Neighborhood Complexity and Kernelization for Nowhere Dense Classes of Graphs*†

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Abstract

We prove that whenever G is a graph from a nowhere dense graph class $\mathcal{C}$, and A is a subset of vertices of G, then the number of subsets of A that are realized as intersections of A with r-neighborhoods of vertices of G is at most $f(r,\varepsilon)\cdot|A|^{1+\varepsilon}$, where r is any positive integer, ε is any positive real, and f is a function that depends only on the class $\mathcal{C}$. This yields a characterization of nowhere dense classes of graphs in terms of *neighborhood complexity*, which answers a question posed by Reidl et al. [26]. As an algorithmic application of the above result, we show that for every fixed integer r, the parameterized DISTANCE-r DOMINATING SET problem admits an almost linear kernel on any nowhere dense graph class. This proves a conjecture posed by Drange et al. [9], and shows that the limit of parameterized tractability of DISTANCE-r DOMINATING SET on subgraph-closed graph classes lies exactly on the boundary between nowhere denseness and somewhere denseness.

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1 Introduction

Sparse graphs. The notion of nowhere denseness was introduced by Nešetřil and Ossona de Mendez [23, 24] as a general model of *uniform sparseness* of graphs. Many familiar classes of sparse graphs, like planar graphs, graphs of bounded treewidth, graphs of bounded degree, and, in fact, all classes that exclude a fixed (topological) minor, are nowhere dense. Notably, classes of bounded average degree or bounded degeneracy are not necessarily nowhere dense. In an algorithmic context this is reasonable, as every graph can be turned into a graph of degeneracy at most 2 by subdividing every edge once; however, the structure of the graph is essentially preserved under this operation.

▶ **Definition 1.** A *minor model* of a graph H in G is a family $(I_u)_{u\in V(H)}$ of pairwise vertex-disjoint connected subgraphs of G such that whenever $\{u, v\}$ is an edge in H, there are $u' \in I_u$ and $v' \in I_v$ for which $\{u', v'\}$ is an edge in G. The graph H is a *depth-r minor* of G, denoted $H \preccurlyeq_r G$, if there is a minor model $(I_u)_{u\in V(H)}$ of H in G such that each subgraph I_u has radius at most r.

▶ **Definition 2.** A class $\mathcal{C}$ of graphs is *nowhere dense* if there is a function $t\colon \mathbb{N} \to \mathbb{N}$ such that $K_{t(r)} \not\preccurlyeq_r G$ for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$.

Nowhere denseness turns out to be a very robust concept with several seemingly unrelated natural characterizations. These include characterizations by the density of shallow (topological) minors [23, 24], quasi-wideness [24] (a notion introduced by Dawar [6] in his study of homomorphism preservation properties), low tree-depth colorings [20], generalized coloring numbers [29], sparse neighborhood covers [16, 17], by a game called the splitter game [17] and by the model-theoretic concepts of stability and independence [1]. For a broader discussion we refer to the book of Nešetřil and Ossona de Mendez [25].

An important and related concept is the notion of a graph class of *bounded expansion* [20, 21, 22]. Precisely, a class of graphs $\mathcal{C}$ has bounded expansion if for any $r \in \mathbb{N}$, the ratio between the numbers of edges and vertices in any r-shallow minor of a graph from $\mathcal{C}$ is bounded by a constant depending on r only. Obviously, every class of bounded expansion is also nowhere dense, but the converse is not always true.

Domination problems. In the parameterized Dominating Set problem we are given a graph G and an integer parameter k, and the task is to determine the existence of a subset $D \subseteq V(G)$ of size at most k such that every vertex u of G is *dominated* by D, that is, u either belongs to D or has a neighbor in D. More generally, for fixed $r \in \mathbb{N}$ we can consider the Distance-r Dominating Set problem, where we are asked to determine the existence of a subset $D \subseteq V(G)$ of size at most k such that every vertex $u \in V(G)$ is within distance at most r from a vertex from D. The Dominating Set problem plays a central role in the theory of parameterized complexity, as it is a prime example of a W[2]-complete problem again with k as the paramenter, thus considered intractable in full generality from the parameterized point of view. For this reason, Dominating Set and Distance-r Dominating Set have been extensively studied in restricted graph classes, including the sparse setting.

The study of parameterized algorithms for Dominating Set on sparse and topologically constrained graph classes has a long history, and, arguably, it played a pivotal role in the development of modern parameterized complexity. A point of view that was particularly fruitful, and most relevant to our work, is *kernelization*. Recall that a kernelization algorithm is a polynomial-time preprocessing algorithm that transforms a given instance into an equivalent one whose size is bounded by a function of the parameter only, independently of the overall input size. We are mostly interested in kernelization algorithms whose output guarantees are polynomial in the parameter, or maybe even linear. For Dominating Set on topologically restricted graph classes, linear kernels were given for planar graphs [2], bounded genus graphs [3], apex-minor-free graphs [12], graphs excluding a fixed minor [13], and graphs excluding a fixed topological minor [14]. All these results relied on applying tools of topological nature, most importantly deep decomposition theorems for graphs excluding a fixed (topological) minor. Notably, the research on kernelization for Dominating Set directly led to the introduction of the technique of *meta-kernelization* [3], which applies to a much larger family of problems on bounded-genus and H-minor-free graph classes.

Dawar and Kreutzer [7] showed that for every $r \in \mathbb{N}$ and every nowhere dense class $\mathcal{C}$, Distance-r Dominating Set is fixed-parameter tractable on $\mathcal{C}$. As far as polynomial kernelization is concerned, Drange et al. [9] gave a linear kernel for Distance-r Dominating Set on any graph class of bounded expansion[1], for every $r \in \mathbb{N}$, and an almost linear kernel for Dominating Set on any nowhere dense graph class; that is, a kernel of size $f(\varepsilon) \cdot k^{1+\varepsilon}$ for some function f. Drange et al. could not extend their techniques to larger domination radii r on nowhere dense classes, however, they conjectured that this should be possible. An important step was made recently by a subset of the authors [18], who gave a polynomial kernel for Distance-r Dominating Set on any nowhere dense class $\mathcal{C}$.

Nowhere dense classes are the limit for the fixed-parameter tractability of the problem: Drange et al. [9] showed that whenever $\mathcal{C}$ is a somewhere dense class closed under taking subgraphs, there is some $r \in \mathbb{N}$ for which Distance-r Dominating Set is W[2]-hard on $\mathcal{C}$.

Neighborhood complexity. One of the crucial ideas in the work of Drange et al. [9] was to focus on the *neighborhood complexity* in sparse graph classes. For an integer $r \in \mathbb{N}$, a graph G, and a subset $A \subseteq V(G)$ of vertices of G, the r*-neighborhood complexity* of A, denoted $\nu_r(G, A)$, is defined as the number of different subsets of A that are of the form $N_r[u] \cap A$ for some vertex u of G; here, $N_r^G[u]$ denotes the ball of radius r around u. That is,

$$\nu_r(G, A) = |\{N_r^G[u] \cap A \ : \ u \in V(G)\}|.$$

It was proved by Reidl et al. [26] that linear neighborhood complexity exactly characterizes subgraph-closed classes of bounded expansion. More precisely, a subgraph-closed class $\mathcal{C}$ has bounded expansion if and only if for each $r \in \mathbb{N}$ there is a constant c_r such that $\nu_r(G, A) \leq c_r \cdot |A|$ for all graphs $G \in \mathcal{C}$ and vertex subsets $A \subseteq V(G)$. They posed as an open problem whether nowhere denseness can be similarly characterized by almost linear neighborhood complexity. The lack of a neighborhood complexity theorem for nowhere dense classes was a major, however not the only, obstacle preventing Drange et al. [9] from extending their kernelization results to Distance-r Dominating Set on any nowhere dense class. The neighborhood complexity result for nowhere dense classes for $r = 1$ was given by Gajarský et al. [15], and this result was used by Drange et al. [9] in their kernelization algorithm for Dominating Set (distance $r = 1$) on nowhere dense graph classes.

Conversely, if $\mathcal{C}$ is somewhere dense and closed under taking subgraphs, then for some $r \in \mathbb{N}$ it contains the exact r-subdivision of every graph [24]. In this case it is easy to see that the r-neighborhood complexity of a vertex subset A can be as large as $2^{|A|}$.

[1] Precisely, the kernelization algorithm of Drange et al. [9] outputs an instance of an annotated problem where some vertices are not required to be dominated; this will be the case in this paper as well.

Our results. In this paper we resolve in the affirmative both outlined conjectures. Let us first focus on neighborhood complexity.

▶ **Theorem 3.** *Let $\mathcal{C}$ be a graph class closed under taking subgraphs. Then $\mathcal{C}$ is nowhere dense if and only if there exists a function $f_{\mathrm{nei}}(r,\varepsilon)$ such that $\nu_r(G,A) \leq f_{\mathrm{nei}}(r,\varepsilon) \cdot |A|^{1+\varepsilon}$ for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G \in \mathcal{C}$, and $A \subseteq V(G)$.*

To prove the above, we carefully analyze the argument of Reidl et al. [26] for linear neighborhood complexity in classes of bounded expansion. This argument is based on the analysis of vertex orderings certifying the constant upper bound on the *weak coloring number* of any graph from a fixed class of bounded expansion. In the nowhere dense setting, we only have an n^{ε} upper bound on the weak coloring number, and therefore the reasoning breaks whenever one tries to use a bound that is exponential in this number. We circumvent this issue by applying tools based on model-theoretic properties of nowhere dense classes of graphs. More precisely, we use the fact that every nowhere dense class is *stable* in the sense of Shelah, and hence every graph H which is obtained from a graph G from a nowhere dense class via a first-order interpretation has bounded VC-dimension [1]. Then exponential blow-ups can be reduced to polynomial using the Sauer-Shelah Lemma. These tools were recently used by a subset of the authors to give polynomial bounds for uniform quasi-wideness [18], a fact that also turns out to be useful in our proof.

We remark that we were informed by Micek, Ossona de Mendez, Oum, and Wood [19] that they have independently proved the statement of Theorem 3 using different methods.

Having the almost linear neighborhood complexity for any nowhere dense class of graphs, we can revisit the argumentation of Drange et al. [9] and prove the following result.

▶ **Theorem 4.** *Let $\mathcal{C}$ be a fixed nowhere dense class of graphs, let r be a fixed positive integer, and let $\varepsilon > 0$ be any fixed real. Then there is a polynomial-time algorithm that, given a graph $G \in \mathcal{C}$ and a positive integer k, returns a subgraph $G' \subseteq G$ and a vertex subset $Z \subseteq V(G')$ with the following properties:*

- *there is a set $D \subseteq V(G)$ of size at most k which r-dominates G if and only if there is a set $D' \subseteq V(G')$ of size at most k which r-dominates Z in G'; and*
- *$|V(G')| \leq f_{\mathrm{ker}}(r,\varepsilon) \cdot k^{1+\varepsilon}$, for some function $f_{\mathrm{ker}}(r,\varepsilon)$ depending only on the class $\mathcal{C}$.*

Just as in Drange et al. [9], the obtained triple (G', Z, k) is formally not an instance of Distance-r Dominating Set, but of an annotated variant of this problem where some vertices (precisely $V(G') \setminus Z$) are not required to be dominated. This is an annoying formal detail, however it can be addressed almost exactly as in Drange et al. by additional gadgeteering of annotations; see the full version for details.

Our proof of Theorem 4 revisits the line of reasoning of Drange et al. [9] for bounded expansion classes, and improves it in several places where the arguments could not be immediately lifted to the nowhere dense setting. The key ingredient is, of course, the usage of the newly proven almost linear neighborhood complexity for nowhere dense classes (Theorem 3), however, this was not the only piece missing. Another issue was that the algorithm of Drange et al. starts by iteratively applying the constant-factor approximation algorithm for Distance-r Dominating Set of Dvořák [10] in order to expose a certain structure in the instance. This part does not carry over to the nowhere dense setting, but we are able to circumvent it by using the new polynomial bounds for uniform quasi-wideness [18].

All proofs which are omitted in this extended abstract are marked with ($\star$). The full version of the paper can be found as an arxiv preprint [11].

2 Preliminaries

We use standard graph notation; see e.g. [8] for reference. All graphs considered in this paper are finite, simple, and undirected. For a graph G, by $V(G)$ and $E(G)$ we denote the vertex and edge sets of G, respectively.

Nowhere denseness (see Definition 2) admits several equivalent definitions, see [25] for a wider discussion. We next recall the ones used in this paper, as well as related concepts.

Weak coloring mumbers. For a graph G we let $\Pi(G)$ denote the set of all linear orders of $V(G)$. For $L \in \Pi(G)$, $u, v \in V(G)$, and any $r \geq 0$, we say that u is *weakly r-reachable* from v with respect to L, if there is a path P of length at most r connecting u and v such that u is the smallest among the vertices of P with respect to L. By $\mathrm{WReach}_r[G, L, v]$ we denote the set of vertices that are weakly r-reachable from v with respect to L. For any subset $A \subseteq V(G)$, we let $\mathrm{WReach}_r[G, L, A] = \bigcup_{v \in A} \mathrm{WReach}_r[G, L, v]$. The *weak r-coloring number* $\mathrm{wcol}_r(G)$ of G is defined as

$$\mathrm{wcol}_r(G) \quad = \quad \min_{L \in \Pi(G)} \max_{v \in V(G)} \big|\mathrm{WReach}_r[G, L, v]\big|.$$

As proved by Zhu [29], the weak coloring numbers can be used to characterize bounded expansion and nowhere dense classes of graphs.

▶ **Theorem 5** ([29]). *Let $\mathcal{C}$ be a nowhere dense class of graphs. There is a function $f_{\mathrm{wcol}}(r, \varepsilon)$ such that $\mathrm{wcol}_r(H) \leq f_{\mathrm{wcol}}(r, \varepsilon) \cdot |V(H)|^{\varepsilon}$ for every $r \in \mathbb{N}$, $\varepsilon > 0$, and $H \subseteq G \in \mathcal{C}$.*

Quasi-wideness. A set $B \subseteq V(G)$ is called *r-independent* in G if for all distinct $u, v \in B$ we have $\mathrm{dist}_G(u, v) > r$.

▶ **Definition 6.** A class $\mathcal{C}$ of graphs is *uniformly quasi-wide* if there are functions $N \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ such that for all $r, m \in \mathbb{N}$ and all subsets $A \subseteq V(G)$ for $G \in \mathcal{C}$ of size $|A| \geq N(r, m)$ there is a set $S \subseteq V(G)$ of size $|S| \leq s(r)$ and a set $B \subseteq A \setminus S$ of size $|B| \geq m$ which is r-independent in $G - S$. The functions N and s are called the *margins* of the class $\mathcal{C}$.

It was shown by Nešetřil and Ossona de Mendez [24] that a class $\mathcal{C}$ of graphs is nowhere dense if and only if it is uniformly quasi-wide. For us it will be important that the margins N and s can be assumed to be polynomial in r and that the sets B and S can be efficiently computed. This was proved only recently by Kreutzer et al. [18].

▶ **Theorem 7** ([18]). *Let $\mathcal{C}$ be a nowhere dense class of graphs and let $t \colon \mathbb{N} \to \mathbb{N}$ be a function such that $K_{t(r)} \not\preccurlyeq_r G$ for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$. For every $r \in \mathbb{N}$ there exist constants $p(r)$ and $s(r) \leq t(r)$ such that for all $m \in \mathbb{N}$, all $G \in \mathcal{C}$, and all sets $A \subseteq V(G)$ of size at least $m^{p(r)}$, there is a set $S \subseteq V(G)$ of size at most $s(r)$ such that there is a set $B \subseteq A \setminus S$ of size at least m which is r-independent in $G - S$. Furthermore, there is an algorithm that, given an n-vertex graph $G \in \mathcal{C}$, $\varepsilon > 0$, $r \in \mathbb{N}$, and $A \subseteq V(G)$ of size at least $m^{p(r)}$, computes sets S and $B \subseteq A$ as described above in time $\mathcal{O}(r \cdot t \cdot |A|^{t+1} \cdot n^{1+\varepsilon})$.*

We remark that the running time of the algorithm of Theorem 7 is stated in the SODA version [18] only as $\mathcal{O}(r \cdot t \cdot n^{t+6})$. A finer analysis with the running times as stated above can be found in the arXiv version of that paper.

VC-dimension. Let $\mathcal{F} \subseteq 2^A$ be a family of subsets of a set A. For a set $X \subseteq A$, we denote $X \cap \mathcal{F} = \{X \cap F : F \in \mathcal{F}\}$. The set X is *shattered by* $\mathcal{F}$ if $X \cap \mathcal{F} = 2^X$. The *Vapnik-Chervonenkis dimension*, short *VC-dimension*, of $\mathcal{F}$ is the maximum size of a set X that is shattered by $\mathcal{F}$. Note that if X is shattered by $\mathcal{F}$, then also every subset of X is shattered by $\mathcal{F}$.

The following theorem was first proved by Vapnik and Chervonenkis [5], and rediscovered by Sauer [27] and Shelah [28]. It is often called the Sauer-Shelah Lemma in the literature.

▶ **Theorem 8** (Sauer-Shelah Lemma). *If $|A| \leq n$ and $\mathcal{F} \subseteq 2^A$ has VC-dimension d, then $|\mathcal{F}| \leq \sum_{i=0}^{d} \binom{n}{i} \in \mathcal{O}(n^d)$.*

Note that in the interesting cases $d \geq 2, n \geq 2$ it holds that $\sum_{i=0}^{d} \binom{n}{i} \leq n^d$, and in general it holds that $\sum_{i=0}^{d} \binom{n}{i} \leq 2 \cdot n^d$. For a graph G, the VC-dimension of G is defined as the VC-dimension of the family $\{N[v] \colon v \in V(G)\}$ of sets over the set $V(G)$.

VC-dimension and nowhere denseness. Adler and Adler [1] have proved that any nowhere dense class $\mathcal{C}$ of graphs is *stable*, which in particular implies that any class of structures obtained from $\mathcal{C}$ by means of a first-order interpretation has VC-dimension bounded by a constant depending only on $\mathcal{C}$ and the interpretation. In particular, the following is an immediate corollary of the results of Adler and Adler [1].

▶ **Corollary 9.** *Let $\mathcal{C}$ be a nowhere dense class of graphs and let $r \in \mathbb{N}$. For $G \in \mathcal{C}$, let $G_{=r}$ be the graph with the same vertex set as G and an edge $\{u, v\} \in E(G_{=r})$ if and only if $\mathrm{dist}_G(u, v) = r$. Define the graph $G_{\leq r}$ in the same manner, but putting an edge $\{u, v\}$ into $E(G_{\leq r})$ if and only if $\mathrm{dist}_G(u, v) \leq r$. Then there is an integer $c(r)$ such that both $G_{=r}$ and $G_{\leq r}$ have VC-dimension at most $c(r)$ for every $G \in \mathcal{C}$.*

By combining Corollary 9 with the Sauer-Shelah Lemma we infer the following.

▶ **Corollary 10.** *Let $\mathcal{C}$ be a nowhere dense class of graphs and $r \in \mathbb{N}$. Then $\nu_r(G, A) \leq |A|^{c(r)}$ for every graph $G \in \mathcal{C}$ and $A \subseteq V(G)$, where $c(r)$ is the constant given by Corollary 9.*

Thus, a polynomial bound on the neighborhood complexity for any nowhere dense class, and, in fact, for any stable class, already follows from known tools. Our goal in the next section will be to show that with the assumption of nowhere denseness we can prove an almost linear bound, as described in Theorem 3.

Distance profiles. In our reasoning we will need a somewhat finer view of the neighborhood complexity. More precisely, we would like to partition the vertices of the graph not only with respect to their r-neighborhood in a fixed set A, but also with respect to what are the exact distances of the elements of this r-neighborhood from the considered vertex. With this intuition in mind, we introduce the notion of a *distance profile*.

Let G be a graph and let $A \subseteq V(G)$ be a subset of its vertices. For a vertex $u \in V(G)$, the *r-distance profile* of u, denoted $\pi_r^G[u, A]$, is a function mapping vertices of A to $\{0, 1, \ldots, r, \infty\}$ defined as follows:

$$\pi_r^G[u, A](v) = \begin{cases} \mathrm{dist}_G(u, v) & \text{if } \mathrm{dist}_G(u, v) \leq r, \\ \infty & \text{otherwise.} \end{cases}$$

We say that a function $f \colon A \to \{0, 1, \ldots, r, \infty\}$ is *realized as an r-distance profile on A* if there is $u \in V(G)$ such that $f = \pi_r^G[u, A]$. We may drop the superscript if the graph is clear from the context.

Similarly to the neighborhood complexity, we define the *distance profile complexity* of a vertex subset $A \subseteq V(G)$ in a graph G, denoted $\widehat{\nu}_r(G, A)$, as the number of different functions realized as r-distance profiles on A in G. Clearly it always holds that $\nu_r(G, A) \leq \widehat{\nu}_r(G, A)$, thus Theorem 3 will follow directly from the following result, which will be proved in the next section.

▶ **Theorem 11.** *Let $\mathcal{C}$ be a nowhere dense class of graphs. Then there is a function $f_{\mathrm{nei}}(r, \varepsilon)$ such that for every $r \in \mathbb{N}$, $\varepsilon > 0$, graph $G \in \mathcal{C}$, and vertex subset $A \subseteq V(G)$, it holds that $\widehat{\nu}_r(G, A) \leq f_{\mathrm{nei}}(r, \varepsilon) \cdot |A|^{1+\varepsilon}$.*

Let us observe that the polynomial bound of Corollary 10 carries over to distance profiles.

▶ **Lemma 12** ($\star$). *Let $\mathcal{C}$ be a nowhere dense class of graphs. Then there is an integer $d(r)$ such that for every $r \in \mathbb{N}$, $\varepsilon > 0$, graph $G \in \mathcal{C}$, and vertex subset $A \subseteq V(G)$ with $|A| \geq 2$, it holds that $\widehat{\nu}_r(G, A) \leq |A|^{d(r)}$.*

3 Neighborhood complexity of nowhere dense classes

In this section we prove Theorem 11, which directly implies Theorem 3, as explained in the previous section. Our approach is to carefully analyze the proof of Reidl et al. [26] for bounded expansion classes, and to fix parts that break down in the nowhere dense setting using tools derived, essentially, from the stability of nowhere dense classes.

We first prove the following auxiliary lemma. We believe it may be of independent interest, as it seems very useful for the analysis of weak coloring numbers in the nowhere dense setting.

▶ **Lemma 13.** *Let $\mathcal{C}$ be a nowhere dense class of graphs and let $G \in \mathcal{C}$. For $r \geq 0$ and a linear order $L \in \Pi(G)$, let*

$$\mathcal{W}_{r,L} = \{\mathrm{WReach}_r[G, L, v] \ : \ v \in V(G)\}.$$

Then there is a constant $x(r)$, depending only on $\mathcal{C}$ and r (and not on G and L), such that $\mathcal{W}_{r,L}$ has VC-dimension at most $x(r)$.

Proof. Since $\mathcal{C}$ is nowhere dense, according to Theorem 5, it is uniformly quasi-wide, say with margins N and s. We fix a number m to be determined later, depending only on r and $\mathcal{C}$. Let $x = x(r) = N(2r, m)$ and $s = s(r)$. Assume towards a contradiction that there is a set $A \subseteq V(G)$ of size x which is shattered by $\mathcal{W}_{r,L}$. Fix sets $S \subseteq V(G)$ and $B \subseteq A \setminus S$ such that $|B| = m$, $|S| \leq s$, and B is $2r$-independent in $G - S$. We will treat L also as a linear order on the vertex set of $G - S$.

As a subset of A, the set B is also shattered by $\mathcal{W}_{r,L}$. That is, for every $X \subseteq B$ there is a vertex $v_X \in V(G)$ such that $X = \mathrm{WReach}_r[G, L, v_X] \cap B$. Note that since B is $2r$-independent in $G - S$, so is X, and we have $|\mathrm{WReach}_r[G - S, L, v_X] \cap B| \leq 1$.

For a vertex $\sigma \in S$, number $\rho \in \{0, \ldots, r-1\}$, and vertex $v \in V(G)$, let us consider the set $\mathcal{P}_{v,\sigma,\rho}$ of all paths of length (exactly) ρ that connect σ and v. We define $b_{\sigma,\rho}(v)$ to be the largest (with respect to L) vertex $b \in B$, for which there exists a path $P \in \mathcal{P}_{v,\sigma,\rho}$ such that every vertex on P is strictly larger than b with respect to L. If no such vertex in B exists, we put $b_{\sigma,\rho}(v) = \perp$. The *signature* of a vertex $v \in V(G)$ is defined as

$$\chi(v) = \big(b_{\sigma,\rho}(v)\big)_{\sigma \in S,\, 0 \leq \rho \leq r-1}.$$

It now follows that the number of possible signatures is small.

▶ **Claim 14** (⋆). *The set* $\{\chi(v) \colon v \in V(G)\}$ *has size at most* $(m+1)^{r \cdot s}$.

The next claim intuitively shows that for a vertex v, the signature of v plus the set of vertices of B that are weakly reachable from v in $G - S$ provide enough information to deduce precisely the set of vertices of B that are weakly reachable from v in G.

▶ **Claim 15** (⋆). *Suppose* $v, w \in V(G)$ *are such that*

$$\chi(v) = \chi(w) \quad \text{and} \quad \mathrm{WReach}_r[G - S, L, v] \cap B = \mathrm{WReach}_r[G - S, L, w] \cap B.$$

Then $\mathrm{WReach}_r[G, L, v] \cap B = \mathrm{WReach}_r[G, L, w] \cap B$.

By Claim 14 there are at most $(m+1)^{r \cdot s}$ possible signatures, while we argued that the intersection $\mathrm{WReach}_r[G - S, L, v] \cap B$ is always of size at most 1, hence there are at most $(m+1)$ possibilities for it. Thus, by Claim 15 we conclude that only at most $(m+1)^{r \cdot s} \cdot (m+1)$ subsets of B are realized as $B \cap W$ for some $W \in \mathcal{W}_{r,L}$. To obtain a contradiction with B being shattered by $\mathcal{W}_{r,L}$, it suffices to select m so that $(m+1)^{r \cdot s+1} < 2^m$. Since $s = s(r)$ is a constant depending on $\mathcal{C}$ and r only, we may choose m depending on $\mathcal{C}$ and r so that the above inequality holds. ◀

With Lemma 13 in hand, we now are ready to prove Theorem 11.

Proof of Theorem 11. Fix $r \geq 1$ and $\varepsilon > 0$. We also use small constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$, which will be determined in the course of the proof.

The first step is to reduce the problem to the case when the size of the graph is bounded polynomially in $|A|$. Without loss of generality assume $|A| \geq 2$. According to Lemma 12, there is an integer $d(r)$ such that there are at most $|A|^{d(r)}$ different r-distance profiles on A. We therefore classify the elements of $V(G)$ according to their r-distance profiles on A, that is, we define an equivalence relation $\sim$ on $V(G)$ as follows:

$$v \sim w \quad \text{if and only if} \quad \pi_r^G[v, A] = \pi_r^G[w, A].$$

Construct a set A' by taking A and, for each equivalence class κ of $\sim$, adding an arbitrary element v_κ to A'. Then, construct a set A'' by starting with A', and, for each distinct $u, v \in A''$, performing the following operation: if $\mathrm{dist}_G(u, v) \leq r$, then add the vertex set of any shortest path between u and v to A''. Finally, let $G' = G[A'']$.

▶ **Claim 16** (⋆). *It holds that* $|A''| \leq |A|^{2 \cdot d(r)+3}$.

▶ **Claim 17** (⋆). *It holds that* $\widehat{\nu}_r(G', A) \geq \widehat{\nu}_r(G, A)$.

The gain from this step is that the size of G' is bounded polynomially in terms of $|A|$, hence we can use better bounds on the weak coloring numbers, as explained next.

According to Theorem 5, there is a function f_{wcol} such that

$$\mathrm{wcol}_{2r}(G') \leq f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A''|^{\varepsilon_4} = f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A|^{(2 \cdot d(r)+3) \cdot \varepsilon_4} = f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A|^{\varepsilon_3},$$

where $\varepsilon_4 = \varepsilon_3/(2 \cdot d(r)+3)$. Let L be a linear order of $V(G')$ with $|\mathrm{WReach}_{2r}[G', L, v]| \leq f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A|^{\varepsilon_3}$. For each $v \in V(G')$, let us define the following set:

$$Y[v] = \mathrm{WReach}_r[G', L, v] \cap \mathrm{WReach}_r[G', L, A].$$

In other words, $Y[v]$ comprises all vertices that are weakly r-reachable both from v and from some vertex of A. Since $Y[v] \subseteq \mathrm{WReach}_r[G', L, v]$ for each $v \in V(G')$, we have $|Y[v]| \leq |\mathrm{WReach}_r[G', L, v]| \leq f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A|^{\varepsilon_3}$. Furthermore, as for each $v \in V(G')$ we have $Y[v] \subseteq \mathrm{WReach}_r[G, L, A] = \bigcup_{w \in A} \mathrm{WReach}_r[G, L, w]$, we have $\left|\bigcup_{v \in V(G')} Y[v]\right| \leq |A| \cdot \max_{w \in V(G')} |\mathrm{WReach}_r[G, L, w]| \leq f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A|^{1+\varepsilon_3}$.

We now classify the vertices $v \in V(G')$ according to their distance profiles $\pi_r^{G'}[v, Y[v]]$. More precisely, let $\equiv$ be the equivalence relation on $V(G')$ defined as follows:

$$v \equiv w \quad \text{if and only if} \quad Y[v] = Y[w] \text{ and } \pi_r^{G'}[v, Y[v]] = \pi_r^{G'}[w, Y[w]].$$

We next show that the equivalence relation $\equiv$ refines the standard partitioning according to r-distance profiles on A.

▶ **Claim 18** ($\star$). *For every $v, w \in V(G)$, if $v \equiv w$, then $\pi_r^{G'}[v, A] = \pi_r^{G'}[w, A]$.*

Claim 18 suggests the following approach to bounding $\widehat{\nu}_r(G', A)$: first give an upper bound on the number of possible sets of the form $Y[v]$, and then for each such set, bound the number of r-distance profiles on it. We deal with the second part first, as it essentially follows from Lemma 12. Let us set $\varepsilon_2 = \varepsilon_3 \cdot d(r)$, where $d(r)$ is the constant given by Lemma 12.

▶ **Claim 19** ($\star$). *There is $g(r, \varepsilon_4)$ such that for all $v \in V(G)$, we have $\widehat{\nu}_r(G, Y[v]) \leq g(r, \varepsilon_4)|A|^{\varepsilon_2}$.*

It remains to give an upper bound on the number of distinct sets $Y[v]$. Let us set $\varepsilon_1 = \varepsilon_3 \cdot x(r)$, where $x(r)$ is the constant given by Lemma 13.

▶ **Claim 20.** *It holds that $|\{Y[v] \colon v \in V(G')\}| \leq 1 + 2 \cdot f_{\mathrm{wcol}}(2r, \varepsilon_4)^{x(r)+1} \cdot |A|^{1+\varepsilon_1+\varepsilon_3}$.*

Proof. Let $\mathcal{Y} = \{Y[v] \colon v \in V(G')\} \setminus \{\emptyset\}$ be the family of all non-empty sets of the form $Y[v]$ for $v \in V(G')$; it suffices to show that $|\mathcal{Y}| \leq f_{\mathrm{wcol}}(2r, \varepsilon_4)^{x(r)+1} \cdot |A|^{1+\varepsilon_1+\varepsilon_3}$. Define mapping $\gamma \colon \mathcal{Y} \to V(G')$ as follows: for $Z \in \mathcal{Y}$, $\gamma(Z)$ is the largest element of Z with respect to L.

Take any $v \in V(G')$ with $Y[v] \neq \emptyset$, and recall that every vertex in $Y[v]$ is weakly r-reachable from v. Observe that every vertex $w \in Y[v]$ is weakly $2r$-reachable from $\gamma(Y[v])$. To see this, concatenate the two paths of length at most r that certify that $w \in \mathrm{WReach}_r[G', L, v]$ and $\gamma(Y[v]) \in \mathrm{WReach}_r[G', L, v]$, and note that this path of length at most $2r$ certifies that $w \in \mathrm{WReach}_{2r}[G', L, \gamma(Y[v])]$. Consequently, for every $y \in \gamma(\mathcal{Y})$, we have

$$\bigcup \gamma^{-1}(y) \subseteq \mathrm{WReach}_{2r}[G', L, y].$$

Hence the union of all sets of $\mathcal{Y}$ that choose the same y via γ has size at most $\mathrm{wcol}_{2r}(G')$.

Fix any $y \in \gamma(\mathcal{Y})$ and denote $S_y = \bigcup \gamma^{-1}(y)$. Let us count how many distinct subsets of S_y belong to $\mathcal{Y}$. Every such $Z = Y[v]$, as a subset of S_y, satisfies $Y[v] \cap S_y = Y[v] = \mathrm{WReach}_r[G'L, v] \cap \mathrm{WReach}_r[G', L, A]$. As for all v the set $\mathrm{WReach}_r[G', L, A]$ is the same, this means that the number of different $Y[v] \in \mathcal{Y}$ that are mapped to a fixed y is not larger than the number of different sets $\mathrm{WReach}_r[G', L, v] \cap S_y$, for $v \in V(G')$.

By Lemma 13, the set $\mathcal{W}_{r,L} = \{\mathrm{WReach}_r[G', L, v] \colon v \in V(G')\}$ has VC-dimension at most $x(r)$, and so has the subfamily $\{S_y \cap \mathrm{WReach}_r[G', L, v] \colon v \in V(G')\}$. Hence, by the Sauer-Shelah Lemma, we infer that $|\{S_y \cap \mathrm{WReach}_r[G', L, v] \colon v \in V(G')\}| \leq 2 \cdot |S_y|^{x(r)}$.

Finally, we observe that $\gamma(\mathcal{Y}) \subseteq \bigcup \mathcal{Y}$ and recall that $|\bigcup \mathcal{Y}| \leq f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A|^{1+\varepsilon_3}$, hence

$$\begin{aligned}
|\mathcal{Y}| &\leq \sum_{y \in \gamma(\mathcal{Y})} |\{S_y \cap \mathrm{WReach}_r[G', L, v] \colon v \in V(G')\}| \\
&\leq 2 \cdot \sum_{y \in \gamma(\mathcal{Y})} |S_y|^{x(r)} \leq 2 \cdot |\gamma(\mathcal{Y})| \cdot (\mathrm{wcol}_{2r}(G'))^{x(r)} \\
&\leq 2 \cdot |\bigcup \mathcal{Y}| \cdot (f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A|^{\varepsilon_3})^{x(r)} \\
&\leq 2 \cdot f_{\mathrm{wcol}}(2r, \varepsilon_4) \cdot |A|^{1+\varepsilon_3} \cdot f_{\mathrm{wcol}}(2r, \varepsilon_4)^{x(r)+1} \cdot |A|^{\varepsilon_1} \\
&\leq 2 \cdot f_{\mathrm{wcol}}(2r, \varepsilon_4)^{x(r)+1} \cdot |A|^{1+\varepsilon_1+\varepsilon_3}.
\end{aligned}$$

◀

By combining Claim 17, Claim 18, Claim 19, and Claim 20, we conclude that

$$\begin{aligned}
\widehat{\nu}_r(G, A) &\leq \widehat{\nu}_r(G', A) \leq \mathrm{index}(\equiv) \\
&\leq |\{Y[v] \ : \ v \in V(G')\}| \cdot g(r, \varepsilon_4) \cdot |A|^{\varepsilon_2} \\
&\leq (1 + 2 \cdot f_{\mathrm{wcol}}(2r, \varepsilon_4)^{x(r)+1} \cdot |A|^{1+\varepsilon_1+\varepsilon_3}) \cdot g(r, \varepsilon_4) \cdot |A|^{\varepsilon_2} \\
&\leq 3 \cdot f_{\mathrm{wcol}}(2r, \varepsilon_4)^{x(r)+1} \cdot g(r, \varepsilon_4) \cdot |A|^{1+\varepsilon_1+\varepsilon_2+\varepsilon_3}.
\end{aligned}$$

Now fix $\varepsilon_4 > 0$ so that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 < \varepsilon$ and set $f_{\mathrm{nei}}(r, \varepsilon) = 3 \cdot f_{\mathrm{wcol}}(2r, \varepsilon_4)^{x(r)+1} \cdot g(r, \varepsilon_4)$. ◀

4 Kernelization for distance-r dominating sets

In this section we use the neighborhood complexity tools to prove Theorem 4. Throughout the section we fix a nowhere dense class $\mathcal{C}$. Whenever we say that the running time of some algorithm on a graph G is *polynomial*, we mean that it is of the form $\mathcal{O}((|V(G)| + |E(G)|)^{\alpha})$, where α is a universal constant that is independent of $\mathcal{C}$, r, ε, or any other constants defined in the context. However, the constants hidden in the $\mathcal{O}(\cdot)$-notation may depend on $\mathcal{C}$, r, and ε.

Projections and projection profiles. Let $G \in \mathcal{C}$ be a graph and let $A \subseteq V(G)$ be a subset of vertices. For vertices $v \in A$ and $u \in V(G) \setminus A$, a path P connecting u and v is called *A-avoiding* if none of its vertices apart from v belong to A. For a positive integer r, the *r-projection* of any $u \in V(G) \setminus A$ on A, denoted $M_r^G(u, A)$ is the set of all vertices $v \in A$ that can be connected to u by an A-avoiding path of length at most r. The *r-projection profile* of a vertex $u \in V(G) \setminus A$ on A is a function $\rho_r^G[u, A]$ mapping vertices of A to $\{0, 1, \ldots, r, \infty\}$, defined as follows: for every $v \in A$, the value $\rho_r^G[u, A](v)$ is the length of a shortest A-avoiding path connecting u and v, and ∞ in case this length is larger than r. Similarly as for r-neighborhoods and r-distance profiles, we define

$$\mu_r(G, A) = |\{M_r^G(u, A) \colon u \in V(G) \setminus A\}| \quad \text{and} \quad \widehat{\mu}_r(G, A) = |\{\rho_r^G[u, A] \colon u \in V(G) \setminus A\}|$$

to be the number of different r-projections and r-projection profiles realized on A, respectively. Clearly, again it always holds that $\mu_r(G, A) \leq \widehat{\mu}_r(G, A)$. The following lemma is a simple consequence of the results of the previous section.

▶ **Lemma 21** ($\star$). *Suppose $\mathcal{C}$ is a nowhere dense class of graphs. Then there is a function $f_{\mathrm{proj}}(r, \varepsilon)$ such that for every $r \in \mathbb{N}$, $\varepsilon > 0$, graph $G \in \mathcal{C}$, and vertex subset $A \subseteq V(G)$, it holds that $\widehat{\mu}_r(G, A) \leq f_{\mathrm{proj}}(r, \varepsilon) \cdot |A|^{1+\varepsilon}$.*

We next recall the main tool for projections proved by Drange et al. [9], namely the *Closure Lemma*. Intuitively, it says that any vertex subset $A \subseteq V(G)$ can be "closed" to a set $\mathrm{cl}_r(A)$ that is not much larger than A, such that all r-projections on $\mathrm{cl}_r(A)$ are small. The next lemma follows from a straightforward adaptation of the proof of Drange et al.

▶ **Lemma 22** ($\star$, Lemma 2.9 of [9], adjusted). *There is a function $f_{\mathrm{cl}}(r,\varepsilon)$ and a polynomial-time algorithm that, given $G \in \mathcal{C}$, $X \subseteq V(G)$, $r \in \mathbb{N}$, and $\varepsilon > 0$, computes the r-*closure *of X, denoted $\mathrm{cl}_r(X)$ with the following properties.*
- *$X \subseteq \mathrm{cl}_r(X) \subseteq V(G)$;*
- *$|\mathrm{cl}_r(X)| \leq f_{\mathrm{cl}}(r,\varepsilon) \cdot |X|^{1+\varepsilon}$; and*
- *$|M_r^G(u, \mathrm{cl}_r(X))| \leq f_{\mathrm{cl}}(r,\varepsilon) \cdot |X|^{\varepsilon}$ for each $u \in V(G) \setminus \mathrm{cl}_r(X)$.*

We need another lemma from Drange et al., called the *Short Paths Closure Lemma*.

▶ **Lemma 23** ($\star$, Lemma 2.11 of [9], adjusted). *There is a function $f_{\mathrm{pth}}(r,\varepsilon)$ and a polynomial-time algorithm which on input $G \in \mathcal{C}$, $X \subseteq V(G)$, $r \in \mathbb{N}$, and $\varepsilon > 0$, computes a superset $X' \supseteq X$ of vertices with the following properties:*
- *whenever $\mathrm{dist}_G(u,v) \leq r$ for $u, v \in X$, then $\mathrm{dist}_{G[X']}(u,v) = \mathrm{dist}_G(u,v)$; and*
- *$|X'| \leq f_{\mathrm{pth}}(r,\varepsilon) \cdot |X|^{1+\varepsilon}$.*

For the rest of this section let us fix constants $r \in \mathbb{N}$ and $\varepsilon > 0$; they will be used implicitly in the proofs. Let us recall some terminology from Drange et al. [9], which is essentially also present in the approach of Dawar and Kreutzer [7]. For a graph G, and vertex subset $Z \subseteq V(G)$, we say that a subset of vertices D is a *(Z,r)-dominator* if $Z \subseteq N_r^G(D)$. We write $\mathbf{ds}_r(G,Z)$ for the smallest (Z,r)-dominator in G and $\mathbf{ds}_r(G)$ for the smallest $(V(G),r)$-dominator in G. The crux of the approach of Drange et al. [9] is to perform kernelization in two phases: first compute a small *r-domination core*, and then reduce the size of the graph in one step.

▶ **Definition 24.** An *r-domination core* of G is a subset $Z \subseteq V(G)$ such that every minimum size (Z,r)-dominator is also a distance-r dominating set in G.

We shall prove the following analogue of Theorem 4.11 of [9].

▶ **Lemma 25.** *There exists a function $f_{\mathrm{core}}(r,\varepsilon)$ and a polynomial-time algorithm that, given a graph $G \in \mathcal{C}$ and integer $k \in \mathbb{N}$, either correctly concludes that G cannot be r-dominated by k vertices, or finds an r-domination core $Z \subseteq V(G)$ of G of size at most $f_{\mathrm{core}}(r,\varepsilon) \cdot k^{1+\varepsilon}$.*

Starting with $Z = V(G)$, which is clearly an r-domination core of G, we try to iteratively remove vertices from Z while preserving the property that Z is an r-domination core of G. More precisely, we show the following lemma, which is the analogue of Theorem 4.12 of [9] and of Lemma 11 of [7].

▶ **Lemma 26.** *There exists a function $f_{\mathrm{core}}(r,\varepsilon)$ and a polynomial-time algorithm that, given a graph $G \in \mathcal{C}$, an integer $k \in \mathbb{N}$, and an r-domination core Z of G with $|Z| > f_{\mathrm{core}}(r,\varepsilon) \cdot k^{1+\varepsilon}$, either correctly concludes that Z cannot be r-dominated by k vertices, or finds a vertex $z \in Z$ such that $Z \setminus \{z\}$ is still an r-domination core of G.*

Observe that Lemma 25 follows by applying Lemma 26 iteratively until the size of the core is reduced to at most $f_{\mathrm{core}}(r,\varepsilon) \cdot k^{1+\varepsilon}$. This iteration is performed at most n times leading to an additional factor n in the running time in Lemma 25.

The first step of the proof of Lemma 26 is to find a suitable approximation of a (Z,r)-dominator. For this, we can rely on the classic $\mathcal{O}(\log \mathsf{OPT})$-approximation of Brönnimann and Goodrich [4] for the Hitting Set problem in set families of bounded VC-dimension, as explained in the next lemma.

▶ **Lemma 27** ($\star$). *There is a function $f_{\mathrm{apx}}(r,\varepsilon)$ and a polynomial-time algorithm that, given $G \in \mathcal{C}$ and $Z \subseteq V(G)$, computes a (Z,r)-dominator of size at most $f_{\mathrm{apx}}(r,\varepsilon) \cdot k^{1+\varepsilon}$, where k is the size of a minimum r-dominating set of Z.*

We are now ready to prove Lemma 26.

Proof of Lemma 26. The function $f_{\mathrm{core}}(r,\varepsilon)$ will be defined in the course of the proof. For convenience, for now we assume that $|Z| > f_{\mathrm{core}}(r,\varepsilon) \cdot k^{1+C\varepsilon}$ for some large constant C, for at the end we will rescale ε accordingly.

We first apply Lemma 27, to either conclude that there is no (Z,r)-dominator of size at most k or to compute a (Z,r)-dominator X of size at most $f_{\mathrm{apx}}(r,\varepsilon) \cdot k^{1+\varepsilon}$. This application takes polynomial time. In the first case we can reject the instance, hence assume that we are in the second case.

We apply the algorithm of Lemma 22 to the set X and distance parameter $3r$, thus computing its closure $\mathrm{cl}_{3r}(X)$, henceforth denoted by X_{cl}. By Lemma 22, we have

$$|X_{\mathrm{cl}}| \quad \leq \quad f_{\mathrm{cl}}(3r,\varepsilon) \cdot |X|^{1+\varepsilon} \leq f_{\mathrm{cl}}(3r,\varepsilon) \cdot f_{\mathrm{apx}}(r,\varepsilon)^{1+\varepsilon} \cdot k^{1+3\varepsilon},$$

and, for every $u \in V(G) \setminus X_{\mathrm{cl}}$,

$$|M^G_{3r}(u,X_{\mathrm{cl}})| \quad \leq \quad f_{\mathrm{cl}}(3r,\varepsilon) \cdot |X|^{\varepsilon} \leq f_{\mathrm{cl}}(3r,\varepsilon) \cdot f_{\mathrm{apx}}(r,\varepsilon)^{\varepsilon} \cdot k^{3\varepsilon}.$$

We now classify the elements of $Z \setminus X_{\mathrm{cl}}$ according to their $3r$-projection profiles on X_{cl}. More precisely, let us define an equivalence relation $\sim$ on $Z \setminus X_{\mathrm{cl}}$ as follows:

$$u \sim v \quad \text{if and only if} \quad \rho^G_{3r}[u,X_{\mathrm{cl}}] = \rho^G_{3r}[v,X_{\mathrm{cl}}].$$

By Lemma 21, the number of equivalence classes of $\sim$ is bounded as follows:

$$\mathrm{index}(\sim) \quad \leq \quad f_{\mathrm{nei}}(3r,\varepsilon) \cdot |X_{\mathrm{cl}}|^{1+\varepsilon} \leq f_{\mathrm{nei}}(3r,\varepsilon) \cdot f_{\mathrm{cl}}(3r,\varepsilon)^{1+\varepsilon} \cdot f_{\mathrm{apx}}(r,\varepsilon)^{(1+\varepsilon)^2} \cdot k^{1+7\varepsilon}.$$

Note that the partition of $V(G)$ into the equivalence classes of $\sim$ can be computed in polynomial time, by just computing the $3r$-projection profile for each vertex using breadth-first search, and then comparing the profiles pairwise.

Let $t(r)$, $p(r)$, and $s(r)$ be the functions provided by Theorem 7 for the class $\mathcal{C}$. Denote $\alpha = f_{\mathrm{cl}}(3r,\varepsilon) \cdot f_{\mathrm{apx}}(r,\varepsilon)^{\varepsilon} \cdot k^{3\varepsilon} + s(2r) + 1$ and $\beta = \alpha \cdot (r+2)^{s(r)} + 1$. From the above inequalities on $|X_{\mathrm{cl}}|$ and $\mathrm{index}(\sim)$, it follows that setting $C = 7 + 3 \cdot p(2r)$, we can fix the function $f_{\mathrm{core}}(r,\varepsilon)$ so that the following inequality is always satisfied:

$$f_{\mathrm{core}}(r,\varepsilon) \cdot k^{1+C\varepsilon} \geq |X_{\mathrm{cl}}| + \mathrm{index}(\sim) \cdot \beta^{p(2r)}.$$

Since we assumed that $|Z| > f_{\mathrm{core}}(r,\varepsilon) \cdot k^{1+C\varepsilon}$, it follows that $|Z \setminus X_{\mathrm{cl}}| > \mathrm{index}(\sim) \cdot \beta^{p(2r)}$. Hence, by the pigeonhole principle, there exists an equivalence class κ of $\sim$ that contains more than $\beta^{p(2r)}$ vertices. By applying the algorithm of Theorem 7 to any subset of κ of size exactly $\beta^{p(2r)}$, we find sets $S \subseteq V(G)$ and $L \subseteq \kappa \setminus S$ such that $|S| \leq s(r)$, $|L| \geq \beta$, and L is $2r$-independent in $G - S$. This application takes time $\mathcal{O}(r \cdot t(2r) \cdot \beta^{p(2r)\cdot(t(r)+1)} \cdot |V(G)|^{1+\varepsilon})$. Provided ε satisfies $3 \cdot p(2r) \cdot (t(2r)+1) \cdot \varepsilon < 1$, which we can assume without loss of generality, we have that $\beta^{p(2r)\cdot(t(2r)+1)} \leq \mathcal{O}(k)$; here, the constants hidden in the $\mathcal{O}(\cdot)$-notation may depend on $\mathcal{C}$. Since we can assume that $k \leq |V(G)|$, this application of the algorithm of Theorem 7 takes then time $\mathcal{O}(|V(G)|^{2+\varepsilon})$, which is polynomial with the degree independent of $\mathcal{C}$.

We now classify the elements of L according to their r-distance profiles on S. Note that $|L| \leq \beta$ and that the number of r-distance profiles on S is bounded by $(r+2)^{|S|} \leq (r+2)^{s(r)}$.

Since $\beta > \alpha \cdot (r+2)^{s(r)}$, by the pigeonhole principle we infer that there is a subset $R \subseteq L$ of size $|R| > \alpha$ such that

$$\pi_r^G(v,S) = \pi_r^G(w,S) \quad \text{for all} \quad v,w \in R.$$

At this point the situation is almost exactly as in Lemma 3.8 of [9]. More precisely, every vertex of R is an irrelevant dominatee, i.e., it can be excluded from Z without spoiling the property that Z is a domination core. The proof of the following claim is based on an exchange argument.

▶ **Claim 28** ($\star$). *The set Z' is also a domination core of G.*

Claim 28 ensures us that vertex z is an irrelevant dominatee that can be returned by the algorithm. Note that we were able to find z provided $|Z| > f_{\mathrm{core}}(r,\varepsilon) \cdot k^{1+C\varepsilon}$ for some constant C depending on $\mathcal{C}$ and r. Hence, we conclude the proof by rescaling ε to ε/C throughout the reasoning. ◀

Finally, having computed a suitably small domination core, we can construct a kernel. This part of reasoning, encapsulated in the following lemma, is exactly the same as in [9].

▶ **Lemma 29** ($\star$). *There exists a function $f_{\mathrm{fin}}(r,\varepsilon)$ and a polynomial-time algorithm that, given a graph $G \in \mathcal{C}$ and an r-domination core $Z \subseteq V(G)$ of G, computes a graph G' with at most $f_{\mathrm{fin}}(r,\varepsilon) \cdot |Z|^{1+\varepsilon}$ vertices such that $Z \subseteq V(G')$ and $\mathbf{ds}_r(G',Z) = \mathbf{ds}_r(G,Z)$.*

Theorem 4 follows by combining Lemma 25 and LEmma 29, and rescaling ε by factor 3.

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