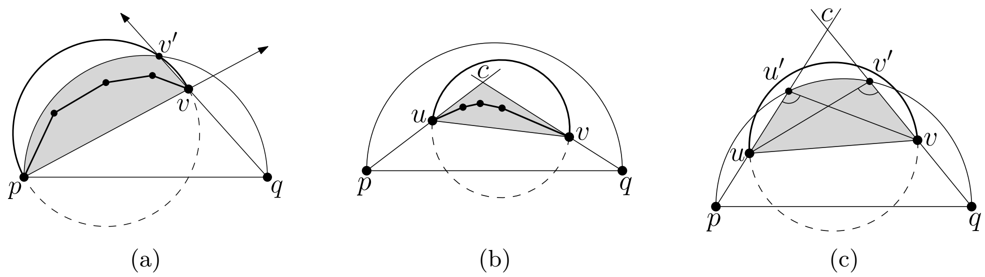


Figure 2 Proof of Theorem 10: the path $\delta_C(u, v)$ is inside the shaded regions.

▶ **Theorem 10.** *Let C be a convex chain with endpoints p and q. If C is in $D(p, q)$, then the stretch factor of C is at most $\frac{\pi}{2}$.*

Proof. Since C is convex, it is contained in a half-disk of $D(p, q)$, i.e., a half-disk with diameter pq. Let u and v be any two points of C. Let $\delta_C(u, v)$ be the path between u and v in C. We show that $\delta_C(u, v)$ is in $D(u, v)$. Then, by Theorem 2 the length of $\delta_C(u, v)$ is at most the length of the half-arc of $D(u, v)$, which is $\frac{\pi}{2}|uv|$. Without loss of generality assume that pq is horizontal, p is to the left of q, and C is above pq. Assume that u appears before v while traversing C from p to q. See Figure 2. We consider the following four cases.

- $u = p$ and $v = q$. Then $\delta_C(p, q) = C$ is in $D(p, q)$ by the hypothesis.
- $u = p$ and $v \neq q$. Let v' be the intersection point of $R(q \rightarrow v)$ with the boundary of $D(p, q)$. See Figure 2(a). Observe that $\angle pv'v = \angle pv'q = \frac{\pi}{2}$. Thus, v' is on the boundary of $D(p, v)$. Since two circles can intersect in at most two points, p and v' are the only intersection points of the boundaries of $D(p, q)$ and $D(p, v)$. Thus, the clockwise arc $\widehat{pv'}$ on the boundary of $D(p, q)$ is inside $D(p, v)$. Because of convexity, no point of $\delta_C(p, v)$ is to the right of $R(p \rightarrow v)$ or $R(q \rightarrow v)$. It follows that $\delta_C(p, v)$ is in $D(p, v)$.
- $u \neq p$ and $v = q$. The proof of this case is similar to the proof of the previous case.
- $u \neq p$ and $v \neq q$. Let c be the intersection point of $R(p \rightarrow u)$ and $R(q \rightarrow v)$. Because of convexity, $\delta_C(u, v)$ is in the triangle $\triangle ucv$. We look at two cases:
  - c is inside $D(p, q)$. See Figure 2(b). Note that $\angle ucv \geqslant \frac{\pi}{2}$. This implies that the point c, and consequently the triangle $\triangle ucv$, are in $D(u, v)$. Thus, $\delta_C(u, v)$ is inside $D(u, v)$.
  - c is outside $D(p, q)$. Let u' (resp. v') be the intersection point of $R(p \rightarrow u)$ (resp. $R(q \rightarrow v)$) with $D(p, q)$. Note that $\delta_C(u, v)$ is inside the shaded region of Figure 2(c). Observe that $\angle uv'v > \angle pv'q = \frac{\pi}{2}$, and $\angle uu'v > \angle pu'q = \frac{\pi}{2}$. Thus, both u' and v' are inside $D(u, v)$. Consequently, the clockwise arc $\widehat{u'v'}$ on the boundary of $D(p, q)$ is inside $D(u, v)$. Therefore, $\delta_C(u, v)$ is inside $D(u, v)$. ◀

▶ **Theorem 11** (Proof in the full version of the paper [4]). *Let C be a convex chain with endpoints p and q. If C is in $L(p, q)$, then the stretch factor of C is at most $\frac{2\pi}{3}$.*

4 Non-Uniform Rectangular Grid

In this section we build a plane spanner of degree at most three for the point set of the vertices of a non-uniform rectangular grid. In a finite non-uniform $m \times k$ grid, Λ, the vertices are arranged on the intersections of m horizontal and k vertical lines. The distances between the horizontal lines and the distances between the vertical lines are chosen arbitrary. The total number of vertices of Λ—the number of points of the underlying point set—is $n = m \cdot k$.

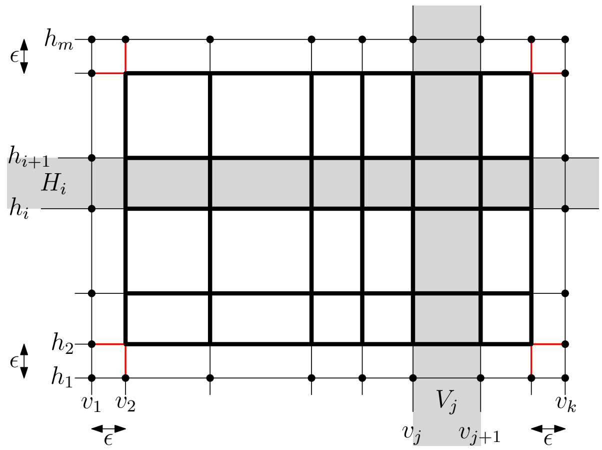


Figure 3 The augmented grid.

If $m \in \{1,2\}$ or $k \in \{1,2\}$ then Λ is a plane spanner whose maximum vertex degree is at most 3 and whose stretch factor is at most $\sqrt{2}$. Assume $m \geqslant 3$ and $k \geqslant 3$. We present an algorithm that constructs a degree-3 plane spanner, G, for the points of Λ. Note that Λ is a finite grid and has boundary vertices. In order to simplify the analysis and the proofs for boundary vertices, we augment Λ in the following way. We add four lines at distance ϵ, to the left, right, above and below Λ. We choose ϵ to be smaller than the distances among all pairs of vertical lines, and all pairs of horizontal lines of Λ. See Figure 3. For simplicity, in the rest of this section, we refer to the augmented lattice as Λ, and assume it has m horizontal and k vertical lines. Based on this assumption, the original lattice has $m-2$ horizontal lines and $k-2$ vertical lines.

Let $h_1, \dots, h_m$ be the horizontal lines of Λ from bottom to top. Similarly, let $v_1, \dots, v_k$ be the vertical lines of Λ from left to right. Note that Λ consists of $m-1$ horizontal slabs (rows) and $k-1$ vertical slabs (columns). Each horizontal slab H_i, with $1 \leqslant i < m$, is bounded by consecutive horizontal lines h_i and h_{i+1}. Each vertical slab V_j, with $1 \leqslant j < k$, is bounded by consecutive vertical lines v_j and v_{j+1}. See Figure 3. For each slab we define the *width* of that slab as the distance between the two parallel lines on its boundary. Let $p_{i,j}$, with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$, be the vertex of Λ that is the intersection point of h_i and v_j. For each H_i, $1 \leqslant i < m$, we define $E(H_i) = \{(p_{i,j}, p_{i+1,j}) : 2 \leqslant j \leqslant k-1\}$ as the set of edges of H_i. Moreover, we define the set of *candidate edges* of H_i as follows:

$$CE(H_i) = \begin{cases} \{(p_{i,j}, p_{i+1,j}) : 2 \leqslant j \leqslant k-1 \text{ and } j \text{ is even}\} & \text{if } i \text{ is even,} \\ \{(p_{i,j}, p_{i+1,j}) : 2 \leqslant j \leqslant k-1 \text{ and } j \text{ is odd}\} & \text{if } i \text{ is odd.} \end{cases}$$

Similarly, for each V_j, $1 \leqslant j < k$, we define $E(V_j) = \{(p_{i,j}, p_{i,j+1}) : 2 \leqslant i \leqslant m-1\}$ as the set of edges of V_j. The set of *candidate edges* of V_j is defined as follows:

$$CE(V_j) = \begin{cases} \{(p_{i,j}, p_{i,j+1}) : 2 \leqslant i \leqslant m-1 \text{ and } i \text{ is even}\} & \text{if } j \text{ is even,} \\ \{(p_{i,j}, p_{i,j+1}) : 2 \leqslant i \leqslant m-1 \text{ and } i \text{ is odd}\} & \text{if } j \text{ is odd.} \end{cases}$$

See Figure 4(a). Informally speaking, the set of edges of each horizontal slab contains $k-2$ vertical edges of Λ that are on $v_2, \dots, v_{k-1}$, and the set of edges of each vertical slab contains $m-2$ horizontal edges of Λ that are on $h_2, \dots, h_{m-1}$. The boundary edges of Λ, i.e., the edges with both their endpoints on the boundary of Λ, do not belong to any of these sets.

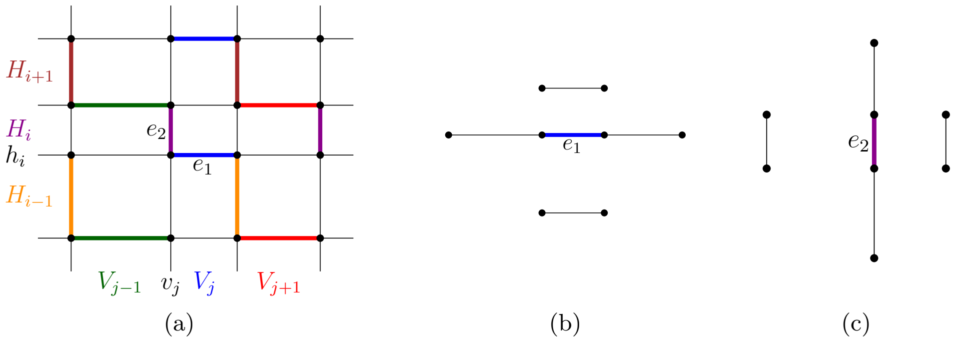


Figure 4 (a) Candidate edges in each slab are shown in their slab label color. The edges that are above, below, to the left, and to the right of a candidate edge, are not candidate in any of the slabs: (b) a horizontal candidate edge, and (c) a vertical candidate edge.

Every second vertical edge in H_i belongs to the set of candidate edges of H_i. The set of candidate edges of H_{i-1} (resp. H_{i+1}) contains every second vertical edge in H_{i-1} (rep. H_{i+1}) that is not adjacent to any candidate edge in H_i. The same observation applies on each V_i. Thus, if e is a candidate edge in some set, then the edges of Λ that are above, below, to the left, and to the right of e, are not candidate edges in any set. See Figures 4(b) and 4(c).

Now we describe the algorithm. We know that Λ is a $\sqrt{2}$-spanner of degree 4. The algorithm consists of two phases. In the first phase it removes some edges from Λ and constructs a graph G' whose largest vertex degree is 3 (G' may have large stretch factor). In the second phase the algorithm adds some edges to G' and constructs a graph G whose maximum degree is 3 and whose stretch factor is $3\sqrt{2}$. Note that $G' \subseteq G \subset \Lambda$. Refer to Figure 5 for an illustration of the two phases.

Phase 1 (Edge Deletion): In this phase, the algorithm iterates over all the slabs, $\{H_1, \dots, H_{m-1}, V_1, \dots, V_{k-1}\}$, in a non-increasing order of their widths. Let S be the current slab. The algorithm considers the candidate edges of S, i.e., all edges of $CE(S)$, from left to right if S is horizontal, and bottom-up if S is vertical (however, this ordering does not matter). The algorithm removes a candidate edge if it has at least one endpoint of degree 4. Let G' be the graph obtained at the end of this phase.

Phase 2 (Edge Insertion): Consider the graph G' obtained at the end of Phase 1. Let E' be the empty set. In the second phase, the algorithm iterates over all the slabs, $\{H_1, \dots, H_{m-1}, V_1, \dots, V_{k-1}\}$, in a non-decreasing order of their widths. Let S be the current slab. The algorithm considers all the edges of S, i.e., all edges of $E(S)$. Let $e = (a, b)$ be the current edge. The algorithms adds e to E' if both endpoints of e have degree 2 in $G' \cup E'$, i.e., $\deg_{G'}(a) + \deg_{E'}(a) = 2$ and $\deg_{G'}(b) + \deg_{E'}(b) = 2$. At the end of this phase, let G be the graph obtained by taking the union of G' and E'.

Consider the graph G obtained at the end of Phase 2. We show that G is a plane $3\sqrt{2}$-spanner of maximum degree 3 for Λ. Since the algorithm considers only the edges of Λ, then G is a subgraph of Λ and hence it is plane. As for the degree constraint, after Phase 1 the maximum degree in G' is 3. In Phase 2 we add edges between some vertices of degree 2 in G' (at most one edge per vertex) and hence no vertex of degree 4 can appear. Thus G has maximum degree 3. It only remains to show that G is a $3\sqrt{2}$-spanner. Before that, we review some properties of G' and G.

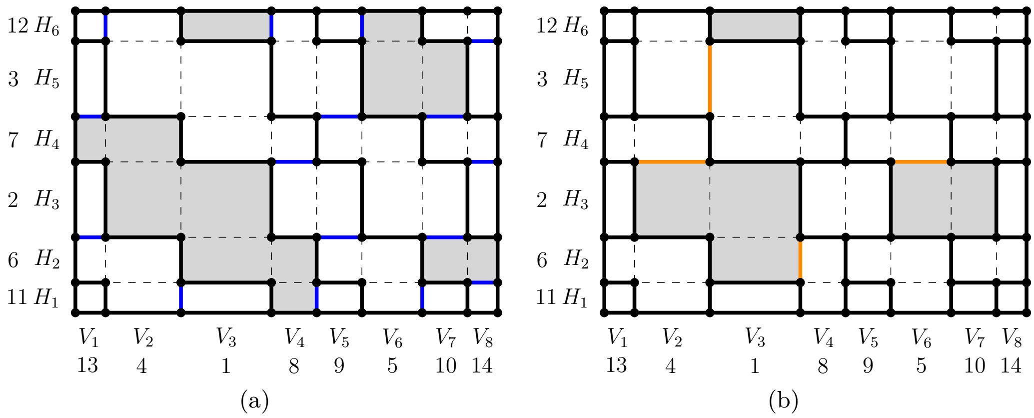


Figure 5 The numbers close to the slab labels show the order in which the slabs are considered in Phase 1. (a) The graph G' obtained at the end of Phase 1; its four types of faces are shaded (the blue edges are the candidate edges that have not been removed in Phase 1). (b) The graph G obtained at the end of Phase 2; its three types of faces are shaded (the orange edges have been added in Phase 2, and belong to E').

The candidate edges form stair-cases in Λ; see Figure 4(a). Moreover, the set of non-candidate edges (black edges of Figure 4(a)) also form stair-cases. Each internal vertex of Λ belongs to a staircase of candidate edges and a stair-case of non-candidate edges. Thus, in G', every vertex is on a stair-case of non-candidate edges that is connected to the boundary edges in both directions. Moreover, each of the stair-cases formed by candidate edges is surrounded by two stair-cases of non-candidate edges. Since G' contains all boundary edges, i.e., the edges with both endpoints on the boundary, each boundary vertex has degree at least 2 and at most 3 in G'. The edge deletion phase ensures that in G' there is no internal vertex of degree 4. Further, each internal vertex is incident on two candidate edges. Thus at the end of Phase 1, each internal vertex looses at most two edges, and hence has degree at least 2 in G'. Therefore we have the following observation.

▶ **Observation 12.** *The graph G' has the following properties.* (1) *G' contains all boundary edges of Λ,* (2) *G' is connected,* (3) *each vertex of G' has degree 2 or 3, and* (4) *each face in G' is either (see the shaded regions of Figure 5(a)):*

- *1-cell: consists of one cell of Λ, or*
- *2-cell: consists of two adjacent cells with the middle edge missing, or*
- *3-cell: consists of three adjacent cells which form an L-shape with the two middle edges missing (this L-shape might also be rotated), or*
- *stair-case: consists of more than three cells which form a stair-case with more than one vertex of degree two.*

We define $\{p_{1,1}, p_{m,1}, p_{1,k}, p_{m,k}\}$ as the set of *corner vertices* of Λ. We also define the set of *corner edges* of Λ as $\{(p_{1,2}, p_{2,2}), (p_{2,2}, p_{2,1}), (p_{1,k-1}, p_{2,k-1}), (p_{2,k-1}, p_{2,k}), (p_{m-1,1}, p_{m-1,2}), (p_{m-1,2}, p_{m,2}), (p_{m-1,k-1}, p_{m,k-1}), (p_{m-1,k-1}, p_{m-1,k})\}$. In Figure 3 the corner edges are in red. Note that each corner edge is adjacent to another corner edge. A *non-corner edge* is an edge of Λ which is not a corner edge.

▶ **Lemma 13** (Proof in the full version of the paper [4]). *All non-corner edges of Λ that are incident to a boundary vertex are in G'.*

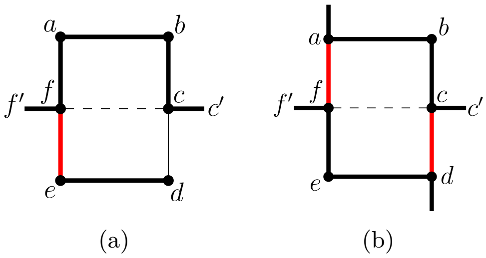


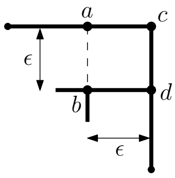
Figure 6 The edge (a, f) is a candidate edge (a) that is in G', and (b) that is not in G'.

By Lemma 13, any non-corner edge e of Λ that is not in G' has both its endpoints in the interior of Λ. That is, both endpoints of e have degree four in Λ. Based on this, and by our choice of candidate edges, we have the following observation; see Figures 4(b) and 4(c).

▶ **Observation 14.** *If a non-corner edge e is not in G', then the edges that are above, below, to the left, and to the right of e are in G', and hence in G.*

▶ **Lemma 15.** *Let (a, b) be a corner edge that is not in G. Then $|ab|_G = 3|ab|$.*

Proof. Recall that each corner edge is adjacent to another corner edge. Let (b, d) be the corner edge that is adjacent to (a, b). Let c be the corner vertex that is adjacent to a and d.



Since (a, c) and (c, d) are boundary edges, both of them are in G. We are going to show, by contradiction, that (b, d) is also in G. Assume $(b, d) \notin G$. If (b, d) was removed before (a, b), then at the moment the algorithm considers (a, b), both a and b have degree less than 4. Hence the algorithm would not remove (a, b); this contradicts the fact that $(a, b) \notin G$. If (a, b) was removed before (b, d) by a similar argument we get a contradiction. Thus $(b, d) \in G$. Note that $|ab| = |ac| = |cd| = |bd| = \epsilon$. Thus the length of the path $\delta = (a, c, d, b)$ is 3 times $|ab|$. ◀

At the end of Phase 2, all the stair-cases that have more than one vertex of degree two, have been broken into 2-cell and 3-cell faces. Thus we have the following observation.

▶ **Observation 16.** *Each face in G is either a 1-cell, a 2-cell, or a 3-cell (see the shaded faces in Figure 5(b)).*

▶ **Lemma 17.** *Let (f, c) be the missing edge of a 2-cell face in G. If (f, c) is a non-corner edge, then $|fc|_G \leqslant 3|fc|$.*

Proof. Let $F = (a, b, c, d, e, f)$ be the 2-cell face of G with the edge (f, c) is missing. Without loss of generality assume that (f, c) is horizontal, f is to the left of c, and a, b, c, d, e, f in the clockwise order of the vertices along F; see Figure 6.

Note that $|fc| = |ab| = |ed|$, $|af| = |bc|$, and $|fe| = |cd|$. Moreover, (a, b), (b, c), (d, e), and (e, f) are not candidate edges, hence they are in G' and in G, while (a, f) and (c, d) are candidate edges. Since (f, c) is a non-corner edge, in both G' and G, f is connected to a point f' and c is connected to a point c' where f' and c' are different from the vertices of F. By Observation 14, (f, f') and (c, c') are in G' and in G. Without loss of generality assume $|af| \leqslant |fe|$. We are going to prove that the length of the path (f, a, b, c) is at most three times $|fc|$. In order to prove this, we show that $|af| \leqslant |fc|$. The proof is by contradiction. Assume $|fc| < |af|$. For an edge $(u, v) \in \Lambda$, let S_{uv} be the slab containing (u, v) in its interior; if (u, v) is horizontal then S_{uv} is vertical, and vice versa. In Phase 1, the slabs S_{fe} and S_{af} are considered before S_{fc}, while in Phase 2, both are considered after S_{fc}. We consider two cases:

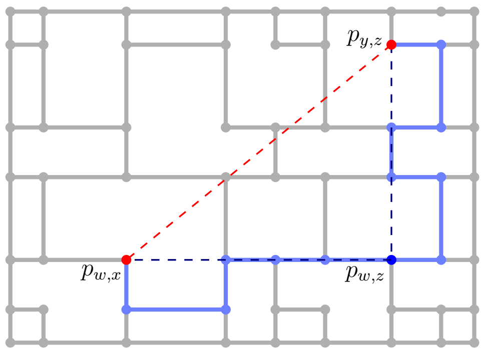
- $(a, f) \in G'$. See Figure 6(a). The reason why (a, f) was not removed is that both a and f had degree less than 4 at the moment the algorithm considered (a, f). At that moment the edge (f, c) was still in the graph. Thus, in order for f to have degree less than four, the edge (e, f) should have been removed before considering (a, f), which contradicts (e, f) being a non-candidate edge.
- $(a, f) \notin G'$. Thus (a, f) is added in Phase 2, and hence, both a and f have degree two in G'. See Figure 6(b). Recall that (c, d) is a candidate edge. Notice that $(c, d) \notin G'$ because at the moment the algorithm considered (c, d), the vertex c had degree 4, and hence (c, d) is removed. Thus (c, d) is added in Phase 2, implying that both c and d have degree two in G'. Therefore a, f, c, and d have degree two in G'. Since, in Phase 2, S_{fc} is considered before both S_{fe} and S_{af}, the edge (f, c) should have been inserted before considering (a, f) and (c, d). This contradicts the fact that (f, c) is not in G.

◀

▶ **Lemma 18** (Proof in the full version of the paper [4]). *Let (b, e) be a missing edge of a 3-cell face in G. If (b, e) is a non-corner edge, then $|be|_G \leqslant 3|be|$.*

▶ **Theorem 19.** *Let Λ be a finite non-uniform rectangular grid. Then, there exists a plane spanner for the point set of the vertices of Λ such that its degree is at most 3 and its stretch factor is at most $3\sqrt{2}$.*

Proof. Assume Λ has m rows and k columns.



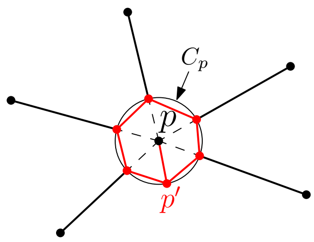
If $m \in \{1, 2\}$ or $k \in \{1, 2\}$, then Λ is a plane spanner whose degree is at most 3 and whose stretch factor is at most $\sqrt{2}$. Assume $m \geqslant 3$ and $k \geqslant 3$. Let $\overline{\Lambda}$ be the augmented lattice obtained from Λ as described in the beginning of this section. Let $\overline{G}$ be the graph obtained by the 2-phase algorithm described in this section. Then $\overline{G}$ is plane and its vertex degree is at most 3. By Lemmas 15, 17, and 18, for any edge $(a, b) \in \overline{\Lambda}$ that is not in $\overline{G}$, there exists a path in $\overline{G}$ whose length is at most 3 times $|ab|$. Now we are going to show that the stretch factor of $\overline{G}$ is at most $3\sqrt{2}$. Let $p_{w,x}$ and $p_{y,z}$ be any two vertices of $\overline{\Lambda}$. Consider the vertex $p_{w,z}$ in $\overline{\Lambda}$. By applying Lemmas 15, 17, and 18, in $\overline{G}$ there exists a path between $p_{w,x}$ and $p_{w,z}$ such that its length is at most 3 times $|p_{w,x}p_{w,z}|$. Similarly, in $\overline{G}$ there exists a path between $p_{y,z}$ and $p_{w,z}$ such that its length is at most 3 times $|p_{y,z}p_{w,z}|$. Since $|p_{w,x}p_{w,z}| + |p_{y,z}p_{w,z}| \leqslant \sqrt{2}|p_{w,x}p_{y,z}|$, we conclude that in $\overline{G}$ there exists a path between $p_{w,x}$ and $p_{y,z}$ that is passing through $p_{w,z}$ and whose length is at most $3\sqrt{2}$ times $|p_{w,x}p_{y,z}|$.

In order to obtain a spanner for Λ, we remove from $\overline{G}$ all the vertices of $\overline{\Lambda}$ that are not in Λ, as well as the edges incident to those vertices. Then we add all the missing boundary edges of Λ to the resulting graph. Let G be the graph that is obtained. As the boundary vertices of Λ have degree at most 3, G has vertex degree at most 3. Since all the boundary edges of Λ are in G, the stretch factor of G is not more than the stretch factor of $\overline{G}$. This completes the proof. ◀

5 Concluding Remarks

In order to obtain plane spanners with small stretch factor, one may think of adding Steiner points[1] to the point set and build a spanner on the augmented point set. In the L_1-metric, a plane 1-spanner of degree 4 can be computed by using $O(n \log n)$ Steiner points (see [14]). Arikati *et al.* [2] showed how to compute, in L_1-metric, a plane $(1+\epsilon)$-spanner with $O(n)$ Steiner points, for any $\epsilon > 0$. Moreover, for the Euclidean metric, they showed how to construct a plane $(\sqrt{2}+\epsilon)$-spanner that uses $O(n)$ Steiner points and has degree 4.

Let S be a set of n points in the plane that is in general position; no three points are collinear. Let G be a plane t-spanner of S. We show that, for any $\epsilon > 0$, there exists a plane $(t+\epsilon)$-spanner G' for S with $O(n)$ Steiner points whose vertex degree is at most 3. We show how to construct such a spanner. Without loss of generality we assume that ϵ is smaller than the closest pair distance in S, otherwise we pick an ϵ' smaller than the closest pair distance, and construct a $(t+\epsilon')$-spanner, which is also a $(t+\epsilon)$-spanner.



For each point p of the point set S, consider a circle C_p with radius $\frac{\epsilon}{\pi n}$ that is centered at p. Introduce a Steiner point on each intersection point of C_p with the edges of G that are incident to p. Also, introduce a Steiner point p' on C_p that is different from these intersection points. Delete the part of the edges of G inside each circle C_p (each edge e of G turns into an edge e' of G' with endpoints on C_p). Add an edge from p to p', and add a cycle whose edges connect consecutive Steiner points on the boundary of C_p. This results in a degree-3 geometric plane graph G'. For each vertex of degree k in G, we added $k+1$ Steiner points in G'. Since G is planar, its total vertex degree is at most $6n-12$. Thus, the number of Steiner points is $7n-12$, in total (by a different construction of G' this can be reduced to $5n-12$).

A path δ_{uv} between two vertices u and v in G can be turned into a path δ'_{uv} in G' as follows. For each point p in S corresponding to an internal vertex of δ_{uv} incident to two edges e_1 and e_2 of δ_{uv}, replace the part of e_1 and e_2 inside C_p by the shorter of the two paths along C_p connecting the corresponding Steiner points. Also, for each of point $p \in \{u, v\}$

[1] Ssome points in the plane that do not belong to the input point set.

incident to an edge e of δ_{uv} replace the part of e inside C_p with edge (p, p') together with the shorter of the two paths along C_p connecting p' and the Steiner point corresponding to e.

Note that $\frac{|\delta_{uv}|}{|uv|} \leqslant t$. Since the Steiner points are located at distance $\frac{\epsilon}{\pi n}$ from points of S, the length of the path along C_p replacing each vertex p of δ_{uv} is at most $\frac{\epsilon}{n}$. Since δ_{uv} has at most n vertices, the length of δ'_{uv} in G' is at most $|\delta_{uv}| + n \cdot \frac{\epsilon}{n}$. Thus, $\frac{|\delta'_{uv}|}{|uv|} \leqslant \frac{|\delta_{uv}|+\epsilon}{|uv|} \leqslant t + \epsilon$, is valid because ϵ is smaller than the closest pair distance in S, and hence smaller than $|uv|$.

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