On Planar Greedy Drawings of 3-Connected Planar Graphs*

Giordano Da Lozzo[1], Anthony D'Angelo[2], and Fabrizio Frati[3]

1 **Department of Computer Science, University of California, Irvine, CA, USA**
gdalozzo@uci.edu
2 **School of Computer Science, Carleton University, Ottawa, Canada**
anthonydangelo@cmail.carleton.ca
3 **Department of Engineering, Roma Tre University, Rome, Italy**
frati@dia.uniroma3.it

Abstract

A graph drawing is *greedy* if, for every ordered pair of vertices (x, y), there is a path from x to y such that the Euclidean distance to y decreases monotonically at every vertex of the path. Greedy drawings support a simple geometric routing scheme, in which any node that has to send a packet to a destination "greedily" forwards the packet to any neighbor that is closer to the destination than itself, according to the Euclidean distance in the drawing. In a greedy drawing such a neighbor always exists and hence this routing scheme is guaranteed to succeed.

In 2004 Papadimitriou and Ratajczak stated two conjectures related to greedy drawings. The *greedy embedding conjecture* states that every 3-connected planar graph admits a greedy drawing. The *convex greedy embedding conjecture* asserts that every 3-connected planar graph admits a planar greedy drawing in which the faces are delimited by convex polygons. In 2008 the greedy embedding conjecture was settled in the positive by Leighton and Moitra.

In this paper we prove that every 3-connected planar graph admits a *planar* greedy drawing. Apart from being a strengthening of Leighton and Moitra's result, this theorem constitutes a natural intermediate step towards a proof of the convex greedy embedding conjecture.

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1 Introduction

Geographic routing is a family of routing protocols for *ad-hoc networks*, which are networks with no fixed infrastructure – such as routers or access points – and with dynamic topology [15, 27, 28]. In a geographic routing scheme each node of the network actively sends, forwards, and receives packets; further, it does so by only relying on the knowledge of its own geographic coordinates, of those of its neighbors, and of those of the packet destination. *Greedy routing* is the simplest and most renowned geographic routing scheme. In this protocol, a node that has to send a packet simply forwards it to any neighbor that is closer – according to the Euclidean distance – to the destination than itself. The greedy routing scheme might fail

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to deliver packets because of the presence of a *void* in the network; this is a node with no neighbor closer to the destination than itself. For this reason, several variations of the greedy routing scheme have been proposed; see, e.g., [6, 19, 20].

Apart from its failure in the presence of voids, the greedy routing protocol has two disadvantages which limit its applicability. First, in order for the protocol to work, each node of the network has to be equipped with a GPS, which might be expensive and might consume excessive energy. Second, two nodes that are close geographically might be unable to communicate with each other because of the presence of topological obstructions. Rao *et al.* [26] introduced the following brilliant idea for extending the applicability of geographic routing in order to overcome the above issues. Suppose that a network topology is known; then one can assign *virtual coordinates* to the nodes and use these coordinates instead of the geographic locations of the nodes in the greedy routing protocol. The virtual coordinates can then be chosen so that the greedy routing protocol is guaranteed to succeed.

Assigning the virtual coordinate to the nodes of a network corresponds to the following graph drawing problem: given a graph G, construct a *greedy drawing* of G, that is a drawing in the plane such that, for any ordered pair of vertices (x, y), there is a neighbor of x in G that is closer – in terms of Euclidean distance – to y than x. Equivalently, a greedy drawing of G is such that, for any ordered pair of vertices (x, y), there is a *distance-decreasing* path from x to y, that is, a path $(u_1, \ldots, u_m)$ in G such that $x = u_1$, $y = u_m$, and the Euclidean distance between u_{i+1} and u_m is smaller than the one between u_i and u_m, for any $i = 1, \ldots, m-2$.

Greedy drawings experienced a dramatical surge of popularity in the theory community in 2004, when Papadimitriou and Ratajczak [24] proposed the following two conjectures about greedy drawings of 3-connected planar graphs (the convex greedy embedding conjecture has not been stated in the journal version [25] of their paper [24]).

▶ **Conjecture 1** (Greedy embedding conjecture). *Every 3-connected planar graph admits a greedy drawing.*

▶ **Conjecture 2** (Convex greedy embedding conjecture). *Every 3-connected planar graph admits a convex greedy drawing.*

Papadimitriou and Ratajczak [24, 25] provided several reasons why 3-connected planar graphs are central to the study of greedy drawings. First, there exist non-3-connected planar graphs and 3-connected non-planar graphs that do not admit any greedy drawing. Thus, the 3-connected planar graphs form the largest class of graphs that might admit a greedy drawing, in a sense. Second, all the 3-connected graphs with no $K_{3,3}$-minor admit a 3-connected planar spanning graph, hence they admit a greedy drawing, provided the truth of the greedy embedding conjecture. Third, the preliminary study of Papadimitriou and Ratajczak [24, 25] provided evidence for the mathematical depth of their conjectures.

Leighton and Moitra [21] (and, independently and slightly later, Angelini et al. [4]) settled Conjecture 1 in the affirmative. In this paper we show the following result.

▶ **Theorem 3.** *Every 3-connected planar graph admits a planar greedy drawing.*

Given a 3-connected planar graph G, the algorithms in [4, 21] find a spanning subgraph S of G and construct a (planar) greedy drawing of S; then they embed the edges of G not in S as straight-line segments obtaining a, in general, *non-planar* greedy drawing of G. Thus, Theorem 3 strengthens Leighton and Moitra's and Angelini et al.'s results. Furthermore, *convex* drawings, in which all the faces are delimited by convex polygons, are planar, hence Theorem 3 provides a natural step towards a proof of Conjecture 2.

Our proof employs a structural decomposition for 3-connected planar graphs which finds its origins in a paper by Chen and Yu [7]. This decomposition actually works for a super-class of the 3-connected planar graphs known as *strong circuit graphs*. We construct a planar greedy drawing of a given strong circuit graph G recursively: we apply the structural decomposition to G in order to obtain some smaller strong circuit graphs, we recursively construct planar greedy drawings for them, and then we suitably arrange these drawings together to get a planar greedy drawing of G. For this arrangement to be feasible, we need to ensure that the drawings we construct satisfy some restrictive geometric requirements; these are described in the main technical theorem of the paper – Theorem 8.

Related results. Planar greedy drawings always exist for *maximal planar graphs* [11]. Further, every planar graph G with a *Hamiltonian path* $P = (u_1, \dots, u_n)$ has a planar greedy drawing. Namely, construct a planar straight-line drawing Γ of G such that $y(u_1) < \dots < y(u_n)$; such a drawing always exists [12]; scale Γ down horizontally, so that P is "almost vertical". Then, for any $1 \leq i < j \leq n$, the paths $(u_i, u_{i+1} \dots, u_j)$ and $(u_j, u_{j-1} \dots, u_i)$ are distance-decreasing. A characterization of the *trees* that admit a (planar) greedy drawing is known [22]; indeed, a greedy drawing of a tree is always planar [2].

Algorithms have been designed to construct *succinct* greedy drawings, in which the vertex coordinates are represented with a polylogarithmic number of bits [13, 16, 17]; this has been achieved by allowing the embedding space to be different from the Euclidean plane or the metric to be different from the Euclidean distance.

Planar drawings have been studied in which paths between pairs of vertices are required to exist satisfying properties other than being distance-decreasing. We say that a path $P = (u_1, \dots, u_m)$ in a graph drawing is *self-approaching* [1, 23] if, for any three points a, b, c in this order along P from u_1 to u_m, the Euclidean distance between a and c is larger than the one between b and c – then a self-approaching path is also distance-decreasing. We say that P is *increasing-chord* [1, 10, 23] if it is self-approaching in both directions. We say that P is *strongly monotone* [3, 14, 18] if the orthogonal projections of the vertices of P on the line ℓ through u_1 and u_m appear in the order $u_1, \dots, u_m$. It has been recently proved [14] that every 3-connected planar graph has a planar drawing in which every pair of vertices is connected by a strongly monotone path.

Because of space limitations some proofs are sketched or omitted; they can be found in the complete version of the paper [8].

2 Preliminaries

In this section we introduce some preliminaries.

Subgraphs and connectivity. We denote by $V(G)$ and $E(G)$ the vertex and edge sets of a graph G, respectively. For $U \subseteq V(G)$, we denote by $G - U$ the graph obtained from G by removing the vertices in U and their incident edges. Further, if $e \in E(G)$, we denote by $G - e$ the graph obtained from G by removing the edge e. Let H be a subgraph of G. An *H-bridge* B of G is either an edge of G not in H with both the end-vertices in H (then B is a *trivial H-bridge*), or a connected component of $G - V(H)$ together with the edges from that component to the vertices in $V(H)$ (then B is a *non-trivial H-bridge*); the vertices in $V(H) \cap V(B)$ are the *attachments* of B in H. For a vertex $v \in V(G) - V(H)$, we denote by $H \cup \{v\}$ the subgraph of G composed of H and of the isolated vertex v.

A *vertex k-cut* (in the following simply called *k-cut*) in a connected graph G is a set of k vertices whose removal disconnects G. For $k \geq 2$, a connected graph is *k-connected* if it has no $(k-1)$-cut. A *k-connected component* of a graph G is a maximal k-connected subgraph of G. Given a 2-cut $\{a, b\}$ in a 2-connected graph G, an *$\{a, b\}$-component* is either the edge ab (then the $\{a, b\}$-component is *trivial*) or a subgraph of G induced by a, b, and the vertices of a connected component of $G - \{a, b\}$ (then the $\{a, b\}$-component is *non-trivial*).

Plane graphs and embeddings. A drawing of a graph is *planar* if no two edges cross. A *plane graph* is a planar graph with a plane embedding; a *plane embedding* of a connected planar graph G is an equivalence class of planar drawings of G, where two drawings Γ_1 and Γ_2 are *equivalent* if: (i) the clockwise order of the edges incident to each vertex $v \in V(G)$ coincides in Γ_1 and Γ_2; and (ii) the clockwise order of the edges of the walks delimiting the outer faces of Γ_1 and Γ_2 is the same. When we talk about a planar drawing of a plane graph G, we always mean that it respects the plane embedding of G. We assume that any subgraph H of G is associated with the plane embedding obtained from the one of G by deleting the vertices and edges not in H. A vertex of G is *external* or *internal* depending on whether it is or it is not incident to the outer face of G, respectively. For two external vertices u and v of a 2-connected plane graph G, let $\tau_{uv}(G)$ and $\beta_{uv}(G)$ be the paths composed of the vertices and edges encountered when walking along the boundary of the outer face of G in clockwise and counter-clockwise direction from u to v, respectively. Note that $\tau_{uv}(G)$ and $\beta_{vu}(G)$ have the same vertices and edges, however in reverse linear orders.

Geometry. In this paper every angle is measured in radians. The *slope of a half-line ℓ* is defined as follows. Denote by p the starting point of ℓ and let ℓ' be the vertical half-line starting at p and directed towards decreasing y-coordinates. Then the slope of ℓ is the angle spanned by a counter-clockwise rotation around p bringing ℓ' to coincide with ℓ, minus $\frac{\pi}{2}$. Because of this definition, the slope of any half-line is between -$\frac{\pi}{2}$ (included) and $\frac{3\pi}{2}$ (excluded); in the following there will be very few exceptions to this assumption, which will be however evident from the text. Every angle expressed as $\arctan(\cdot)$ is between -$\frac{\pi}{2}$ and $\frac{\pi}{2}$. We define the *slope of an edge uv* in a graph drawing as the slope of the half-line from u through v. Then the slope of an edge uv is equal to the slope of the edge vu plus or minus π. For a directed line ℓ, we let its slope be equal to the slope of any half-line starting at a point of ℓ and directed as ℓ. We denote by Δpqr a triangle with vertices p, q, r, and we denote by $\measuredangle pqr$ the angle of Δpqr incident to q; note that $\measuredangle pqr$ is between 0 and π.

Let Γ be a drawing of a graph G and let $u, v \in V(G)$. We denote by $d(\Gamma, uv)$ the Euclidean distance between the points representing u and v in Γ. We also denote by $d_V(\Gamma, uv)$ the vertical distance between u and v in Γ, that is, $d_V(\Gamma, uv) = |y(u) - y(v)|$, where the y-coordinates of u and v are taken from Γ; the horizontal distance $d_H(\Gamma, uv)$ between u and v in Γ is defined analogously. With a slight abuse of notation, we will use $d(\Gamma, pq)$, $d_H(\Gamma, pq)$, and $d_V(\Gamma, pq)$ even if p and q are points in the plane (and not vertices of G). A *straight-line* drawing of a graph is such that each edge is represented by a straight-line segment.

The following lemma argues that the planarity and the greediness of a drawing are not lost as a consequence of any sufficiently small perturbation of the vertex positions.

▶ **Lemma 4.** *Let Γ be a planar straight-line drawing of a graph G. There is a value $\varepsilon^*_\Gamma > 0$ such that the following holds. Let Γ' be any straight-line drawing in which, for every vertex $z \in V(G)$, the Euclidean distance between the positions of z in Γ and Γ' is at most ε^*_Γ; then Γ' is planar and any path which is distance-decreasing in Γ is also distance-decreasing in Γ'.*

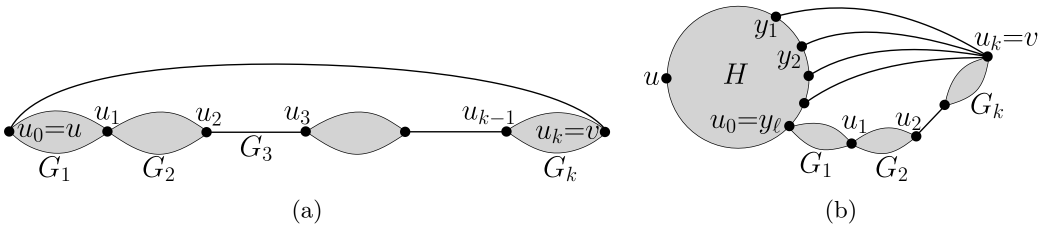

Figure 1 (a) Structure of (G, u, v) in Case A. (b) Structure of (G, u, v) in Case B.

3 Proof of Theorem 3

In this section we prove Theorem 3. Throughout the section, we will work with *plane graphs*. Further, we will deal with a class of graphs, known as *strong circuit graphs* [7], that is wider than 3-connected planar graphs. Strong circuit graphs constitute a subclass of the well-known *circuit graphs*, whose definition is due to Barnette and dates back to 1966 [5]. Here we rephrase the definition of strong circuit graphs as follows.

▶ **Definition 5.** A *strong circuit graph* is a triple (G, u, v) such that either: (i) G is an edge uv or (ii) $|V(G)| \geq 3$ and the following properties are satisfied.

(a) G is a 2-connected plane graph;
(b) u and v are two distinct external vertices of G;
(c) if edge uv exists, then it coincides with the path $\tau_{uv}(G)$; and
(d) for every 2-cut $\{a, b\}$ of G we have that a and b are external vertices of G and at least one of them is an internal vertex of the path $\beta_{uv}(G)$; further, every non-trivial $\{a, b\}$-component of G contains an external vertex of G different from a and b.

Several problems are easier to solve on (strong) circuit graphs than on 3-connected planar graphs. Indeed, (strong) circuit graphs can be easily decomposed into smaller (strong) circuit graphs and hence are suitable for inductive proofs. We now present a structural decomposition for strong circuit graphs whose main ideas can be found in a paper by Chen and Yu [7] (see [9] for an application of this decomposition to *cubic* strong circuit graphs).

Consider a strong circuit graph (G, u, v) such that G is not a single edge. The decomposition distinguishes the case in which the path $\tau_{uv}(G)$ coincides with the edge uv (Case A) from the case in which it does not (Case B).

▶ **Lemma 6.** *Suppose that we are in Case A (refer to Fig. 1(a)). Then the graph $G' = G - uv$ consists of a sequence of graphs $G_1, \dots, G_k$, with $k \geq 1$, such that:*

- *6a: for $i = 1, \dots, k-1$, the graphs G_i and G_{i+1} share a single vertex u_i; further, G_i is in the outer face of G_{i+1} and vice versa in the plane embedding of G;*
- *6b: for $1 \leq i, j \leq k$ with $j \geq i + 2$, the graphs G_i and G_j do not share any vertex; and*
- *6c: for $i = 1, \dots, k$ with $u_0 = u$ and $u_k = v$, (G_i, u_{i-1}, u_i) is a strong circuit graph.*

Given a strong circuit graph (G, u, v) that is not a single edge, the vertex u belongs to one 2-connected component of the graph $G - \{v\}$. Indeed, if it belonged to more than one 2-connected component of $G - \{v\}$, then $\{u\}$ would be a 1-cut of $G - \{v\}$, hence $\{u, v\}$ would be a 2-cut of G, which contradicts Property (d) for (G, u, v). We now present the following.

▶ **Lemma 7.** *Suppose that we are in Case B (refer to Fig. 1(b)). Let H be the 2-connected component of the graph $G - \{v\}$ that contains u; then we have $|V(H)| \geq 3$. Let $H' := H \cup \{v\}$. Then G contains ℓ distinct H'-bridges $B_1, \dots, B_\ell$, for some $\ell \geq 2$, such that:*

- *7a: each H'-bridge B_i has two attachments, namely v and a vertex $y_i \in V(H)$;*
- *7b: the H'-bridges $B_1, \dots, B_{\ell-1}$ are trivial, while B_ℓ might be trivial or not;*
- *7c: any two among $y_1, \dots, y_\ell$ are distinct except, possibly, for $y_{\ell-1}$ and y_ℓ; also if $\ell = 2$, then y_1 and y_2 are distinct;*
- *7d: y_1 is an internal vertex of $\tau_{uv}(G)$; further, B_1 is an edge that coincides with $\tau_{y_1 v}(G)$;*
- *7e: y_ℓ is an internal vertex of $\beta_{uv}(G)$ and $\beta_{uy_1}(H)$; further, B_ℓ contains the path $\beta_{y_\ell v}(G)$;*
- *7f: $B_1, \dots, B_{\ell-1}$ appear in this counter-clockwise order around v and lie in the outer face of B_ℓ in the plane embedding of G;*
- *7g: the triple (H, u, y_1) is a strong circuit graph; and*
- *7h: B_ℓ consists of a sequence of graphs $G_1, \dots, G_k$, with $k \geq 1$, such that:*
  - *for $i = 1, \dots, k-1$, the graphs G_i and G_{i+1} share a single vertex u_i; further, G_i is in the outer face of G_{i+1} and vice versa in the plane embedding of G;*
  - *for $1 \leq i, j \leq k$ with $j \geq i+2$, the graphs G_i and G_j do not share any vertex; and*
  - *for $i = 1, \dots, k$ with $u_0 = y_\ell$ and $u_k = v$, the triple (G_i, u_{i-1}, u_i) is a strong circuit graph.*

We prove that any strong circuit graph (G, u, v) has a planar greedy drawing by exploiting Lemmata 6 and 7 in a natural way. Indeed, if we are in Case A (in Case B) then Lemma 6 (resp. Lemma 7) is applied in order to construct strong circuit graphs (G_i, u_{i-1}, u_i) with $i = 1, \dots, k$ (resp. strong circuit graphs (H, u, y_1) and (G_i, u_{i-1}, u_i) with $i = 1, \dots, k$) for which planar greedy drawings are inductively constructed and combined together in order to get a planar greedy drawing of (G, u, v). The base case of the induction is the one in which G is an edge; then a planar greedy drawing of G is directly constructed. In order to be able to combine the planar greedy drawings for the strong circuit graphs (G_i, u_{i-1}, u_i) (and (H, u, y_1) if we are in Case B) to construct a planar greedy drawing of (G, u, v), we need the inductively constructed drawings to satisfy some restrictive geometric requirements. These are expressed in the following theorem, which is the core of the proof of Theorem 3.

▶ **Theorem 8.** *Let (G, u, v) be a strong circuit graph with at least three vertices and let $0 < \alpha < \frac{\pi}{4}$ be an arbitrary parameter. Let $\beta_{uv}(G) = (u = b_1, b_2, \dots, b_m = v)$. There exists a straight-line drawing Γ of G in the Cartesian plane such that the following holds. For any value $\delta \geq 0$, denote by Γ_δ the straight-line drawing obtained from Γ by moving the position of vertex u by δ units to the left. Then Γ_δ satisfies the following properties (refer to Fig. 2).*

1. *Γ_δ is planar;*
2. *$\tau_{uv}(G)$ lies entirely on a horizontal line ℓ_u with u to the left of v;*
3. *the edge $b_1 b_2$ has slope in the interval $(-\alpha, 0)$ and the edge $b_i b_{i+1}$ has slope in the interval $(0, \alpha)$, for each $i = 2, 3, \dots, m-1$;*
4. *for every vertex $x \in V(G)$ there is a path $P_x = (x = v_1, v_2, \dots, v_p = v)$ from x to v in G such that the edge $v_i v_{i+1}$ has slope in the interval $(-\alpha, \alpha)$ in Γ_δ, for each $i = 1, 2, \dots, p-1$; further, if $x \neq u$, then $u \notin V(P_x)$;*
5. *for every vertex $x \in V(G)$ there is a path $Q_x = (x = w_1, w_2, \dots, w_q = u)$ from x to u in G such that the edge $w_i w_{i+1}$ has slope in the interval $(\pi - \alpha, \pi + \alpha)$ in Γ_δ, for each $i = 1, 2, \dots, q-1$; and*
6. *for every ordered pair of vertices (x, y) in $V(G)$ there is a path P_{xy} from x to y in G such that P_{xy} is distance-decreasing in Γ_δ; further, if $x \neq u$ and $y \neq u$, then $u \notin V(P_{xy})$.*

We comment on the statement of Theorem 8. First, let us set $\delta = 0$ and argue about $\Gamma_0 = \Gamma$. Properties 1 and 6 are those that one would expect, as they state that Γ is planar and greedy, respectively. Properties 2 and 3 state that all the edges incident to the outer face of Γ are "almost" horizontal; indeed, the edges of $\tau_{uv}(G)$ are horizontal (this does not

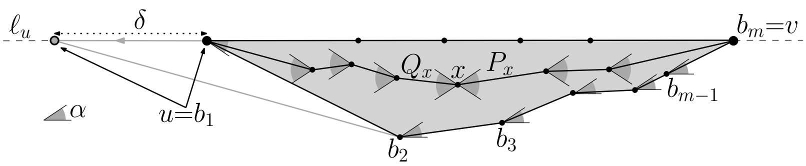

Figure 2 Illustration for the statement of Theorem 8.

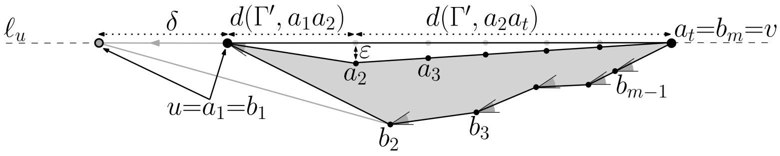

Figure 3 The straight-line drawing Γ of G in Case A if $k = 1$.

compromise the planarity of Γ since G does not contain edges between non-consecutive vertices of $\tau_{uv}(G)$, by Property (d) of (G, u, v)), the edge b_1b_2 has a slightly negative slope, and all the other edges of $\beta_{uv}(G)$ have slightly positive slopes. Then the planarity of Γ implies that Γ is contained in a wedge delimited by two half-lines with slopes 0 and $-\alpha$ starting at u. Properties 4 and 5 argue about the existence of certain paths from any vertex to u and v; these two vertices play an important role in the structural decomposition we employ, since distinct subgraphs are joined on those vertices, and the paths incident to them are inductively combined together in order to construct distance-decreasing paths. Finally, all these properties still hold if u is moved by an arbitrary non-negative amount δ to the left. This is an important feature we exploit in one of our inductive cases.

We now present an inductive proof of Theorem 8. In the **Base Case** the graph G is a single edge. We remark that, although Theorem 8 assumes that the given graph has at least three vertices, for its proof we need to inductively draw certain subgraphs of it; these subgraphs might indeed be single edges. Whenever we need to draw a strong circuit graph (G, u, v) such that G is a single edge uv, we draw it as a horizontal straight-line segment with positive length, with u to the left of v. We remark that, since Theorem 8 assumes that $|V(G)| \geq 3$, we do not need the constructed drawing to satisfy Properties 1–6.

We now discuss the inductive cases. In Case A the path $\tau_{uv}(G)$ coincides with the edge uv, while in Case B it does not. We discuss **Case A** first. Let $G' = G - uv$, where G' consists of a sequence of graphs $G_1, \dots, G_k$, with $k \geq 1$, satisfying the properties described in Lemma 6. Our construction is different if $k = 1$ and $k \geq 2$.

Suppose first that $\mathbf{k = 1}$; by Lemma 6 the triple $(G' = G_1, u, v)$ is a strong circuit graph (and G_1 is not a single edge, as otherwise we would be in the Base Case). Inductively construct a straight-line drawing Γ' of G' with $\frac{\alpha}{2}$ as a parameter. By Property 2 the path $\tau_{uv}(G') = (u = a_1, \dots, a_t = v)$ lies on a horizontal line ℓ_u in Γ' with u to the left of v. Let $Y > 0$ be the minimum distance in Γ' of any vertex strictly below ℓ_u from ℓ_u. Let

$$\varepsilon = \frac{1}{2}\min\{\varepsilon^*_{\Gamma'}, Y, \tan(\alpha) \cdot d(\Gamma', a_1a_2), \tan(\alpha) \cdot d(\Gamma', a_2a_t)\}.$$

We construct a straight-line drawing Γ of G from Γ' as follows; refer to Fig. 3. Decrease the y-coordinate of the vertex a_2 by ε; for $i = 3, \dots, t-1$, decrease the y-coordinate of the vertex a_i so that it ends up on the straight-line segment $\overline{a_2a_t}$. Draw the edge uv as a straight-line segment. We have the following.

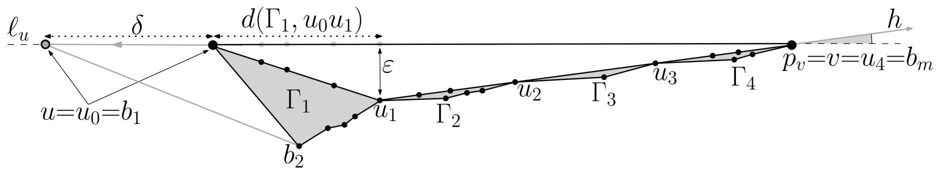

Figure 4 The straight-line drawing Γ of G in Case A if $k \geq 2$. In this example $k = 4$. The gray angle in the drawing is $\frac{\alpha}{2}$.

▶ **Lemma 9.** *For any $\delta \geq 0$, the drawing Γ_δ constructed in Case A if $k = 1$ satisfies Properties 1–6 of Theorem 8.*

Proof Sketch. The planarity of Γ is established due to the inequality $\varepsilon < \varepsilon^*_{\Gamma'}$ and to Lemma 4. Since Γ and Γ_δ coincide, except for the position of u, every crossing in Γ_δ has to involve edges incident to u. The proof that in fact there are no such crossings relies on the fact that Γ'_δ is planar, by induction, and on the inequalities $\varepsilon < \varepsilon^*_{\Gamma'}$ and $\varepsilon < Y$.

The paths P_x and Q_x requested for Properties 4 and 5 are obtained by suitably modifying paths satisfying the same properties for (G', u, v). The paths P_x and Q_x might contain edges in $\tau_{uv}(G')$; however, the slopes of the edges are in the required interval, which is $(-\alpha, \alpha)$ or $(\pi - \alpha, \pi + \alpha)$ depending on whether these edges are traversed towards v or u, respectively. This is due to the inequalities $\varepsilon < \tan(\alpha) \cdot d(\Gamma', a_1a_2)$ and $\varepsilon < \tan(\alpha) \cdot d(\Gamma', a_2a_t)$. ◀

We now discuss the case in which $\mathbf{k} \geq \mathbf{2}$. Refer to Fig. 4. By Lemma 6, for $i = 1, \dots, k$, the triple (G_i, u_{i-1}, u_i) is a strong circuit graph, where $u_0 = u$, $u_k = v$, and u_i is the only vertex shared by G_i and G_{i+1}, for $i = 1, \dots, k-1$.

If G_1 is a single edge, then inductively construct a straight-line drawing Γ_1 of G_1 and define $\varepsilon = \frac{1}{2}\min\{\varepsilon^*_{\Gamma_1}, \tan(\alpha) \cdot d(\Gamma_1, u_0u_1)\}$. If G_1 is not a single edge, then inductively construct a straight-line drawing Γ_1 of G_1 with $\frac{\alpha}{2}$ as a parameter. By Property 2 of Γ_1, the path $\tau_{u_0u_1}(G_1)$ lies on a horizontal line ℓ_u. Let $Y > 0$ be the minimum distance in Γ_1 of any vertex strictly below ℓ_u from ℓ_u. Let $\varepsilon = \frac{1}{2}\min\{\varepsilon^*_{\Gamma_1}, Y, \tan(\alpha) \cdot d(\Gamma_1, u_0u_1)\}$. In both cases, decrease the y-coordinate of u_1 by ε. Further, decrease the y-coordinate of every internal vertex of the path $\tau_{u_0u_1}(G_1)$, if any, so that it ends up on the straight-line segment $\overline{u_0u_1}$.

Now consider a half-line h with slope $s = \frac{\alpha}{2}$ starting at u_1. Denote by p_v the point at which h intersects the horizontal line ℓ_u through u. For $i = 2, \dots, k$, inductively construct a straight-line drawing Γ_i of G_i with $\frac{\alpha}{3}$ as a parameter (if G_i is a single edge, then the parameter does not matter). Uniformly scale the drawings $\Gamma_2, \dots, \Gamma_k$ so that the Euclidean distance between u_{i-1} and u_i in Γ_i is equal to $\frac{d(\Gamma_1, u_1p_v)}{k-1}$. For $i = 2, \dots, k$, rotate the scaled drawing Γ_i around u_{i-1} counter-clockwise by s radians. Translate the scaled and rotated drawings $\Gamma_2, \dots, \Gamma_k$ so that the representations of u_i in Γ_i and Γ_{i+1} coincide, for $i = 1, \dots, k-1$. Finally, draw the edge uv as a straight-line segment. This completes the construction of a drawing Γ of G. We have the following.

▶ **Lemma 10.** *For any $\delta \geq 0$, the drawing Γ_δ constructed in Case A if $k \geq 2$ satisfies Properties 1–6 of Theorem 8.*

Proof Sketch. The fulfillment of Property 3 for Γ_δ is the reason for the asymmetry of the construction, which shifts vertices in Γ_1, while it rotates $\Gamma_2, \dots, \Gamma_k$. Indeed, for $i = 1, \dots, k$, the first edge of $\beta_{u_{i-1}u_i}(G_i)$ has negative slope in Γ_i, while all the other edges have positive slopes; we need to ensure that the same property holds for $\beta_{uv}(G) = \beta_{u_0u_1}(G_1) \cup \dots \cup \beta_{u_{k-1}u_k}(G_k)$ in Γ_δ. For $i = 2, \dots, k$, the counter-clockwise rotation of Γ_i by $s = \frac{\alpha}{2}$ radians

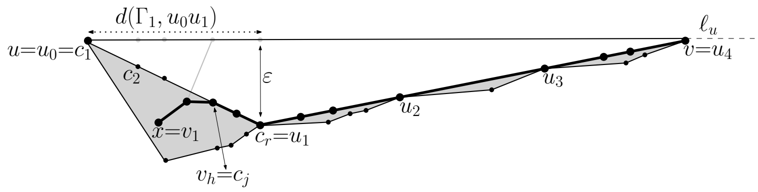

Figure 5 Illustration for the proof that the slope in Γ_δ of every edge in the path P_x is in $(-\alpha, \alpha)$, in the case in which x belongs to G_1. The path P_x is thick.

makes up for the negative slope (at most $\frac{\alpha}{3}$ in absolute value) of the first edge of $\beta_{u_{i-1}u_i}(G_i)$ in Γ_i. On the other hand, the edges of $\beta_{u_0u_1}(G_1)$ do not move when transforming Γ_1 in Γ, except for the edge incident to u_1, which however does not change its slope significantly, due to the inequality $\varepsilon < Y$; hence the slope of the first edge of $\beta_{u_0u_1}(G_1)$ remains negative (or becomes negative if G_1 is a single edge) and the other ones remain positive.

We present a proof that Γ_δ satisfies Property 4. Let $x \in V(G)$. If $x = u$, let $P_x = (u, v)$; then the only edge of P_x has slope $0 \in (-\alpha, \alpha)$ in Γ_δ. If $x = u_i$, for some $i \in \{1, \dots, k-1\}$, then let $P_x = \bigcup_{j=i+1}^{k} \tau_{u_{j-1}u_j}(G_j)$ and observe that all the edges of P_x have slope $s = \frac{\alpha}{2} \in (-\alpha, \alpha)$; further P_x does not pass through u. If $x \neq u_i$, for every $i \in \{0, \dots, k\}$, then x belongs to a unique graph G_i, for some $i \in \{1, \dots, k\}$. Assume that $i = 1$; the case $i \geq 2$ is easier to handle. Refer to Fig. 5. Let $\tau_{u_0u_1}(G_1) = (u_0 = c_1, c_2, \dots, c_r = u_1)$. Since Γ_1 satisfies Property 4, there exists a path $P_x^1 = (x = v_1, v_2, \dots, v_p = u_1)$ from x to u_1 in G_1, not passing through u_0, whose edges have slopes in $(-\frac{\alpha}{2}, \frac{\alpha}{2})$ in Γ_1; let h be the smallest index such that $v_h = c_j$, for some $j \in \{1, \dots, r\}$. Such an index h exists (possibly $h = p$ and $j = r$). Then let P_x consist of the paths $(x = v_1, v_2, \dots, v_h)$, $(v_h = c_j, c_{j+1}, \dots, c_r)$, and $\bigcup_{j=2}^{k} \tau_{u_{j-1}u_j}(G_j)$. Since $u \notin V(P_x^1)$, we have that $u \notin V(P_x)$, hence it suffices to argue about the slopes of the edges of P_x in Γ rather than in Γ_δ.

For $l = 1, \dots, h-2$, the slope of $v_l v_{l+1}$ is in $(-\alpha, \alpha)$ in Γ since it is in $(-\alpha, \alpha)$ in Γ_1 and since neither v_l nor v_{l+1} moves when transforming Γ_1 into Γ. Further, for $l = j, \dots, r-1$, the slope of the edge $c_l c_{l+1}$ in Γ is $-\arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0u_1)}\right)$, which is in the interval $(-\alpha, 0) \subset (-\alpha, \alpha)$, given that $\varepsilon, d(\Gamma_1, u_0u_1) > 0$ and that $\varepsilon < \tan(\alpha) \cdot d(\Gamma_1, u_0u_1)$. Moreover, the edges of $\bigcup_{j=2}^{k} \tau_{u_{j-1}u_j}(G_j)$ have slope $s = \frac{\alpha}{2} \in (-\alpha, \alpha)$. Finally, let σ_1 and σ be the slopes of the edge $v_{h-1}v_h$ in Γ_1 and Γ, respectively. Since $v_{h-1}v_h \in E(P_x^1)$, we have $\sigma_1 \in (-\frac{\alpha}{2}, \frac{\alpha}{2})$; since $\alpha \leq \frac{\pi}{4}$, we have $x(v_{h-1}) < x(v_h)$ in Γ_1 and Γ (note that the x-coordinates of the vertices do not change when transforming Γ_1 into Γ). Further, by Properties 1–4 of Γ_1, we have that v_{h-1} lies below ℓ_u, which contains v_h; hence, $y(v_{h-1}) < y(v_h)$ in Γ_1. Since the vertex v_h moves down (while v_{h-1} stays put) when transforming Γ_1 into Γ, and since $\varepsilon \leq \frac{Y}{2} < d_V(\Gamma_1, v_{h-1}v_h)$, it follows that $0 < \sigma < \sigma_1$; hence $\sigma \in (0, \frac{\alpha}{2}) \subset (-\alpha, \alpha)$.

Turning our attention to Property 6, consider any two vertices $x, y \in V(G)$, and assume that $x \in V(G_i)$ and $y \in V(G_j)$. We prove the existence of a path P_{xy} from x to y in G that is distance-decreasing in Γ_δ in the case in which $2 \leq i < j \leq k$; the other cases can be treated similarly. Let P_{xy} consist of a path P_x^i in G_i from x to u_i whose edges have slopes in $(-\frac{\alpha}{3}, \frac{\alpha}{3})$ in Γ_i, of the path $\bigcup_{l=i+1}^{j-1} \tau_{u_{l-1}u_l}(G_l)$, and of a path $P_{u_{j-1}y}^j$ in G_j that is distance-decreasing in Γ_j. By induction, P_x^i and $P_{u_{j-1}y}^j$ exist since Γ_i and Γ_j satisfy Properties 4 and 6, respectively; further, note that $u \notin V(P_{xy})$. Let $P_{xy} = (z_1, z_2, \dots, z_s)$; we prove that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$, for $h = 1, 2, \dots, s-2$, hence P_{xy} is distance-decreasing in Γ_δ. We distinguish three cases.

If $z_h z_{h+1}$ is in G_j, then $(z_h, z_{h+1}, \dots, z_s)$ is a sub-path of $P_{u_{j-1}y}^j$, hence it is distance-decreasing in Γ_δ since it is distance-decreasing in Γ_j and since the drawing of G_j in Γ_δ is

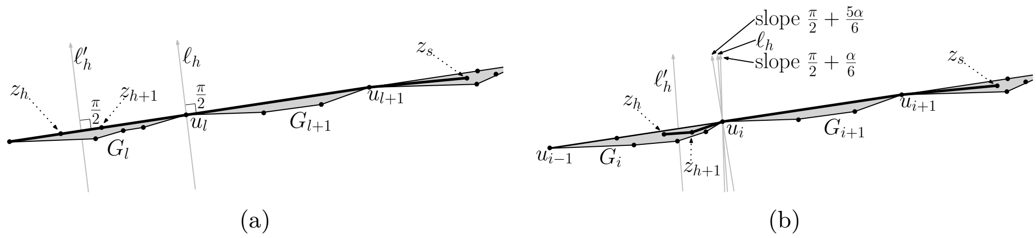

Figure 6 (a) Illustration for the proof that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$ if $z_h z_{h+1}$ is in $\tau_{u_{l-1}u_l}(G_l)$. (b) Illustration for the proof that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$ if $z_h z_{h+1}$ is in P^i_x.

congruent to Γ_j, up to affine transformations (a uniform scaling, a rotation, and a translation), which preserve the property of a path to be distance-decreasing.

If $z_h z_{h+1}$ is in $\tau_{u_{l-1}u_l}(G_l)$, for some $l \in \{i+1, i+2, \dots, j-1\}$, as in Fig. 6(a), then it has slope $s = \frac{\alpha}{2}$. The directed line ℓ_h with slope $\frac{\pi+\alpha}{2}$ through u_l, oriented towards increasing y-coordinates has the drawings of $G_{l+1}, \dots, G_k$ (and in particular the vertex z_s) to its right; this is because by Property 3 of Γ_δ every edge in $\beta_{u_l v}(G)$ has slope in the interval $(0, \alpha)$, where $\frac{-\pi+\alpha}{2} < 0 < \alpha < \frac{\pi+\alpha}{2}$, and because the path $\bigcup_{m=l+1}^{k} \tau_{u_{m-1}u_m}(G_m)$ has slope $s = \frac{\alpha}{2}$, where $\frac{-\pi+\alpha}{2} < \frac{\alpha}{2} < \frac{\pi+\alpha}{2}$. Then the directed line ℓ'_h parallel to ℓ_h, passing through the midpoint of the edge $z_h z_{h+1}$, and oriented towards increasing y-coordinates has ℓ_h to its right, hence it has z_s to its right. Since the half-plane to the right of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$.

If $z_h z_{h+1}$ is in P^i_x, as in Fig. 6(b), then by Property 4 it has slope in $(-\frac{\alpha}{3}, \frac{\alpha}{3})$ in Γ_i. Since Γ_i is counter-clockwise rotated by s radians in Γ_δ, it follows that $z_h z_{h+1}$ has slope in $(s - \frac{\alpha}{3}, s + \frac{\alpha}{3}) = (\frac{\alpha}{6}, \frac{5\alpha}{6})$ in Γ_δ. Consider the directed line ℓ_h that passes through u_i, that is directed towards increasing y-coordinates and that is orthogonal to the line through z_h and z_{h+1}. Denote by s_h the slope of ℓ_h. Then $s_h \in (\frac{\pi}{2} + \frac{\alpha}{6}, \frac{\pi}{2} + \frac{5\alpha}{6})$. We have that ℓ_h has the drawings of $G_{i+1}, \dots, G_k$ to its right; this is because by Property 3 of Γ_δ every edge in $\beta_{u_i v}(G)$ has slope in $(0, \alpha)$ with $s_h - \pi < -\frac{\pi}{2} + \frac{5\alpha}{6} < 0 < \alpha < \frac{\pi}{2} + \frac{\alpha}{6} < s_h$ and because the path $\bigcup_{m=i+1}^{k} \tau_{u_{m-1}u_m}(G_m)$ has slope $s = \frac{\alpha}{2}$, where $s_h - \pi < -\frac{\pi}{2} + \frac{5\alpha}{6} < \frac{\alpha}{2} < \frac{\pi}{2} + \frac{\alpha}{6} < s_h$. Further, ℓ_h has the drawings of $G_2, \dots, G_i$ to its left; this is because by Property 3 of Γ_δ every edge in $\tau_{u_i u_1}(G)$ has slope in $(\pi, \pi+\alpha)$ with $s_h < \frac{\pi}{2} + \frac{5\alpha}{6} < \pi < \pi + \alpha < \frac{3\pi}{2} + \frac{\alpha}{6} < \pi + s_h$ and because the path $\bigcup_{m=2}^{i} \beta_{u_m u_{m-1}}(G_m)$ has slope $s = \pi + \frac{\alpha}{2}$, where $s_h < \frac{\pi}{2} + \frac{5\alpha}{6} < \pi + \frac{\alpha}{2} < \frac{3\pi}{2} + \frac{\alpha}{6} < \pi + s_h$. Now consider the directed line ℓ'_h parallel to ℓ_h, passing through the midpoint of the edge $z_h z_{h+1}$, and oriented towards increasing y-coordinates. This line has ℓ_h to its right, given that the drawing of G_i (and in particular the midpoint of $z_h z_{h+1}$) is to the left of ℓ_h in Γ_δ. Thus, ℓ'_h has the drawings of $G_{l+1}, \dots, G_k$ (and in particular the vertex z_s) to its right. Since the half-plane to the right of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$. ◀

We now discuss **Case B**, in which (G, u, v) is decomposed according to Lemma 7. Refer to Figs. 7 and 8. First, the triple (H, u, y_1) is a strong circuit graph with $|V(H)| \geq 3$. Inductively construct a straight-line drawing Γ_H of H with $\frac{\alpha}{2}$ as a parameter.

Let $\beta_{uy_1}(H) = (u = b_1, \dots, b_m = y_1)$. Let ϕ_i be the slope of the edge $b_i b_{i+1}$ in Γ_H and let $\phi = \min_{i=2,\dots,m-1}\{\phi_i\}$. By Property (c) of (H, u, y_1) if the edge uy_1 belongs to H then it coincides with the path $\tau_{uy_1}(H)$. Hence, $m \geq 3$ and ϕ is well-defined. Further, ϕ is in the interval $(0, \frac{\alpha}{2})$ by Property 3 of Γ_H.

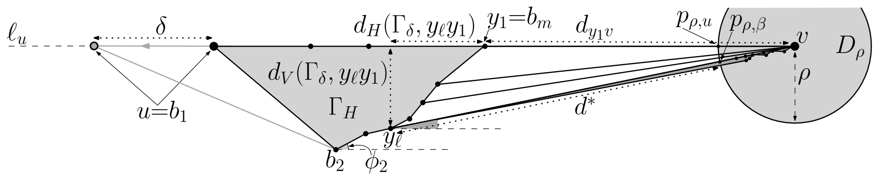

Figure 7 The straight-line drawing Γ of G in Case B. For the sake of readability, ϕ and ρ are larger than they should be. The dark gray angle is β. Fig. 8 shows an enlarged view of D_ρ.

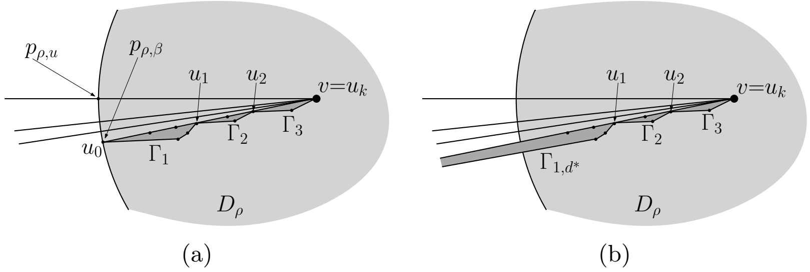

Figure 8 A closer look at D_ρ. Figure (a) represents the drawings $\Gamma_1, \ldots, \Gamma_k$ once they have been uniformly scaled, rotated, and translated, while (b) also has the vertex u_0 moved by d^* units (this movement actually happens before the rotation and translation of Γ_1).

Let $\beta = \frac{1}{2} \min \left\{ \phi, \arctan \left(\frac{d_V(\Gamma_H, y_\ell y_1)}{3 d_V(\Gamma_H, y_\ell y_1) + 3 d_H(\Gamma_H, y_\ell y_1)} \right) \right\}$. Note that $\beta > 0$, given that $\phi, d_V(\Gamma_H, y_\ell y_1) > 0$ and $d_H(\Gamma_H, y_\ell y_1) \geq 0$. In particular, $d_V(\Gamma_H, y_\ell y_1) > 0$ because y_1 is a vertex of $\tau_{uy_1}(H)$ and y_ℓ is an internal vertex of $\beta_{uy_1}(H)$ by Lemma 7, and because of Properties 1–3 of Γ_H. Also note that $\beta < \frac{\alpha}{4}$, given that $\phi < \frac{\alpha}{2}$.

Consider a half-line h_β with slope β starting at y_ℓ. Place the vertex v at the intersection point between h_β and the horizontal line ℓ_u through u. Draw all the trivial $(H \cup \{v\})$-bridges of G as straight-line segments. This concludes the construction if every $(H \cup \{v\})$-bridge of G is trivial. Otherwise, B_ℓ is the only non-trivial $(H \cup \{v\})$-bridge of G. Then B_ℓ consists of k strong circuit graphs (G_i, u_{i-1}, u_i), where $u_0 = y_\ell$ and $u_k = v$. With a slight change of notation, in the remainder of the section we assume that, if the edge $y_\ell v$ exists, then it is an edge of B_ℓ (rather than an individual trivial $(H \cup \{v\})$-bridge $B_{\ell-1}$ of G); in this case (B_ℓ, u_0, u_k) is a strong circuit graph.

We claim that v lies to the right of y_1. The polygonal line representing $\beta_{y_\ell y_1}(H)$ in Γ_H and the straight-line segment $\overline{y_\ell v}$ are both incident to y_ℓ. By definition of ϕ and since Γ_H satisfies Property 3, $\beta_{y_\ell y_1}(H)$ is composed of straight-line segments with slopes in the range $[\phi, \frac{\alpha}{2})$, while $\overline{y_\ell v}$ has slope β. The claim then follows from $0 < \beta < \phi < \frac{\pi}{2}$. Let $d_{y_1 v}$ be the distance between y_1 and v. Let $Y > 0$ be the minimum distance in Γ_H of any vertex strictly below ℓ_u from ℓ_u. Let $\rho = \min\{\frac{d_{y_1 v}}{3}, \frac{Y}{2}\}$ and let D_ρ be the disk with radius ρ centered at v. Let $p_{\rho,\beta}$ ($p_{\rho,u}$) be the intersection point of the boundary of D_ρ with h_β (resp. with ℓ_u) that is closer to y_ℓ (resp. to y_1). Let d^* be the Euclidean distance between y_ℓ and $p_{\rho,\beta}$.

Let $\alpha' = \frac{\beta}{2}$. Since $\beta > 0$, we have $\alpha' > 0$; further, $\alpha' < \frac{\alpha}{8}$, given that $\beta < \frac{\alpha}{4}$. For $i = 1, \ldots, k$, inductively construct a straight-line drawing Γ_i of G_i with α' as a parameter (if G_i is a single edge, then the parameter does not matter). Uniformly scale the drawings $\Gamma_1, \ldots, \Gamma_k$ so that the Euclidean distance between u_{i-1} and u_i is equal to $\frac{\rho}{k}$. Move the vertex u_0 in Γ_1 by d^* units to the left, obtaining a drawing Γ_{1,d^*}. Rotate the drawings $\Gamma_{1,d^*}, \Gamma_2, \ldots, \Gamma_k$ counter-clockwise by β radians. Translate $\Gamma_{1,d^*}, \Gamma_2, \ldots, \Gamma_k$ so that, for $i = 1, \ldots, k-1$, the representations of u_i in Γ_i and Γ_{i+1} (in Γ_{1,d^*} and Γ_2 if $i = 1$) coincide

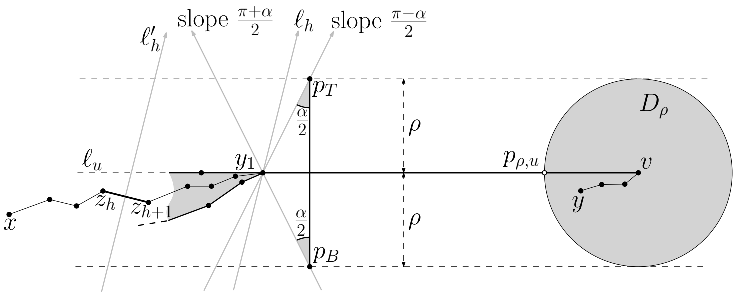

Figure 9 Illustration for the proof that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$ if $z_h z_{h+1}$ is in P_x^H. For the sake of readability, D_ρ is larger than it should be.

and so that the representation of u_0 in Γ_{1,d^*} coincides with the one of y_ℓ in Γ_H. This completes the construction of a straight-line drawing Γ of G. We have the following.

▶ **Lemma 11.** *For any $\delta \geq 0$, the drawing Γ_δ constructed in Case B satisfies Properties 1–6 of Theorem 8.*

Proof Sketch. Let $\Gamma_{H,\delta}$ denote the drawing obtained from Γ_H by moving the vertex u by δ units to the left.

We first prove that every vertex $z \neq u_0$ that belongs to a graph G_i lies inside the disk D_ρ in Γ_δ. A consequence of this statement is a sharp geometric separation between the vertices of G that are in H and those that are not. Consider any drawing Γ_j with $j \in \{1, \ldots, k\}$ (where Γ_1 is considered before moving u_0 by d^* units to the left) and let D_j be the disk centered at u_j with radius $d(\Gamma_j, u_{j-1}u_j)$. By Properties 1 and 2 of Γ_j, the path $\tau_{u_{j-1}u_j}(G_j)$ lies on $\overline{u_{j-1}u_j}$ in Γ_j, hence it lies inside D_j. Further, the edges of $\beta_{u_{j-1}u_j}(G_j)$ have slopes in $(-\alpha', \alpha') \subset (-\frac{\alpha}{8}, \frac{\alpha}{8}) \subset (-\frac{\pi}{32}, \frac{\pi}{32})$; hence $\beta_{u_{j-1}u_j}(G_j)$ also lies inside D_j. By planarity Γ_j lies entirely inside D_j. Hence, u_{j-1} is the farthest vertex of G_j from u_j in Γ_j. This property is true also after the uniform scaling of $\Gamma_1, \ldots, \Gamma_k$; further, after the scaling, the distance between u_{j-1} and u_j is $\frac{\rho}{k}$, by construction. By the triangular inequality, we have that $d(\Gamma_\delta, vz) \leq \sum_{j=i+1}^{k} d(\Gamma_\delta, u_{j-1}u_j) + d(\Gamma_\delta, u_i z)$. Since $d(\Gamma_\delta, u_{j-1}u_j) = \frac{\rho}{k}$ for any $j \in \{2, \ldots, k\}$, and since $d(\Gamma_\delta, u_i z) \leq \frac{\rho}{k}$ (this exploits $z \neq u_0$ and hence $d(\Gamma_\delta, u_i z) = d(\Gamma_i, u_i z)$, where Γ_i is understood as already scaled), we have that $d(\Gamma_\delta, vz) \leq \frac{(k-i+1)\rho}{k} \leq \rho$. Thus z lies inside D_ρ.

We now prove that Property 6 is satisfied by Γ_δ. We devote our attention to the proof of the existence of a distance-decreasing path P_{xy} from a vertex x to a vertex y if: (i) $x \in V(H)$ and $y \in V(G_i)$, for some $i \in \{1, \ldots, k\}$; or (ii) $x \in V(G_i)$, for some $i \in \{1, \ldots, k\}$, and $y \in V(H)$. While the rest of the proof that Property 6 is satisfied by Γ_δ proceeds similarly to the proof of Lemma 10, cases (i) and (ii) above deal with vertices x and y that are "far apart" in Γ_δ, a circumstance that does not occur in the proof of Lemma 10.

In case (i) P_{xy} contains a path P_x^H in H from x to y_1. Assume that $x \neq u$, as the case $x = u$ is easier to handle. By Property 4 of $\Gamma_{H,\delta}$, there is a path $P_x^H = (x = z_1, \ldots, z_s = y_1)$ in H that connects x to y_1, that does not pass through u, and whose edges have slopes in $(-\frac{\alpha}{2}, \frac{\alpha}{2})$ in $\Gamma_{H,\delta}$. We prove that, for $h = 1, \ldots, s-1$, $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$; see Fig. 9. Since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$, $z_h z_{h+1}$ has slope in $(-\frac{\alpha}{2}, \frac{\alpha}{2})$ in Γ_δ. Let ℓ_h be the directed line through y_1 directed towards increasing y-coordinates and orthogonal to the line through z_h and z_{h+1}. Denote by s_h the slope of ℓ_h. Then $s_h \in (\frac{\pi-\alpha}{2}, \frac{\pi+\alpha}{2})$.

We prove that ℓ_h has the disk D_ρ to its right. In order to do that, consider the point p_T on the half-line with slope $\frac{\pi-\alpha}{2}$ starting at y_1 and such that $d_V(\Gamma_\delta, y_1 p_T) = \rho$. Further, consider

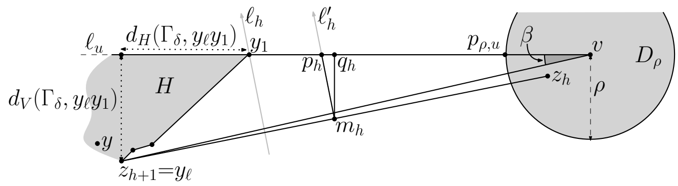

Figure 10 Illustration for the proof that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$ if $z_{h+1} = y_\ell = u_0$.

the point p_B on the half-line with slope $\frac{-\pi+\alpha}{2}$ starting at y_1 and such that $d_V(\Gamma_\delta, y_1 p_B) = \rho$. Note that $\overline{p_T p_B}$ is a vertical straight-line segment with length 2ρ. Consider the infinite closed strip S with height 2ρ that is delimited by the horizontal lines through p_T and p_B. Since D_ρ has its center on ℓ_u and has radius ρ, it lies inside S. The part of ℓ_h inside S is to the left of $\overline{p_T p_B}$, given that $s_h \in (\frac{\pi-\alpha}{2}, \frac{\pi+\alpha}{2})$. Hence, it suffices to show that $p_{\rho,u}$, which is the point of D_ρ with smallest x-coordinate, lies to the right of $\overline{p_T p_B}$. We have $d(\Gamma_\delta, y_1 p_{\rho,u}) = d_{y_1 v} - \rho$. Further, $d_H(\Gamma_\delta, y_1 p_T) = \rho \cdot \tan(\frac{\alpha}{2})$. Hence, it suffices to prove $\rho \cdot \tan(\frac{\alpha}{2}) < d_{y_1 v} - \rho$, that is $\rho < \frac{d_{y_1 v}}{1+\tan(\frac{\alpha}{2})}$; this is true since $\rho < \frac{d_{y_1 v}}{3}$ and $\tan(\frac{\alpha}{2}) < 1$, given that $0 < \alpha < \frac{\pi}{4}$.

The line ℓ_h has $\Gamma_{H,\delta}$ (and in particular the midpoint of $z_h z_{h+1}$) to its left; this is because by Property 2 of $\Gamma_{H,\delta}$ every edge in $\beta_{y_1 u}(H)$ has slope π, where $s_h < \frac{\pi+\alpha}{2} < \pi < \frac{3\pi-\alpha}{2} < \pi + s_h$, and because by Property 3 of $\Gamma_{H,\delta}$ every edge in $\tau_{y_1 u}(H)$ has slope in $(\pi - \frac{\alpha}{2}, \pi + \frac{\alpha}{2})$, where $s_h < \frac{\pi+\alpha}{2} < \pi - \frac{\alpha}{2} < \pi + \frac{\alpha}{2} < \frac{3\pi-\alpha}{2} < \pi + s_h$. Let ℓ'_h be the directed line parallel to ℓ_h, passing through the midpoint of $z_h z_{h+1}$, and oriented towards increasing y-coordinates; ℓ'_h has ℓ_h to its right, as the midpoint of $z_h z_{h+1}$ is to the left of ℓ_h in Γ_δ. Thus, ℓ'_h has D_ρ, and in particular y, to its right. Since the half-plane to the right of ℓ'_h is the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$.

The path P_{xy} also contains the edge $y_1 v$, which "jumps" from H to D_ρ. Since y lies in D_ρ, we have that $d(\Gamma_\delta, vy) \leq \rho \leq \frac{d_{y_1 v}}{3}$. By the triangular inequality, we have that $d(\Gamma_\delta, y_1 y) > d(\Gamma_\delta, y_1 v) - d(\Gamma_\delta, vy) \geq d_{y_1 v} - \rho \geq \frac{2 d_{y_1 v}}{3}$. Hence, $d(\Gamma_\delta, y_1 y) > d(\Gamma_\delta, vy)$. The third sub-path of P_{xy} is a path P_{vy} from v to y in $\bigcup_{l=i}^{k} G_l$ that is distance-decreasing in Γ_δ. The construction of this path proceeds similarly as in the proof of Lemma 10.

In case (ii) we have that $x \in V(G_i)$, for some $i \in \{1, \dots, k\}$, and $y \in V(H)$. While in case (i) the connection between H and D_ρ is done via y_1, here it is done via y_ℓ. In particular, the first part of P_{xy} consists of edges with slopes in the range $(\pi - \alpha, \pi + \alpha)$ inside D_ρ. Similarly to case (i), the orthogonal line through the midpoint of each of these edges separates H from D_ρ; hence traversing the edge decreases the distance to y.

We now argue that traversing an edge that "jumps" from D_ρ to H decreases the distance to y. That is, we show that, for a vertex z_h in D_ρ incident to an edge $z_h z_{h+1}$ with $z_{h+1} = y_\ell = u_0$, it holds $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$. Refer to Fig. 10. We exploit again the fact that the line ℓ_h passing through y_1 and orthogonal to the line through z_h and z_{h+1} has $\Gamma_{H,\delta}$ (and in particular y) to its left; then consider the directed line ℓ'_h parallel to ℓ_h, oriented towards increasing y-coordinates, and passing through the midpoint m_h of $z_h z_{h+1}$. Since the half-plane to the left of ℓ'_h is the locus of the points of the plane that are closer to z_{h+1} than to z_h, it suffices to show that the intersection point p_h of ℓ'_h and ℓ_u lies to the right of y_1 on ℓ_u; in fact, this implies that ℓ'_h has ℓ_h (and hence y) to its left.

Since z_h lies inside D_ρ, we have $x(z_h) \geq x(p_{\rho,u}) = x(y_1) + d_{y_1 v} - \rho$. Moreover, $x(y_1) = x(y_\ell) + d_H(\Gamma_\delta, y_\ell y_1)$. Thus, we have $x(m_h) = \frac{x(y_\ell)+x(z_h)}{2} \geq \frac{x(y_\ell)+(x(y_\ell)+d_H(\Gamma_\delta, y_\ell y_1)+d_{y_1 v}-\rho)}{2} = x(y_\ell) + \frac{d_H(\Gamma_\delta, y_\ell y_1)+d_{y_1 v}-\rho}{2}$. Translate the Cartesian axes so that $x(y_\ell) = 0$. Thus, $x(m_h) = \frac{d_H(\Gamma_\delta, y_\ell y_1)+d_{y_1 v}-\rho}{2}$. By Lemma 7, y_ℓ is an internal vertex of $\beta_{uv}(G)$, hence y_ℓ lies below ℓ_u.

Since $\rho < Y$ and z_h lies in D_ρ, the y-coordinate of y_ℓ is smaller than the one of z_h. Hence, the slope of $z_h z_{h+1}$ is larger than π. Further, z_h and hence m_h lie on or below h_β, thus the slope of $z_h z_{h+1}$ is at most $\pi + \beta$ and the slope s_h of ℓ'_h is in the interval $(\frac{\pi}{2}, \frac{\pi}{2} + \beta)$.

We now derive a lower bound for the x-coordinate of p_h. Let q_h be the projection of m_h on ℓ_u. Consider the triangle $\Delta m_h p_h q_h$. Since the y-coordinate of y_ℓ is smaller than the one of z_h, it is also smaller than the one of m_h. Thus, $d(\Gamma_\delta, m_h q_h) \leq d_V(\Gamma_\delta, y_\ell y_1)$. Since $s_h \in (\frac{\pi}{2}, \frac{\pi}{2} + \beta)$, the angle $\measuredangle p_h m_h q_h$ is at most β. Hence, $d(\Gamma_\delta, p_h q_h) \leq d_V(\Gamma_\delta, y_\ell y_1) \cdot \tan(\beta)$. Thus, $x(p_h) = x(m_h) - d(\Gamma_\delta, p_h q_h) \geq \frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2} - d_V(\Gamma_\delta, y_\ell y_1) \cdot \tan(\beta)$. It remains to prove that this quantity is larger than $d_H(\Gamma_\delta, y_\ell y_1)$, which is the x-coordinate of y_1.

Since $\beta < \frac{\alpha}{4} < \frac{\pi}{16}$, we have $\tan(\beta) \leq 1$, hence $\frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2} - d_V(\Gamma_\delta, y_\ell y_1) \cdot \tan(\beta) \geq \frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2} - d_V(\Gamma_\delta, y_\ell y_1)$. We want to establish $\frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2} - d_V(\Gamma_\delta, y_\ell y_1) > d_H(\Gamma_\delta, y_\ell y_1)$, that is, $d_{y_1 v} > 2 d_V(\Gamma_\delta, y_\ell y_1) + d_H(\Gamma_\delta, y_\ell y_1) + \rho$. Since $\rho \leq \frac{d_{y_1 v}}{3}$, we need to prove that $d_{y_1 v} > \frac{6 d_V(\Gamma_\delta, y_\ell y_1) + 3 d_H(\Gamma_\delta, y_\ell y_1)}{2}$.

We now express $d_{y_1 v}$ as a function of β. This is done by looking at the triangle whose vertices are y_ℓ, v, and the projection of y_ℓ on ℓ_u. Since the angle of this triangle at v is β, we get $d_{y_1 v} = \frac{d_V(\Gamma_\delta, y_\ell y_1)}{\tan(\beta)} - d_H(\Gamma_\delta, y_\ell y_1)$. Substituting this into the previous inequality, we need to have $\frac{d_V(\Gamma_\delta, y_\ell y_1)}{\tan(\beta)} - d_H(\Gamma_\delta, y_\ell y_1) > \frac{6 d_V(\Gamma_\delta, y_\ell y_1) + 3 d_H(\Gamma_\delta, y_\ell y_1)}{2}$, hence $\tan(\beta) < \frac{2 d_V(\Gamma_\delta, y_\ell y_1)}{6 d_V(\Gamma_\delta, y_\ell y_1) + 5 d_H(\Gamma_\delta, y_\ell y_1)}$, which is true since $\beta < \arctan\left(\frac{d_V(\Gamma_H, y_\ell y_1)}{3 d_V(\Gamma_H, y_\ell y_1) + 3 d_H(\Gamma_H, y_\ell y_1)}\right)$. This concludes the proof that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$.

The path P_{xy} continues with a path $P_{y_\ell y}$ from y_ℓ to y in H that is distance-decreasing in $\Gamma_{H,\delta}$ (and hence in Γ_δ, since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$). This concludes the proof of the lemma. ◀

Given a strong circuit graph (G, u, v) such that G is not a single edge, we are in Case A or Case B depending on whether the edge uv exists or not, respectively. Thus, Lemmata 9–11 prove Theorem 8. We show how to use Theorem 8 in order to prove Theorem 3. Consider any 3-connected planar graph G and associate any plane embedding to it; let u and v be two consecutive vertices in the clockwise order of the vertices along the outer face of G. We have that (G, u, v) is a strong circuit graph. Indeed: (a) by assumption G is 2-connected – in fact 3-connected – and associated with a plane embedding; (b) by construction u and v are two distinct external vertices of G; (c) edge uv exists and coincides with $\tau_{uv}(G)$, given that v immediately follows u in the clockwise order of the vertices along the outer face of G; and (d) G does not have any 2-cut, as it is 3-connected. Thus, Theorem 8 can be applied in order to construct a planar greedy drawing of G. This concludes the proof of Theorem 3.

4 Conclusions

In this paper we have shown how to construct planar greedy drawings of 3-connected planar graphs. It is tempting to try to use the graph decomposition we employed in this paper for proving that 3-connected planar graphs admit *convex* greedy drawings. However, despite some efforts in this direction, we have not been able to modify the statement of Theorem 8 in order to guarantee the desired convexities of the angles in the drawings. Thus, proving or disproving the convex greedy embedding conjecture remains an elusive goal.

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