A New Holant Dichotomy Inspired by Quantum Computation*†

Miriam Backens

School of Mathematics, University of Bristol, Bristol, UK
m.backens@bristol.ac.uk

Abstract

Holant problems are a framework for the analysis of counting complexity problems on graphs. This framework is simultaneously general enough to encompass many counting problems on graphs and specific enough to allow the derivation of dichotomy results, partitioning all problems into those which are in FP and those which are #P-hard. The Holant framework is based on the theory of holographic algorithms, which was originally inspired by concepts from quantum computation, but this connection appears not to have been explored before.

Here, we employ quantum information theory to explain existing results in a concise way and to derive a dichotomy for a new family of problems, which we call Holant$^+$. This family sits in between the known families of Holant*, for which a full dichotomy is known, and Holant^c, for which only a restricted dichotomy is known. Using knowledge from entanglement theory – both previously existing work and new results of our own – we prove a full dichotomy theorem for Holant$^+$, which is very similar to the restricted Holant^c dichotomy and may thus be a stepping stone to a full dichotomy for that family.

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1 Introduction

Quantum computation (QC) provided the inspiration for holographic algorithms [30], which in turn inspired the Holant framework [11]. While Holant problems are an area of active research, so far there appear to have been no attempts to apply knowledge from quantum information theory (QIT) or QC to their analysis. Yet, as we show in the following, QIT and QC offer promising new avenues of research into Holant problems.

The Holant framework encompasses a wide range of counting complexity problems on graphs, parameterised by sets of functions $\mathcal{F}$. Here, we consider functions of Boolean inputs taking values in the set of algebraic complex numbers. Each vertex in the graph is assigned a function from $\mathcal{F}$, with each edge incident on the vertex corresponding to an input of the function. This structure is associated with a complex number, the *Holant*, computed by multiplying the function values together and summing over all possible input assignments for each edge (for the full definition, see Section 2). The associated counting problem Holant $(\mathcal{F})$ is the following: given a graph and an assignment of functions from $\mathcal{F}$ to

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vertices, find the value of the Holant [11]. From a QIT perspective, each function can be considered as a tensor with one index for each input, making $\text{Holant}(\mathcal{F})$ the evaluation of a tensor network contraction.

The Holant framework is general enough to include problems such as counting matchings or perfect matchings, counting vertex covers [11], or counting Eulerian orientations [21]. It also encompasses other counting complexity frameworks like counting constraint satisfaction problems (#CSP) or counting graph homomorphisms [11]. On the other hand, the Holant framework is specific enough to allow the derivation of dichotomy theorems, which state that for function sets $\mathcal{F}$ within certain classes, the Holant problem is either in FP or it is #P-hard. By an analogue of Ladner's Theorem about NP-intermediate problems [23], such a dichotomy is not expected to hold for general counting problems [6].

One example of such a dichotomy is that for Holant^*. The problem $\text{Holant}^*(\mathcal{F})$ is equal to $\text{Holant}(\mathcal{F} \cup \mathcal{U})$, where $\mathcal{U}$ is the set of all unary functions [6]. Another example is the dichotomy for symmetric Holant^c, where all function sets considered must contain the unary functions pinning edges to values 0 or 1, respectively. Additionally, all functions are required to be *symmetric*, meaning their value depends only on the Hamming weight of the input [10]. Further dichotomies exist, but these, too, assume the availability of certain functions [12] or restrict the function sets, e.g. to symmetric or real-valued functions only [9, 27]. A full dichotomy for all Holant problems, as well as a full dichotomy for Holant^c, have so far remained elusive.

Here, we use knowledge from QC and QIT to make a step towards a full dichotomy for Holant^c. First, we analyse existing dichotomies in quantum terms, finding natural characterisations of the Holant^* and symmetric Holant^c dichotomies. The former can be described in terms of the entanglement classes of the allowed functions. Entanglement is a core concept in quantum theory: a quantum state of multiple systems is entangled if it cannot be written as a tensor product of states of subsystems. For states of more than two systems, there are different classes of entanglement which can be used for different QIT tasks [28]; their classification is an area of ongoing research [15, 31, 24, 25, 2]. We also find that the tractable class of *affine functions* arising in the dichotomy for symmetric Holant^c (see Section 3.2) is well-known in QIT as *stabilizer states* [20].

Motivated by this, we define a new class of Holant problems, which we call Holant^+. This class encompasses Holant problems where function sets are required to contain four specific unary functions, including the two that are available in Holant^c. In this way, Holant^+ fits between Holant^*, for which there is a full dichotomy, and Holant^c, for which there is no full dichotomy. These four unary functions enable the use of a known result from entanglement theory about producing two-system entangled states from many-system ones via projections [29, 18]: this corresponds to the ability to produce non-degenerate binary functions via gadgets. In fact, we prove an extension of that result about constructing three-qubit entangled states, or equivalently ternary functions. Using this, we derive our dichotomy theorem for Holant^+, whose tractable classes are very similar to those of the dichotomy for symmetric Holant^c [11]. Our dichotomy is the first full Holant dichotomy with no restrictions on the type of functions and where only a finite number of functions are assumed available, except for the dichotomy for $\#\text{R}_3$-CSP [12].

In the following, Section 2 contains a more detailed introduction to the Holant problem and associated concepts. In Section 3, we recap the relevant existing dichotomies and results. The quantum perspective on Holant problems, together with important notions from entanglement theory, is introduced in Section 4. We define and motivate the new family of Holant problems, called Holant^+, and prove the dichotomy theorem in Section 5.

2 Holant problems

Holant problems are a framework for counting complexity problems on graphs, introduced by Cai *et al.* [11], and based on the theory of holographic algorithms developed by Valiant [30]. Let $\mathcal{F}$ be a set of complex-valued functions with Boolean inputs, also called *signatures*, and let $G = (V, E)$ be an undirected graph with vertices V and edges E. Throughout, graphs are allowed to have parallel edges and self-loops. All complex numbers are assumed to be algebraic [7]. A *signature grid* is a tuple $\Omega = (G, \mathcal{F}, \pi)$ where π is a function that assigns to each n-ary vertex $v \in V$ a function $f_v : \{0,1\}^n \to \mathbb{C}$ in $\mathcal{F}$, specifying which edge corresponds to which input. The Holant for a signature grid Ω is:

$$\text{Holant}_\Omega = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v(\sigma|_{E(v)}), \tag{1}$$

where σ is an assignment of Boolean values to each edge and $\sigma|_{E(v)}$ is the restriction of σ to the edges incident on v.

▶ **Definition 1.** The Holant problem for a set of signatures $\mathcal{F}$, denoted by $\text{Holant}(\mathcal{F})$, is defined as follows:

- *Input*: a signature grid $\Omega = (G, \mathcal{F}, \pi)$ over the signature set $\mathcal{F}$,
- *Output*: Holant_Ω.

A *symmetric signature* is a function that depends only on the Hamming weight of the input. An n-ary symmetric signature is often written as $f = [f_0, f_1, \ldots, f_n]$, where f_k is the value f takes on inputs of Hamming weight k for $k \in \{0, \ldots, n\}$. A signature is called *degenerate* if it is a product of unary signatures. Any signature that cannot be expressed as a product of unary signatures is called *non-degenerate*. Multiplying a signature by a non-zero constant does not change the complexity of evaluating the Holant, so we will usually identify functions that are equal up to non-zero scalar factor.

Given a bipartite graph, we can define a *bipartite signature grid* by specifying two signature sets $\mathcal{F}$ and $\mathcal{G}$ and assigning signatures from $\mathcal{F}$ ($\mathcal{G}$) to vertices from the first (second) partition. A bipartite signature grid is denoted by a tuple $(G, \mathcal{F} \mid \mathcal{G}, \pi)$. The corresponding *bipartite Holant problem* is $\text{Holant}(\mathcal{F} \mid \mathcal{G})$. Any signature grid can be made bipartite by inserting a new vertex carrying the binary equality signature in the middle of each edge.

2.1 Signature grids in terms of vectors

As noted in [8], any signature $f : \{0,1\}^n \to \mathbb{C}$ can be considered as a complex vector of 2^n components indexed by $\{0,1\}^n$. Let $\{|x\rangle\}_{x \in \{0,1\}^n}$ be an orthonormal basis[1] for $\mathbb{C}^{2^n}$. The vector corresponding to the signature f is then denoted by $|f\rangle = \sum_{x \in \{0,1\}^n} f(x) |x\rangle$.

Suppose $\Omega = (G, \mathcal{F} \mid \mathcal{G}, \pi)$ is a bipartite signature grid, where $G = (V, W, E)$ has vertex partitions V and W. Then the Holant for Ω can be written as:

$$\text{Holant}_\Omega = \left(\bigotimes_{w \in W} (|g_w\rangle)^T \right) \left(\bigotimes_{v \in V} |f_v\rangle \right) = \left(\bigotimes_{v \in V} (|f_v\rangle)^T \right) \left(\bigotimes_{w \in W} |g_w\rangle \right), \tag{2}$$

where the tensor products are assumed to be ordered such that, in each inner product, two systems associated with the same edge meet.

[1] In using this notation for vectors, called *Dirac notation* and common in QC and QIT, we anticipate the interpretation of the vectors associated with signatures as quantum states, cf. Section 4.

2.2 Reductions

Holographic transformations are the origin of the name 'Holant problems'. Let M be a 2 by 2 complex matrix. For any $f : \{0,1\}^n \to \mathbb{C}$, write $M \circ f$ for the function corresponding to the vector $M^{\otimes n} |f\rangle$. Furthermore, for any signature set $\mathcal{F}$, write $M \circ \mathcal{F} := \{M \circ f \mid f \in \mathcal{F}\}$.

▶ **Theorem 2** (Valiant's Holant Theorem, [30]). *Suppose $\mathcal{F}$ and $\mathcal{G}$ are two sets of signatures, M an invertible 2 by 2 complex matrix, and $\Omega = (G, \mathcal{F} \mid \mathcal{G}, \pi)$ a signature grid. Let $\Omega' = (G, M \circ \mathcal{F} \mid (M^{-1})^T \circ \mathcal{G}, \pi')$ be the signature grid resulting from Ω by replacing each f_v or g_w by $M \circ f_v$ or $(M^{-1})^T \circ g_w$, respectively. Then $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$ and therefore $\text{Holant}(\mathcal{F} \mid \mathcal{G}) \equiv_T \text{Holant}\left(M \circ \mathcal{F} \mid (M^{-1})^T \circ \mathcal{G}\right)$.*

Here, $\equiv_T$ means the two problems have the same complexity. For non-bipartite signature grids, Theorem 2 implies that $\text{Holant}(\mathcal{F}) \equiv_T \text{Holant}(O \circ \mathcal{F})$, where O is any orthogonal 2 by 2 complex matrix [30]. Going from a signature set $\mathcal{F} \mid \mathcal{G}$ to $M \circ \mathcal{F} \mid (M^{-1})^T \circ \mathcal{G}$ or from $\mathcal{F}$ to $O \circ \mathcal{F}$ is a *holographic reduction*.

A *gadget* over a signature set $\mathcal{F}$ (also called $\mathcal{F}$-gate) is a fragment of a signature grid with some 'dangling' edges. Any gadget can be assigned an effective signature g. If g is the effective signature of some gadget over $\mathcal{F}$, g is said to be *realisable over* $\mathcal{F}$.

▶ **Lemma 3** ([6]). *Suppose $\mathcal{F}$ is some signature set and g is realisable over $\mathcal{F}$. Then $\text{Holant}(\mathcal{F} \cup \{g\}) \equiv_T \text{Holant}(\mathcal{F})$.*

Following [27], we define for any signature set $\mathcal{F}$, $S(\mathcal{F}) = \{g \mid g \text{ is realisable over } \mathcal{F}\}$. Then Lemma 3 implies that $\text{Holant}(S(\mathcal{F})) \equiv_T \text{Holant}(\mathcal{F})$.

If $g \notin S(\mathcal{F})$, in certain cases it is nevertheless possible to show a result like Lemma 3 by analysing a family of signature grids that differ in specific ways. This process is called *polynomial interpolation* and will not be used here, though it is a crucial ingredient in some of the results we build upon. The interested reader can find a discussion of polynomial interpolation in [11].

3 Existing results about the Holant problem

We now introduce the existing families of Holant problems and the associated dichotomy results. Gadget constructions, which are at the heart of many reductions, are easier the more signatures are known to be available. As a result, several families of Holant problems have been defined, in which certain sets of signatures are freely available – i.e. have to be included in any set $\mathcal{F}$ – and can thus be used in gadget constructions and polynomial interpolation.

3.1 Holant*

The Holant problem in which all unary signatures are freely available is $\text{Holant}^*(\mathcal{F}) = \text{Holant}(\mathcal{F} \cup \mathcal{U})$, where $\mathcal{U}$ is the set of all unary signatures [11, 6].

We begin with some definitions. Given a bit string x, let $\bar{x}$ be its bit-wise complement. Denote by $\langle \mathcal{F} \rangle$ the closure of a signature set $\mathcal{F}$ under tensor products. Furthermore, let:

- $\mathcal{T}$ be the set of all binary signatures,
- $\mathcal{E}$ the set of signatures which are non-zero only on two inputs x and $\bar{x}$, and
- $\mathcal{M}$ the set of signatures which are non-zero only on inputs of Hamming weight at most 1.

Finally, define $K = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ and $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The matrix K satisfies $K^T K \doteq X$, where $\doteq$ denotes equality up to non-zero scalar factor. In fact, up to multiplication by a diagonal matrix or by X itself, K is the only solution to this equation (see the full version of this paper at [1]).

▶ **Theorem 4** ([6]). *Let $\mathcal{F}$ be any set of complex valued functions in Boolean variables. The problem* $\#\mathrm{HOLANT}^*(\mathcal{F})$ *is polynomial time computable if:*

- $\mathcal{F} \subseteq \langle \mathcal{T} \rangle$, *or*
- $\mathcal{F} \subseteq \langle O \circ \mathcal{E} \rangle$, *where O is a complex orthogonal 2 by 2 matrix, or*
- $\mathcal{F} \subseteq \langle K \circ \mathcal{E} \rangle$, *or*
- $\mathcal{F} \subseteq \langle K \circ \mathcal{M} \rangle$ *or* $\mathcal{F} \subseteq \langle KX \circ \mathcal{M} \rangle$.

In all other cases, $\#\mathrm{HOLANT}^*(\mathcal{F})$ *is #P-hard. The dichotomy is still valid even if the inputs are restricted to planar graphs.*

3.2 Holant^c

$\#\mathrm{HOLANT}^c$ is the Holant problem in which only the unary signatures pinning edges to 0 or 1 are freely available [11, 10], i.e. $\#\mathrm{HOLANT}^c(\mathcal{F}) = \#\mathrm{HOLANT}(\mathcal{F} \cup \{\delta_0, \delta_1\})$ with $\delta_0 = [1, 0]$ and $\delta_1 = [0, 1]$. There is no full dichotomy for $\#\mathrm{HOLANT}^c$ yet, though there is a dichotomy that applies to sets of symmetric signatures only. This dichotomy features a new family of tractable signatures, which do not appear in the $\#\mathrm{HOLANT}^*$ dichotomy.

▶ **Definition 5.** A signature $f : \{0,1\}^n \to \mathbb{C}$ is called *affine* if it has the form:

$$f(x) = ci^{l(x)}(-1)^{q(x)}\chi_{Ax=b}(x), \tag{3}$$

where $c \in \mathbb{C}$, $i^2 = -1$, $l : \{0,1\}^n \to \mathbb{Z}_2$ is a linear function, $q : \{0,1\}^n \to \mathbb{Z}_2$ is a quadratic function, A is an m by n matrix with Boolean entries for some $0 \leq m \leq n$, $b \in \{0,1\}^m$, and χ is an indicator function which takes value 1 on inputs satisfying $Ax = b$, and 0 otherwise.

The set of all affine signatures is denoted by $\mathcal{A}$; this is already closed under tensor products. For the reader familiar with quantum information theory, the affine signatures correspond – up to a scalar factor – to stabilizer states (cf. Section 4.2).

▶ **Theorem 6** ([10]). *Let $\mathcal{F}$ be a set of complex symmetric signatures.* $\#\mathrm{HOLANT}^c(\mathcal{F})$ *is #P-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case it is in FP:*

- $\#\mathrm{HOLANT}^*(\mathcal{F})$ *is polynomial-time computable (cf. Theorem 4), or*
- *there exists a $T \in \mathcal{I}$ such that $\mathcal{F} \subseteq T \circ \mathcal{A}$, where:*

$$\mathcal{I} = \left\{ T \,\middle|\, \left(T^{-1}\right)^{\otimes T} \circ \{=_2, \delta_0, \delta_1\} \subset \mathcal{A} \right\}. \tag{4}$$

3.3 Other Holant problems

Complex-weighted Boolean #CSP (the counting constraint satisfaction problem) corresponds to a Holant problem in which equality functions of any arity are freely available. Formally, $\#\mathrm{CSP}(\mathcal{F}) = \#\mathrm{HOLANT}(\mathcal{F} \cup \mathcal{G})$, where $\mathcal{G} = \{=_1, =_2, =_3, \ldots\}$ with $=_1$ being the function that is equal to 1 on both inputs [11, 10, 12].

▶ **Theorem 7** ([12]). *Suppose $\mathcal{F}$ is a class of functions mapping Boolean inputs to complex numbers. If $\mathcal{F} \subseteq \mathcal{A}$ or $\mathcal{F} \subseteq \langle \mathcal{E} \rangle$, then* $\#\mathrm{CSP}(\mathcal{F})$ *is computable in polynomial time. Otherwise,* $\#\mathrm{CSP}(\mathcal{F})$ *is #P-hard.*

The same dichotomy also holds for $\#\mathrm{R}_3$-CSP, which corresponds to the bipartite Holant problem $\#\mathrm{HOLANT}(\mathcal{F} \mid \{=_1, =_2, =_3\})$ [12]. This dichotomy follows immediately from that for #CSP if $\mathcal{F}$ contains the binary (or indeed any non-unary) equality function, but it is non-trivial if $\mathcal{F}$ does not contain any non-unary equality functions.

In the case of $\#\mathrm{HOLANT}$ with no free signatures, there exists a dichotomy for complex-valued symmetric signatures [9] and a dichotomy for (not necessarily symmetric) signatures taking non-negative real values [27]. We will not explore those results in any detail here.

3.4 Results about ternary symmetric signatures

The hardness of problems of the form $\text{Holant}\,(\{[y_0,y_1,y_2]\} \mid \{[x_0,x_1,x_2,x_3]\})$ has been fully determined. If $[x_0,x_1,x_2,x_3]$ is degenerate, the problem is tractable by the first case of Theorem 4. If $[x_0,x_1,x_2,x_3]$ is non-degenerate, it can always be mapped to $[1,0,0,1]$ or $[1,1,0,0]$ by a holographic transformation [10]. By Theorem 2, it thus suffices to consider the cases $\{[y_0,y_1,y_2]\} \mid \{[1,0,0,1]\}$ and $\{[y_0,y_1,y_2]\} \mid \{[1,1,0,0]\}$.

There are holographic transformations which leave the signature $[1,0,0,1]$ invariant: in particular, $\left(\begin{smallmatrix}1 & 0\\ 0 & \omega\end{smallmatrix}\right) \circ [1,0,0,1] = [1,0,0,1]$ if $\omega^3 = 1$ [10]. Thus, by Theorem 2:

$$\text{Holant}\,(\{[y_0,y_1,y_2]\} \mid \{[1,0,0,1]\}) \equiv_T \text{Holant}\left(\{[y_0,\omega y_1,\omega^2 y_2]\} \mid \{[1,0,0,1]\}\right). \quad (5)$$

This relationship can be used to reduce the number of symmetric binary signatures to be considered. Following [10], a signature of the form $[y_0,y_1,y_2]$ is called *ω-normalised* if $y_0 = 0$, or there does not exist a primitive $(3t)$-th root of unity λ, where $gcd(t,3) = 1$, such that $y_2 = \lambda y_0$. Similarly, a unary signature $[a,b]$ is ω-normalised if $a = 0$, or there does not exist a primitive $(3t)$-th root of unity λ, where $gcd(t,3) = 1$, such that $b = \lambda a$.

▶ **Theorem 8** ([10]). *Let $\mathcal{G}_1, \mathcal{G}_2$ be two sets of signatures and let $[y_0,y_1,y_2]$ be a ω-normalised and non-degenerate signature. In the case of $y_0 = y_2 = 0$, further assume that $\mathcal{G}_1$ contains a unary signature $[a,b]$ which is ω-normalised and satisfies $ab \neq 0$. Then:*

$$\text{Holant}\,(\{[y_0,y_1,y_2]\} \cup \mathcal{G}_1 \mid \{[1,0,0,1]\} \cup \mathcal{G}_2) \equiv_T \#\text{CSP}(\{[y_0,y_1,y_2]\} \cup \mathcal{G}_1 \cup \mathcal{G}_2). \quad (6)$$

More specifically, $\text{Holant}\,(\{[y_0,y_1,y_2]\} \cup \mathcal{G}_1 \mid \{[1,0,0,1]\} \cup \mathcal{G}_2)$ *is #P-hard unless:*

- $\{[y_0,y_1,y_2]\} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \subseteq \langle \mathcal{E} \rangle$, *or*
- $\{[y_0,y_1,y_2]\} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \subseteq \mathcal{A}$,

in which cases the problem is in FP.

▶ **Theorem 9** ([10]). $\text{Holant}\,(\{[y_0,y_1,y_2]\} \mid \{[x_0,x_1,x_2,x_3]\})$ *is #P-hard unless $[y_0,y_1,y_2]$ and $[x_0,x_1,x_2,x_3]$ satisfy one of the following conditions, in which case the problem is in FP:*

- $[x_0,x_1,x_2,x_3]$ *is degenerate, or*
- *there is a 2 by 2 matrix M such that:*
  - $[x_0,x_1,x_2,x_3] = M \circ [1,0,0,1]$ *and* $M^T \circ [y_0,y_1,y_2]$ *is in* $\mathcal{A} \cup \langle \mathcal{E} \rangle$,
  - $[x_0,x_1,x_2,x_3] = M \circ [1,1,0,0]$ *and* $[y_0,y_1,y_2] = (M^{-1})^T \circ [0,a,b]$ *for some* $a,b \in \mathbb{C}$.

4 The quantum state perspective on signature grids

In Section 2.1, we introduced the idea of considering signatures as complex vectors. This perspective is useful for proving Valiant's Holant Theorem, which is at the heart of the theory of Holant problems. It also gives a connection to the theory of QC.

In QC and QIT, the basic system of interest is a *qubit* (quantum bit), which takes the place of the usual bit in standard computer science. The state of a qubit is described by a vector[2] in $\mathbb{C}^2$. State spaces compose by tensor product, i.e. the state of n qubits is described by a vector in $\left(\mathbb{C}^2\right)^{\otimes n}$, which is isomorphic to $\mathbb{C}^{2^n}$. Thus, the vector associated with an n-ary signature can be considered to be an (unnormalised) quantum state of n qubits.

[2] Strictly speaking, vectors only describe *pure* quantum states: there are also *mixed* states, which need to be described differently; but we do not consider those here.

Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis for $\mathbb{C}^2$. We call this the *computational basis*. The induced basis on $\left(\mathbb{C}^2\right)^{\otimes n}$ is labelled by $\{|x\rangle\}_{x\in\{0,1\}^n}$ as a short-hand, e.g. we write $|00\ldots0\rangle$ instead of $|0\rangle\otimes|0\rangle\otimes\ldots\otimes|0\rangle$. This is exactly the same as the basis introduced in Section 2.1.

Holographic transformations also have a natural interpretation in quantum information theory: going from an n-qubit state $|f\rangle$ to $M^{\otimes n}|f\rangle$, where M is some invertible 2 by 2 matrix, is a 'stochastic local operation with classical communication' (SLOCC) [4, 15]. These are physical operations that can be applied locally (without needing access to more than one qubit at a time) using classical (i.e. non-quantum) communication between the sites where the different qubits are held, and which succeed with non-zero probability. Unlike holographic transformations, SLOCC operations do not need to be symmetric under interchange of the qubits: the most general SLOCC operation on an n-qubit state is given by $M_1\otimes M_2\otimes\ldots\otimes M_n$, where $M_1, M_2, \ldots M_n$ are invertible complex 2 by 2 matrices [15].

From now on, we will use standard Holant terminology (or notation) and quantum terminology (or notation) interchangeably, and sometimes mix the two.

4.1 Entanglement and its classification

One major difference between quantum theory and preceding theories of physics (known as 'classical physics') is the possibility of *entanglement* in states of multiple systems.

▶ **Definition 10.** A state of multiple systems is *entangled* if it cannot be written as a tensor product of states of individual systems.

Where a state involves more than two systems, it is possible for some of the systems to be entangled with each other and for other systems to be in a product state with respect to the former. We sometimes use the term *genuinely entangled state* to refer to a state in which no subsystem is in a product state with the others. The term *multipartite entanglement* refers to entangled states in which more than two qubits are mutually entangled. Non-degenerate signatures correspond to (not necessarily genuinely) entangled states.

Entanglement is an important resource in QC, where it has been shown that quantum speedups are impossible without the presence of unboundedly growing amounts of entanglement [22]. Similarly, it is a resource in QIT [28], featuring in protocols such as quantum teleportation [3] and quantum key distribution [16]. Many QIT protocols have the property that two quantum states can be used to perform the same task if one can be transformed into the other by SLOCC, motivating the following equivalence relation.

▶ **Definition 11.** Two n-qubit states are *equivalent under SLOCC* if one can be transformed into the other using SLOCC. More formally: suppose $|f\rangle$ and $|g\rangle$ are two n-qubit states. Then $|f\rangle \sim_{SLOCC} |g\rangle$ if and only if there exist invertible complex 2 by 2 matrices $M_1, M_2, \ldots, M_n$ such that $(M_1\otimes M_2\otimes\ldots\otimes M_n)|f\rangle = |g\rangle$.

The equivalence classes of this relation are called *entanglement classes* or *SLOCC classes*.

For two qubits, there is only one class of entangled states, i.e. all entangled two-qubit states are equivalent to $|00\rangle + |11\rangle$ under SLOCC. For three qubits, there are two classes of genuinely entangled states [15], called the GHZ class and the W class. The former contains states that are equivalent under SLOCC to the GHZ state $|\mathrm{GHZ}\rangle := |000\rangle + |111\rangle$, the latter those equivalent to the W state $|W\rangle := |001\rangle + |010\rangle + |100\rangle$. Given an arbitrary three-qubit state expressed in the computational basis, it is straightforward to determine its entanglement class [26]. For more than three qubits, there are infinitely many SLOCC classes. It is possible to partition these into families which share similar properties. Yet, so

far, there is no consensus on how to partition the classes: there are different schemes for partitioning even the four-qubit entanglement classes, yielding different families [31, 25, 2].

It is sometimes useful to generalise the definitions of GHZ and W states to n-qubit states. The generalised GHZ state on n qubits is $|\mathrm{GHZ}_n\rangle := |0\rangle^{\otimes n} + |1\rangle^{\otimes n}$, i.e. it is the state corresponding to the n-ary equality signature. The generalised W state on n qubits is defined as $|W_1\rangle := |1\rangle$ and $|W_n\rangle := |1\rangle \otimes |0\rangle^{\otimes n-1} + |0\rangle \otimes |W_{n-1}\rangle$ for $n > 1$, i.e. $|W_n\rangle$ corresponds to the n-ary indicator function for inputs of Hamming weight 1. We sometimes drop the word 'generalised' when talking about generalised GHZ or W states. It should be clear from context whether or not we mean the three-qubit state specifically.

4.2 The existing results in the quantum picture

Several of the existing dichotomies have straightforward descriptions in the quantum picture. The tractable cases of the Holant* dichotomy (cf. Section 3.1) can be described as follows:

- either there is no multipartite entanglement – this corresponds to the case $\mathcal{F} \subseteq \langle \mathcal{T} \rangle$, or
- there is GHZ-type multipartite entanglement but it is impossible to realise W-type multipartite entanglement – this corresponds to the cases $\mathcal{F} \subseteq \langle O \circ \mathcal{E} \rangle$ or $\mathcal{F} \subseteq \langle K \circ \mathcal{E} \rangle$, or
- there is W-type multipartite entanglement and it is impossible to realise GHZ-type multipartite entanglement – this corresponds to the case $\mathcal{F} \subseteq \langle K \circ \mathcal{M} \rangle$ or $\mathcal{F} \subseteq \langle KX \circ \mathcal{M} \rangle$.

By GHZ-type entanglement we mean states that are equivalent to generalised GHZ states under SLOCC, and similarly for W-type entanglement.

The tractable case of Holant^c (cf. Section 3.2) that does not appear in Holant* also has a natural description: in QIT, the states corresponding to affine signatures are known as *stabilizer states* [13]. These states and the associated operations play an important role in the context of quantum error-correcting codes [20] and are thus at the core of most attempts to build large-scale quantum computers [14]. The fragment of quantum theory consisting of stabilizer states and operations that preserve the set of stabilizer states can be efficiently simulated on a classical computer [20]; this result is known as the Gottesman-Knill theorem.

Thus, the Holant problem and QIT are linked not only by quantum algorithms being an inspiration for holographic ones: instead, the known tractable signature sets of various Holant problems correspond to state sets that are of independent interest in QC and QIT.

The restriction to algebraic numbers is not a problem from the quantum perspective, not even when considering the question of universal QC: there exist (approximately) universal sets of quantum operations where each operation can be described using algebraic complex coefficients. One such example is the Clifford+T gate set [5, 19].

5 Holant$^+$

Our new family of Holant problems, called Holant$^+$, sits in between Holant* and Holant^c. It has a small number of freely available signatures, which are all unary. Yet, using results from QIT, these can be shown to be sufficient for constructing the gadgets required to reduce to the dichotomies in Section 3.4. Formally:

$$\text{Holant}^+ (\mathcal{F}) = \text{Holant} (\mathcal{F} \cup \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}), \tag{7}$$

where $|+\rangle := |0\rangle + |1\rangle$ corresponds to the 'unary equality function' and $|-\rangle := |0\rangle - |1\rangle$ is a vector that is orthogonal to $|+\rangle$. In quantum theory, the set $\{|+\rangle, |-\rangle\}$ is known as the *Hadamard basis*, since they are related to the computational basis vectors by a Hadamard transformation (up to scalar factor): $\{|+\rangle, |-\rangle\} \doteq H \circ \{|0\rangle, |1\rangle\}$, where $H = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

5.1 Why these free signatures?

The definition of $\#\text{HOLANT}^+$ is motivated by the following result from quantum theory.

▶ **Theorem 12** ([29],[18]). *Let $|\Psi\rangle$ be an n-system entangled state. For any two of the n systems, there exists a projection, onto a tensor product of states of the other $(n-2)$ systems, that leaves the two systems in an entangled state.*

Here, 'projection' means a (partial) inner product between $|\Psi\rangle$ and the tensor product of single-system states. The original proof of this statement in [29] was flawed but it was recently corrected [18]. The following corollary is not stated explicitly in either paper, but can be seen to hold by inspecting the proof in [18].

▶ **Corollary 13.** *Let $|\Psi\rangle$ be an n-qubit entangled state. For any two of the n qubits, there exists a projection of the other $(n-2)$ qubits onto a tensor product of computational and Hadamard basis states that leaves the two qubits in an entangled state.*

In other words, Theorem 12 holds when the systems are restricted to qubits and the projectors are restricted to products of computational and Hadamard basis states. Here, it is crucial to have projectors taken from two bases that are linked by the Hadamard transformation: the corollary works only in that case. We extend this result as follows.

▶ **Theorem 14.** *Let $|\Psi\rangle$ be an n-qubit entangled state with $n \geq 3$. There exists some choice of three of the n qubits and a projection of the other $(n-3)$ qubits onto a tensor product of computational and Hadamard basis states that leaves the three qubits in a genuinely entangled state.*

Proof (Sketch). If $n = 3$, $|\Psi\rangle$ itself is the desired state. For larger n, the theorem is proved inductively: we show that, given an n-qubit entangled state with $n > 3$, it is possible to project $(n-3)$ qubits in the desired way, assuming the same holds for all k-qubit genuinely entangled states with $3 \leq k \leq n$. The induction step is then proved by contradiction, employing the assumption that Theorem 14 does not hold for $(n+1)$-qubit states while Theorem 12 does. The full proof can be found in [1]. ◀

This result, which was not previously known in the QIT literature, is stronger than Theorem 12 in that we construct entangled three-qubit states rather than two-qubit ones. On the other hand, our result may not hold for all choices of three qubits: all we show is that there exists some choice of three qubits for which it does hold. The original proof of Theorem 14 in an earlier version of this paper was long and involved; this new shorter proof was suggested by Gachechiladze and Gühne [17].

5.2 The dichotomy theorem

Using Theorem 14, as well as Theorems 8 and 9, we prove our main result: a dichotomy for $\#\text{HOLANT}^+$ applying to complex, not necessarily symmetric signatures.

▶ **Theorem 15.** *Let $\mathcal{F}$ be a set of complex signatures. $\#\text{HOLANT}^+(\mathcal{F})$ is in FP if $\mathcal{F}$ satisfies one of the following conditions:*

- *$\#\text{HOLANT}^*(\mathcal{F})$ is in FP, or*
- *$\mathcal{F} \subseteq \mathcal{A}$.*

In all other cases, the problem is #P-hard.

The tractable cases are almost the same as those for symmetric Holant^c (Theorem 6), now without the symmetry restriction. The only difference is that the holographic transformations allowed in the affine case of the Holant^c dichotomy are trivial in the case of Holant$^+$: any transformation that maps $\{|=_2\rangle, |0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ to a subset of $\mathcal{A}$ must itself be in $\mathcal{A}$.

The tractability proof follows immediately by reduction to Holant* or #CSP, respectively. For the hardness proof, we use Theorem 14 to construct signatures corresponding to three-qubit entangled states. We then show that, unless we are in one of the tractable cases, it is possible to construct ternary gadgets with non-degenerate symmetric signatures. If the ternary symmetric signature is in the GHZ class, Theorem 8 applies. If the ternary symmetric signature is in the W class but not in $K \circ \mathcal{M}$ or $KX \circ \mathcal{M}$, we use Theorem 9. Finally, if the ternary symmetric signature is contained in $K \circ \mathcal{M}$, then by assumption the set of available signatures $\mathcal{F}$ must contain some signature that is not in $K \circ \mathcal{M}$ – otherwise, the problem is already known to be tractable. We show how to use such a signature to construct a binary symmetric signature that is not in $K \circ \mathcal{M}$. Then the desired result follows by Theorem 9. An analogous result holds with $KX \circ \mathcal{M}$ instead.

The gadget constructions for ternary symmetric signatures are given in Section 5.3. The gadget construction for a symmetric binary signature that is not in $K \circ \mathcal{M}$ (or $KX \circ \mathcal{M}$) follows in Section 5.4. Section 5.5 contains the hardness proof itself.

5.3 Symmetrising ternary signatures

The dichotomies given in Section 3.4 apply to symmetric ternary entangled signatures. The signatures constructed according to Theorem 14 are ternary and entangled, but they are not generally symmetric. Yet, these general ternary entangled signatures can be used to realise symmetric ones, possibly with the help of an additional binary non-degenerate signature. We prove this by distinguishing cases according to whether the ternary entangled signature constructed using Theorem 14 is in the GHZ or the W entanglement class.

First, consider a general GHZ-class state $|\psi\rangle$. By definition, there exist invertible complex 2 by 2 matrices A, B, C such that $|\psi\rangle = (A \otimes B \otimes C) |\mathrm{GHZ}\rangle$. We can draw the signature associated with $|\psi\rangle$ as the 'virtual gadget' shown in Figure 1a. The 'boxes' denoting the matrices are non-symmetric to indicate that A, B, C are not in general symmetric. The white dot represents the GHZ state. This notation is not meant to imply that the signatures A, B, C or the ternary equality signature are available on their own. Thinking of the signature as such a composite will simply make future arguments more straightforward. A similar argument can be applied if $|\psi\rangle$ is a W-class state, in which case the white dots in Figure 1 should be thought of as having signature $|W\rangle$.

In both cases, three vertices with the same ternary entangled signature can be connected to form the rotationally symmetric gadget shown in Figure 1b. In fact, the signature for that gadget is fully symmetric: its value depends only on the Hamming weight of the inputs. On the other hand, it may not be entangled or it may have the all-zero signature. For a general non-symmetric $|\psi\rangle$ there are three such symmetric gadgets that can be constructed by permuting the roles of A, B, and C in Figure 1b – in particular, which of the three ends up on the external edge of the gadget. This idea leads to the following lemmas.

▶ **Lemma 16.** *Let $|\psi\rangle$ be a three-qubit GHZ-class state, i.e. $|\psi\rangle = (A \otimes B \otimes C) |\mathrm{GHZ}\rangle$ for some invertible 2 by 2 matrices A, B, C. Then at least one of the three possible symmetric gadgets resulting from permutations of A, B, C in Figure 1b is non-degenerate unless $|\psi\rangle \in K \circ \mathcal{E}$ and is furthermore already symmetric.*

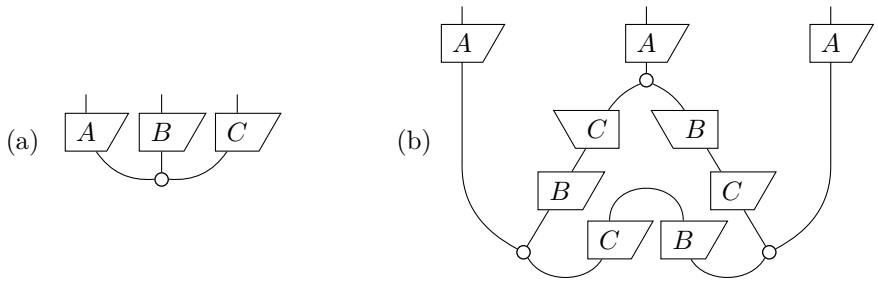

Figure 1 (a) A 'virtual gadget' for an entangled ternary signature based on the idea of SLOCC classes. (b) A symmetric gadget constructed from three copies of that ternary signature.

▶ **Lemma 17.** *Let $|\psi\rangle$ be a three-qubit W-class state, i.e. $|\psi\rangle = (A \otimes B \otimes C)|W\rangle$ for some invertible 2 by 2 matrices A, B, C. If $|\psi\rangle \in K \circ \mathcal{M}$ (or $|\psi\rangle \in KX \circ \mathcal{M}$), assume that we also have a two-qubit entangled state $|\phi\rangle$ that is not in $K \circ \mathcal{M}$ (or $KX \circ \mathcal{M}$, respectively). Then we can realise a symmetric three-qubit entangled state.*

5.4 Constructing binary signatures

We have shown in the previous section that it is possible to realise a non-degenerate ternary symmetric signature under some mild assumptions. Now, we show that if the full signature set $\mathcal{F}$ is not a subset of $K \circ \mathcal{M}$ (or $KX \circ \mathcal{M}$), it is possible to construct a symmetric binary gadget over $\mathcal{F} \cup \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ whose signature is not in $K \circ \mathcal{M}$ (or $KX \circ \mathcal{M}$, respectively). This signature can be used in Lemma 17, and a symmetric signature realised from it can also be used for a hardness proof according to Theorem 9.

▶ **Lemma 18.** *Suppose $|\psi\rangle$ is a genuinely entangled n-qubit state with $n \geq 2$, and $|\psi\rangle \notin K \circ \mathcal{M}$. Then there exists a non-degenerate binary gadget over $\{|\psi\rangle, |0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ with signature $|\varphi\rangle \notin K \circ \mathcal{M}$.*

The binary signature required in Lemma 17 is not required to be symmetric, only non-degenerate. The one in Theorem 9, on the other hand, does need to be symmetric.

▶ **Lemma 19.** *Suppose $|\psi\rangle \in K \circ \mathcal{M}$ is a three-qubit symmetric entangled state and $|\phi\rangle \notin K \circ \mathcal{M}$ is a two-qubit entangled state. Then there exists a gadget over $\{|\psi\rangle, |\phi\rangle, |0\rangle, |1\rangle, |\pm\rangle\}$ such that its signature $|\varphi\rangle$ is a two-qubit symmetric entangled state and $|\varphi\rangle \notin K \circ \mathcal{M}$.*

An analogous argument holds with KX instead of K. Hence, we can construct a non-degenerate symmetric binary signature satisfying the required properties whenever needed.

5.5 Sketch of the hardness proof

Suppose $\mathcal{F}$ is not in one of the tractable cases. Then, in particular, $\mathcal{F} \not\subseteq \langle \mathcal{T} \rangle$, i.e. $\mathcal{F}$ must contain multipartite entanglement (cf. Section 3.1). We can therefore use Theorem 14 to realise a ternary entangled signature. The quantum state associated with this signature must be in either the GHZ or the W SLOCC class.

In the GHZ case, either the state is already symmetric or it is possible to realise a non-degenerate symmetric ternary signature by Lemma 16. In the W case, if the ternary signature is not in $K \circ \mathcal{M}$ or $KX \circ \mathcal{M}$, it can be used to realise a non-degenerate ternary symmetric signature by Lemma 17. If the ternary signature is in $K \circ \mathcal{M}$, by Lemma 18, we can realise a binary signature that is not in $K \circ \mathcal{M}$ since by assumption $\mathcal{F} \not\subseteq K \circ \mathcal{M}$; and

similarly with KX instead of K. This then enables the use of Lemma 17. Hence if $\mathcal{F}$ is not one of the tractable sets, it is always possible to realise a non-degenerate symmetric ternary signature. Again, the quantum state associated with this signature must be in either the GHZ or the W SLOCC class.

If it is a GHZ class state, use the following lemma and corollary to reduce the problem to Theorem 8. This theorem yields #P-hardness unless $\mathcal{F}$ is a subset of $\langle O \circ \mathcal{E} \rangle$ or $\mathcal{A}$, which we assumed it was not.

▶ **Lemma 20.** *Let f be a signature and $\mathcal{G}$ a set of signatures. Then:*

$$\text{Holant}(\{f\} \cup \mathcal{G}) \equiv_T \text{Holant}(\{f, [1,0,1]\} \mid \mathcal{G} \cup \{[1,0,1]\}). \tag{8}$$

▶ **Corollary 21.** *Let f be a signature and $\mathcal{G}$ a set of signatures, and let M be an invertible 2 by 2 matrix. Then:*

$$\text{Holant}(\{M \circ f\} \cup \mathcal{G}) \equiv_T \text{Holant}\left(\left\{f, M^{-1} \circ [1,0,1]\right\} \,\middle|\, (\mathcal{G} \cup \{[1,0,1]\}) \circ M^T\right). \tag{9}$$

The corollary follows immediately from Lemma 20 and Theorem 2.

If the non-degenerate symmetric ternary signature $|\psi\rangle$ realised according to Section 5.3 is in the W class, then, by Theorem 9, the problem is #P-hard unless the signature is in $K \circ \mathcal{M}$ (or $KX \circ \mathcal{M}$). In the latter case, as by assumption $\mathcal{F} \not\subseteq K \circ \mathcal{M}$ (or $\mathcal{F} \not\subseteq KX \circ \mathcal{M}$), we can use Lemmas 18 and 19 to construct a symmetric binary signature $|\varphi\rangle$ that is not in $K \circ \mathcal{M}$ (or $KX \circ \mathcal{M}$, respectively).

Now, $\text{Holant}(\{|\varphi\rangle\} \mid \{|\psi\rangle\}) \leq_T \text{Holant}(\{|\varphi\rangle, |\psi\rangle\} \cup \mathcal{G})$ for any set $\mathcal{G}$. But if $|\psi\rangle \in K \circ \mathcal{M}$ and $|\varphi\rangle \notin K \circ \mathcal{M}$, then $\text{Holant}(\{|\varphi\rangle\} \mid \{|\psi\rangle\})$ is #P-hard by Theorem 9, and similarly with KX instead of K. Thus $\text{Holant}^+(\mathcal{F})$ is #P-hard whenever such $|\psi\rangle$ and $|\varphi\rangle$ are realisable over $\mathcal{F}$.

This concludes the investigation of all cases. We have therefore shown that Holant^+ is #P-hard in all but the listed cases. A full proof of this result can be found in [1].

6 Conclusions

Applying knowledge from QIT to Holant problems, we find that several tractable classes of existing dichotomies have concise descriptions in the framework of quantum entanglement. Motivated by this and by existing results in entanglement theory, we define a new Holant family, Holant^+, fitting between the known families Holant^* and Holant^c. We derive a full dichotomy for this family, which is closely related to the dichotomy for symmetric Holant^c [10]. It may therefore be a useful stepping stone towards a full Holant^c dichotomy, and thus to a full dichotomy for all Holant problems.

We also prove a new result in entanglement theory: given any n-qubit genuinely entangled state, it is possible to find some subset of $(n-3)$ qubits and a projector which is a tensor product of $(n-3)$ computational and Hadamard basis states such that the projection leaves the remaining three qubits in a genuinely entangled state. This is a generalisation of a similar result about constructing two-qubit entangled states [29, 18], though our result is slightly weaker in some aspects, which it may be possible to strengthen in future work.

We expect that further analysis of Holant problems using methods from QIT and QC will lead to further new insights, both into the complexity of Holant problems and into entanglement or other areas of quantum theory.

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