# Fault Tolerant Subgraphs with Applications in Kernelization

**William Lochet**
University of Bergen, Norway
william.lochet@uib.no

**Daniel Lokshtanov**
University of California, Santa Barbara, USA
daniello@ucsb.edu

**Pranabendu Misra**
Max Planck Institute for Informatics, Saarbrücken, Germany
pmisra@mpi-inf.mpg.de

**Saket Saurabh**
Institute of Mathematical Sciences, HBNI and IRL 2000 ReLaX, Chennai, India
University of Bergen, Bergen, Norway
saket@imsc.res.in

**Roohani Sharma**
Institute of Mathematical Sciences, HBNI, India
roohani@imsc.res.in

**Meirav Zehavi**
Ben-Gurion University of the Negev, Beer-Sheva, Israel
meiravze@bgu.ac.il

**Abstract**

In the past decade, the design of *fault tolerant data structures* for networks has become a central topic of research. Particular attention has been given to the construction of a subgraph $H$ of a given digraph $D$ with as fewest arcs/vertices as possible such that, after the failure of any set $F$ of at most $k \geq 1$ arcs, testing whether $D - F$ has a certain property $\mathcal{P}$ is equivalent to testing whether $H - F$ has that property. Here, reachability (or, more generally, distance preservation) is the most basic requirement to maintain to ensure that the network functions properly. Given a vertex $s \in V(D)$, Baswana et al. [STOC'16] presented a construction of $H$ with $\mathcal{O}(2^k n)$ arcs in time $\mathcal{O}(2^k nm)$ where $n = |V(D)|$ and $m = |E(D)|$ such that for any vertex $v \in V(D)$: if there exists a path from $s$ to $v$ in $D - F$, then there also exists a path from $s$ to $v$ in $H - F$. Additionally, they gave a tight matching lower bound. While the question of the improvement of the dependency on $k$ arises for special classes of digraphs, an arguably more basic research direction concerns the dependency on $n$ (for reachability between a pair of vertices $s, t \in V(D)$) – which are the largest classes of digraphs where the dependency on $n$ can be made sublinear, logarithmic *or even constant*? Already for the simple classes of directed paths and tournaments, $\Omega(n)$ arcs are mandatory. Nevertheless, we prove that "almost acyclicity" suffices to eliminate the dependency on $n$ *entirely* for a broad class of dense digraphs called *bounded independence digraphs*. Also, the dependence in $k$ is only a polynomial factor for this class of digraphs. In fact, our sparsification procedure extends to preserve parity-based reachability. Additionally, it finds notable applications in Kernelization: we prove that the classic Directed Feedback Arc Set (DFAS) problem as well as Directed Edge Odd Cycle Transversal (DEOCT) (which, in sharp contrast to DFAS, is W[1]-hard on general digraphs) admit polynomial kernels on bounded independence digraphs. In fact, for any $p \in \mathbb{N}$, we can design a polynomial kernel for the problem of hitting all cycles of length $\ell$ where ($\ell \mod p = 1$). As a complementary result, we prove that DEOCT is NP-hard on tournaments by establishing a combinatorial identity between the minimum size of a feedback arc set and the minimum size of an edge odd cycle transversal. In passing, we also improve upon the running time of the sub-exponential FPT algorithm for DFAS in digraphs of bounded independence number given by Misra et at. [FSTTCS 2018], and give the first sub-exponential FPT algorithm for DEOCT in digraphs of bounded independence number.


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## 1 Introduction

In most real-life applications, even the most reliable networks are highly prone to unexpected failures of a small number of links that connect their nodes. In the past decade, the design of *fault tolerant data structures* for networks has become a central topic of research [7, 9, 13, 47, 44, 12, 16, 17, 18, 11, 26, 45, 46]. Generally, the scenario under study concerns the design of a structure that, after the failure of any set $F$ of at most $k \geq 1$ arcs (representing links) in a given digraph $D$ (representing a network), should provide a fast answer to certain types of queries that address the properties of $D - F$. The most common queries of this form address the *reachability* between two vertices, or, more generally, the length of a shortest path existent, if any, between them. Indeed, reachability (or, more generally, distance preservation) is the most basic requirement to maintain to ensure that the network functions properly. In this context, particular attention has been given to the case where the data structure should consist of a subgraph or a minor of $D$ with as fewest arcs/vertices as possible [7, 47, 9, 11, 8, 45, 13]. Then, queries can be answered by standard means as the usage of BFS or Dijkstra's algorithm. In particular, these simple data structures are of interest as they also double as *sparsifiers*. The study of various graph sparsifiers – such as *flow-sparsifiers* [38] which are closely related to the aforementioned data structures – is a fundamental, active area of research in computer science and structural graph theory [22, 5, 29, 38, 15].

More concretely, in the Fault-Tolerance $(S,T)$-Reachability problem (or FTR$(S,T)$ for short), we are given a digraph $D$, two (not necessarily disjoint) terminals sets $S, T \subseteq V(D)$, and a positive integer $k$. The objective is to construct a subgraph $H$ of $D$ with minimum number of arcs/vertices such that, after the failure of any set of at most $k$ arcs in $D$, the following property is preserved for any two vertices $s \in S$ and $t \in T$: if there still exists a directed path from $s$ to $t$ in $D$, then there also still exists a directed path from $s$ to $t$ in $H$. Clearly, a trivial lower bound on the number of arcs in $H$ is $m = \Omega(n^2)$. For the case where $|S| = 1$ and $T = V(D)$, Baswana et al. [9] presented a construction of a subgraph $H$ with $\mathcal{O}(2^k n)$ arcs in time $\mathcal{O}(2^k nm)$ where $n = |V(D)|$ and $m = |E(D)|$. Additionally, they gave a tight matching lower bound: for any $n, k \in \mathbb{N}$ where $n \geq 2^k$, there exists a digraph on $n$ vertices where $H$ must have $\Omega(2^k n)$ arcs.

Naturally, the question of the improvement of the dependency on $k$ arises for special classes of digraphs. However, an arguably more radical research direction to pursue concerns the dependency on $n$.

Which are the largest classes of digraphs for which $\mathrm{FTR}(S,T)$ admits subgraphs whose size dependency on $n$ can be made sublinear, logarithmic *or even constant*?

At first glance, when we consider the simplest sparsest digraph existent, this pursuit seems futile. Indeed, already in the case where $S = \{s\}$, $T = \{t\}$, $k = 1$ and $D$ is a directed path from $s$ to $t$, the only solution is to choose $H = D$. At second glance, when we consider the simplest densest digraph existent, again we reach a dead-end: for $S$, $T$ and $k$ as before, define $D$ as the tournament obtained by adding, to a directed path $s = v_1 \to v_2 \to \ldots \to v_n = t$, all arcs going from $v_i$ to $v_j$ for every $j + 1 < i$; then, to construct $H$, we must select the entire path.

We show that "almost acyclicity" suffices to eliminate the dependency on $n$ *entirely* for a broad class of dense digraphs called *bounded independence number digraphs.* Furthermore, one can achieve a polynomial dependence in terms of $k$ for this digraph class.

To step beyond the strict confinement of tournaments where *all* relations (arcs) between the input entities (vertices) must be both *present* and *known*, Fradkin and Seymour [33] initiated the study of bounded independence digraphs. Formally, for any integer $\alpha \geq 1$, the class of $\alpha$-bounded independence digraphs, denoted by $\mathcal{D}_\alpha$, is defined as follows.

$$\mathcal{D}_\alpha = \{D \mid D \text{ is a digraph and the maximum size of an independent set in } D \text{ is at most } \alpha\}.$$

chitnis2014directedFor this class of digraphs, Fradkin and Seymour [33] studied the $k$-Disjoint Paths problem, and showed that it admits a polynomial time algorithm for any fixed value of $k$. Observe that $\mathcal{D}_\alpha$ is *hereditary*, and for $\alpha = 1$, it coincides with the class of tournaments. Furthermore, even for $\alpha = 2$, it contains digraphs with a linear fraction of vertex pairs that have no arc between them – thus, it can accommodate the lack of a large number of links/relations.

Our main technical contribution is the following combinatorial lemma.

▶ **Lemma 1.1.** *Given a digraph $D \in \mathcal{D}_\alpha$, positive integers $k$ and $\ell$, and $S \subseteq V(D)$ such that every strongly connected component of $D - S$ has at most $\ell$ vertices, the* Fault-Tolerance *$(S,S)$-*Reachability *($\mathrm{FTR}(S,S)$) problem admits a solution $H$ on $|S|^2(k\ell)^{\mathcal{O}(4^{\alpha\ell^2})}$ vertices. Furthermore, such a solution $H$ can be found in polynomial time.*

In particular, when $D - S$ is acyclic, $\ell = 1$. Thus, if $|S|$ and $\ell$ are independent of $n$ (such as the case where $|S| = |T| = \ell = 1$ discussed earlier), the dependency on $n$ is eliminated. (We remark that a solution for Fault-Tolerance $(S,T)$-Reachability where $S \neq T$ is subsumed by a solution for Fault-Tolerance $(S \cup T, S \cup T)$-Reachability.) Note that we extend the class of digraphs dealt with beyond acyclicity at two fronts: enabling $S$ to be a modulator, thus $D - S$ rather than $D$ should be "almost acyclic"; enabling the strongly connected components to be of size that is ("small" but) larger than 1.

In fact, our result generalizes to *parity reachability.* More precisely, in the Fault-Tolerance $(S,T)$-Parity Reachability problem, we are given a digraph $D$, two terminal sets $S, T \subseteq V(D)$, positive integers $k$ and $p$, and a non-negative integer $r$. The objective is to construct a subgraph $H$ of $D$ with as few arcs/vertices as possible, such that, after the failure of any set of at most $k$ arcs in $D$, the following property is preserved for any two vertices $s \in S$ and $t \in T$: if there exists a directed path from $s$ to $t$ in $D$ whose length $q$ satisfies ($q \mod p = r$), then there also exists a directed path from $s$ to $t$ in $H$ whose length $q'$ satisfies ($q' \mod p = r$). For this problem, we prove the following combinatorial lemma.

▶ **Lemma 1.2.** *Given a digraph $D \in \mathcal{D}_\alpha$, positive integers $k, \ell, p$, a non-negative integer $r$, and $S \subseteq V(D)$ such that every strongly connected component of $D - S$ has at most $\ell$ vertices, the* Fault-Tolerance $(S,S)$-Parity Reachability *problem admits a solution $H$ on $(|S|\alpha\ell pk)^{\mathcal{O}(4^{\alpha\ell^2})}$ vertices. Furthermore, such a solution $H$ can be found in polynomial time.*

## 1.1 Applications in Kernelization

**Directed Feedback Arc Set.** From the perspective of Parameterized Complexity, with the exception of Directed Multicut, the Directed Feedback Arc/Vertex Set (DFA/VS) problem is the most well studied parameterized problem on digraphs. (On general digraphs, the vertex and arc versions of the problem are equivalent [23].) Formally, this problem is defined as follows.

| Directed Feedback Arc Set (DFAS) | **Parameter:** $k$ |
|---|---|
| **Input:** A digraph $D$ and a non-negative integer $k$. | |
| **Question:** Does there exist $S \subseteq E(D)$ of size at most $k$ such that $D - S$ is a DAG? | |

We remark that this problem is among Karp's 21 original NP-complete problems [35]. Already a decade ago, the DFAS problem has been shown to be *fixed-parameter tractable (FPT)* parameterized by the solution size $k$ [19]. Specifically, Chen et al. [19] developed an algorithm that solves DFAS in time $\mathcal{O}(k!4^k k^4 mn)$, based on the powerful machinery of important separators [23]. Since then, the quest to assert the existence of a polynomial kernel for this problem has been unfruitful. Over the years, it has been repeatedly posed as a major challenge in the subfield of Kernelization [23, 28, 42, 41] (also see [1] for a number of workshops and schools where it was posed as an open problem). In fact, the two specific problems whose polynomial kernelization complexity is completely unknown and their resolution is raised most frequently are DFAS and Multiway Cut [23, 28]. At the front of parameterized algorithms, the recent work by Lokshtanov et al. [39] improved upon the polynomial factor of the aforementioned algorithm by the design of an $\mathcal{O}(k!4^k k^5(m+n))$-time algorithm. It is known that unless the Exponential Time Hypothesis (ETH) is false, parameterized by the treewidth $\mathsf{tw}$ of the underlying undirected graph, DFAS cannot be solved in time $2^{o(\mathsf{tw}\log\mathsf{tw})} \cdot n^{\mathcal{O}(1)}$. However, it is unknown whether DFAS is solvable in time $2^{o(k\log k)} \cdot n^{\mathcal{O}(1)}$. In this regard, the only lower bound known is of $2^{\Omega(k)} \cdot n^{\mathcal{O}(1)}$ under the ETH [23, 39].

Particular attention has been given to the parameterized complexity of DFAS on tournaments. The classical complexity (NP-hardness) of DFAS on tournaments has a curious history. More than two decades ago, this problem was conjectured to be NP-hard by Bang-Jensen and Thomassen [6]. In 2008, Ailon et al. [2] proved that this problem does not admit a polynomial-time algorithm unless NP$\subseteq$BPP. Later, the reduction of Ailon et al. [2] was derandomized independently by Alon [3] and Charibt et al. [14], to prove that DFAS on tournaments is NP-hard. With respect to Parameterized Complexity, Alon et al. [4] proved that DFAS on tournaments admits a sub-exponential time parameterized algorithm (with running time $2^{\mathcal{O}(\sqrt{k}\log^2 k)} \cdot n^{\mathcal{O}(1)}$), to which end they introduced the method of chromatic coding. Later, the $\log^2 k$ factor in the exponent was shaved in independent works by Feige [30] and Karpinski and Schudy [36]. Fomin and Pilipczuk [32] presented a general approach, based on a bound on the number of $k$-cuts in transitive tournaments, to achieve the same running time for DFAS on tournaments. Based on this approach, Misra et al. [43] developed a sub-exponential time parameterized algorithm for DFAS on digraphs in $\mathcal{D}_\alpha$, with running time $2^{\mathcal{O}(\alpha^2\sqrt{k}\log(\alpha k))} \cdot n^{\mathcal{O}(\alpha)}$. Yet, the (arguably more) intriguing question

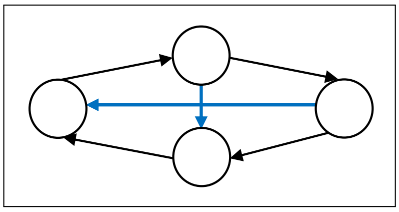

**Figure 1** A directed edge odd cycle transversal (in blue) that is not a directed feedback arc set.

of the existence of a polynomial kernel for DFAS on digraphs in $\mathcal{D}_\alpha$ remained unsolved. On tournaments, Bessy et al. [10] have proved that DFAS admits a linear-vertex kernel (improving upon polynomial kernels given in [4, 27]). Based on our combinatorial lemma (Lemma 1.1), we establish the following theorem.

▶ **Theorem 1.3.** DFAS *on* $\mathcal{D}_\alpha$ *admits a kernel of size* $k^{\mathcal{O}(4^\alpha)}$.

In addition to its rich history in theoretical studies, the elimination of directed feedback loops is highly relevant to rank aggregation, Voting Theory, the resolution of inconsistencies in databases, and the prevention of deadlocks [48, 10, 34, 37, 19, 31]. While in a wide-variety of applications, *most* relations between the entities in a network are both present and known, it is generally unrealistic (in real-world partial and noisy data) that *all* relations will be so. Then, the usage of a bounded independence digraphs naturally comes into play. In passing, using Theorem 1.3, we also improve the running time for DFAS on digraphs in $\mathcal{D}_\alpha$, given by Misra et al. [43], by eliminating the dependence of $\alpha$ in the exponent of $n$. That is, we have the following theorem.

▶ **Theorem 1.4.** DFAS *on* $\mathcal{D}_\alpha$ *can be solved in* $2^{f(\alpha)\sqrt{k}\log k} \cdot n^{\mathcal{O}(1)}$, *where* $f(\alpha)$ *is some function of* $\alpha$ *and* $n$ *is the number of vertices in* $D$.

**Directed Edge Odd Cycle Transversal.** The Directed Edge Odd Cycle Transversal (DEOCT) problem is the parity-based version of DFAS, formally defined as follows. (On general digraphs, the vertex and arc versions of the problem are equivalent [40]).

| Directed Edge Odd Cycle Transversal (DEOCT) | **Parameter:** $k$ |
|---|---|
| **Input:** A digraph $D$ and a non-negative integer $k$. | |
| **Question:** Does there exist $S \subseteq E(D)$ of size at most $k$ such that $D - S$ has no odd cycle? | |

Observe that a tournament has no directed cycle if and only if it has no directed triangle (a cycle on three vertices). In turn, this simple observation implies that, given a tournament $D$, any subset $S$ of the *vertices* of $D$ has the following property: $D - S$ is a DAG if and only if it has no directed odd cycle. Thus, the vertex versions of DFAS and DEOCT on tournaments are equivalent. However, for DFAS and DEOCT the situation is not so clear. Indeed, it is not difficult to come up with a tournament $D$ and a subset of *arcs* $S$ of $D$ such that $D - S$ is not a DAG, yet it has no directed odd cycle (see, e.g., Fig. 1). Nonetheless, we are able to prove that given a tournament $D$ and a subset $S$ of the *arcs* of $D$ such that $D - S$ has no directed odd cycle, there exists a subset of arcs $S'$ of $D$ such that $D - S'$ is a DAG and $|S'| \leq |S|$. In particular, we thus establish the following result.

▶ **Theorem 1.5.** DEOCT *on tournaments is* NP-*hard.*

The question of the parameterized complexity of DEOCT was explicitly stated as an open problem [24] for the first time in 2007, immediately after the announcement of the first parameterized algorithm for DFAS. Since then, the problem has been re-stated several times [20, 21, 42, 41]. Recently, Lokshtanov et al. [40] proved that DEOCT is W[1]-hard. Specifically, this means that DEOCT is highly unlikely to be FPT or admit a kernel of any size (even exponential in $k$). Based on the parity-based generalization of our combinatorial lemma (Lemma 1.2), we establish a polynomial kernel for DEOCT on $\mathcal{D}_\alpha$, which stands in sharp contrast to its aforementioned status on general digraphs.

▶ **Theorem 1.6.** DEOCT *on* $\mathcal{D}_\alpha$ *admits a kernel of size* $(\alpha k)^{\mathcal{O}(4^{4\alpha^3})}$.

In fact, we present combinatorial results stronger than Lemma 1.2 that yield a polynomial kernel for a more general version of DEOCT, where instead of hitting directed odd cycles, the objective is to hit directed cycles whose length $\ell$ satisfies ($\ell \mod p = 1$) for an integer $p \in \mathbb{N}$ given as input.[1]

| Modulo $p$ Directed Cycle Transversal (mod($p$)-DCT) **Parameter:** $k$ |
|---|
| **Input:** A digraph $D$ and non-negative integers $k$ and $p$. |
| **Question:** Does there exist $S \subseteq E(D)$ of size at most $k$ such that $D - S$ has no cycle of length $1 \mod p$? |

▶ **Theorem 1.7.** mod($p$)-DCT *on* $\mathcal{D}_\alpha$ *admits a kernel of size* $(p\alpha k)^{\mathcal{O}(4^{\alpha^3 p^2})}$.

Having Theorem 1.6 at hand, we also show how to employ the general approach of Fomin and Pilipczuk [32] to derive a sub-exponential time parameterized algorithm for DEOCT on digraphs in $\mathcal{D}_\alpha$.

▶ **Theorem 1.8.** DEOCT *on* $\mathcal{D}_\alpha$ *admits an algorithm with running time* $2^{\mathcal{O}(f(\alpha)\sqrt{k}\log k)} \cdot n^{\mathcal{O}(1)}$*, where* $f(\alpha)$ *is a function of* $\alpha$ *and* $n$ *is the number of vertices in* $D$.

## 1.2 Towards the proof of Lemmas 1.1: Cut Preserving Sets

The most central notion in this paper is of a *cut preserving set*. Informally, for a digraph $D$, a pair of vertices $s, t$ and an integer $k$, a set $\mathcal{Z} \subseteq V(D)$ is called a $k$-cut preserving set[2] for $(s, t)$ in $D$ if it preserves all $(s, t)$-arc cuts of size at most $k$. That is, $A$ is an $(s, t)$-arc cut with at most $k$ arcs in $D$ if and only if $A$ is a such a cut in $D[\mathcal{Z}]$. Observe that the graph induced on such a $k$-cut preserving set $\mathcal{Z}$ is a candidate solution for FTR($\{s\}, \{t\}$) problem. Clearly $V(D)$ is a $k$-cut preserving set for any pair of vertices $s, t$. The intent is to have such a set of "small" size. Towards this, let us discuss some properties that suffice for $\mathcal{Z}$ to be a $k$-cut preserving set for $(s, t)$ in $D$.

Since $\mathcal{Z} \subseteq V(D)$, any $(s, t)$-arc cut of $D$ is an $(s, t)$-arc cut of $D[\mathcal{Z}]$. For the other direction, we need the property that, for any $A \subseteq E(D)$ of size at most $k$, the existence of an $(s, t)$-path in $D - A$ implies the existence of an $(s, t)$-path in $D[\mathcal{Z}] - A$. Let us now see which properties suffice to imply the above property. We begin with a special case. Suppose there is a "large" flow from $s$ to $t$ in $D$. In particular, suppose there are at least $k + 1$ internally vertex-disjoint $(s, t)$-paths in $D$. Then, in $\mathcal{Z}$ it is enough to keep the vertices of some $k + 1$ vertex-disjoint $(s, t)$-paths, as no arc set of size at most $k$ can hit all these paths. The more involved case occurs when the flow from $s$ to $t$ in $D$ is at most $k$. Consider any $(s, t)$-path $P$ in $D$. Ideally (if we did not have a size constraint on $\mathcal{Z}$) we would have preserved all the

[1] Note that a fundamental difference between this result and Lemma 1.2 is that the latter only works for any modulo and not just 1.

[2] This is not the way it is defined later. However, for the sake of exposition, we start with this definition and refine it to have properties that also guarantee this property implicitly.

vertices of $P$ in $\mathcal{Z}$. Clearly, this can be expensive in terms of the size of $\mathcal{Z}$. Nevertheless, we can merge the ideas above (the "large-flow idea" and the "keep-full-path idea") to get the desired result. To see this, let $P$ be a $(s,t)$-path in $D$. Let $\mathcal{Z}$ be a set of vertices such that, either all the vertices of $P$ are in $\mathcal{Z}$ or if the vertices of a $(u,v)$-subpath of $P$ are not in $\mathcal{Z}$, then there are $k+1$ internally vertex-disjoint $(u,v)$-paths in $D[\mathcal{Z}]$. That is, if the vertices of a subpath are missing in $\mathcal{Z}$, then $\mathcal{Z}$ contains a witness of a large flow for the endpoints of this subpath. Observe that such a set $\mathcal{Z}$ suffices to be a $k$-cut preserving set for $(s,t)$ in $D$. This is because if $P$ is an $(s,t)$-path in $D - A(A \subseteq E(D)$ and $|A| \leq k)$, then either all the vertices of $P$ are in $\mathcal{Z}$ or for any missing $(u,v)$-subpath of $P$, since there are $k+1$ vertex-disjoint $(u,v)$-paths in $D[\mathcal{Z}]$, at least one still remains in $D[\mathcal{Z}] - A$. Thus, in $D[\mathcal{Z}] - A$, one can find an $(s,t)$-path: for the missing subpaths of $P$ in $\mathcal{Z}$, there exists some (other) path between the same endpoints in $D[\mathcal{Z}] - A$ which together yield an $(s,t)$-walk (and hence an $(s,t)$-path) in $D[\mathcal{Z}] - A$. These properties are formalized in Definition 3.1.

### 1.2.1 About Computing $k$-Cut Preserving Sets

Next we give an intuition for how one can compute such $k$-cut preserving sets for a digraph $D \in \mathcal{D}_\alpha$, each of whose strongly connected component has size at most $\ell$. For exposition purposes, consider (for now), only the case where $D$ is acyclic (i.e. $\ell = 1$). With a certain technical argument, the general case reduces to this one. Moreover, we use the definition of a $k$-cut preserving set from the beginning of this section for this illustration as it allows us to convey our ideas in a clearer manner.

The proof will use induction on $\alpha$. As the base case, consider the case when $\alpha = 1$, that is, $D$ is a transitive tournament. As $D$ is transitive, there exists a topological ordering of the vertices of $D$. Consider the set $S$ of vertices between $s$ and $t$ in this ordering. Note that any path from $s$ to $t$ only uses vertices in $S$. So, either $S$ is smaller than $k+1$, and then $S \cup \{s,t\}$ is a $k$-cut preserving set for $(s,t)$, or it can be seen that there is no arc-cut for $(s,t)$ of size at most $k$. In the latter case, the union of $\{s,t\}$ and any subset of $k+1$ vertices of $S$ is a $k$-cut preserving set for $(s,t)$; indeed, in the subgraph induced by the union there is still no arc-cut for $(s,t)$ of size at most $k$.

Now, let us hint at how the inductive step of the proof works. First, we note that, if $P_1, \dots, P_{k+1}$ are $k+1$ internally vertex-disjoint $(s,t)$-paths, then $\mathcal{Z} = \cup_{i \in [k+1]} P_i$ is a $k$-cut preserving set for the pair $(s,t)$, because there is no arc-cut of $(s,t)$ in both $D$ and $D[\mathcal{Z}]$ of size at most $k$. Moreover, since $D$ is acyclic and $D \in \mathcal{D}_\alpha$, if these paths exist, then Observation 2.1 implies that we can assume that all these paths are shorter than $2\alpha + 1$ and thus $|\mathcal{Z}| \leq k(2\alpha + 1)$.

The last argument means that we can assume the existence of a $(s,t)$-vertex cut of size at most $k$. For simplicity, suppose that $\{c_1, c_2\}$ is a minimal $(s,t)$-vertex -cut. Since $\{c_1, c_2\}$ is a vertex cut, any path from $s$ to $t$ in $D$ can be decomposed as a path from $s$ to $c_i$, a path from $c_i$ to $c_j$ and then a path from $c_j$ to $t$, where $i$ and $j$ are two indices (possibly equal) in $\{1,2\}$. Here, we mean that none of the three paths contains $c_i$ (or $c_j$) as an internal vertex. For $i \in \{1,2\}$, let $S_i$ be the union of the set of vertices of the paths from $s$ to $c_i$ that intersect $\{c_1, c_2\}$ only on the last vertex, and $T_i$ be the union of the set of vertices of the paths from $c_i$ to $t$ that intersect $\{c_1, c_2\}$ only on the first vertex. Finally, for distinct $i, j \in \{1,2\}$, let $C_{i,j}$ be the union of the set of vertices of the paths from $c_i$ to $c_j$. Because of the last remark on how any path from $s$ to $t$ can be decomposed, taking the union of six $k$-cut preserving sets-namely, for each $i, j \in \{1,2\}$, $i \neq j$, for $(s, c_i)$ in $D[S_i]$, $(c_i, t)$ in $D[T_i]$ and $(c_i, c_j)$ in $D[C_{i,j}]$- gives a $k$-cut preserving set for $(s,t)$ in $D$. Now, the question is how to use the induction hypothesis to find a $k$-cut preserving set for each of these pairs. Consider first the digraph induced by the vertices in $S_1$. Because $\{c_1, c_2\}$ is a minimal $(s,t)$-vertex cut, the only vertices of $S_1$ that can possibly have "outgoing arcs towards" $t$ in $S_1$ are $s$ and $c_1$. Moreover, since $\{c_1, c_2\}$ is a minimal $(s,t)$-vertex cut, there exists a path from $c_1$ to $t$ in $D$ and thus $t$ is reachable from any vertex of $S_1$. However, since $D$ is acyclic, this means that there is no arc from $t$ to any of the vertices of $S_1$, else we would get a closed walk and thus a cycle. This implies that $D[S_1 \setminus \{s, c_1\}] \in \mathcal{D}_{\alpha-1}$ as any independent set of $S_1 \setminus \{s, c_1\}$ can be extended with $t$. We cannot apply the induction hypothesis to find a $k$-cut preserving set for $(s, c_1)$ in $S_1$ because the independence number of $D[S_1]$ could be equal to $\alpha$, however the above shows the spirit of the arguments that will be used to find subgraphs with smaller independence number where we can apply the induction hypothesis. A similar argument would also give that the independence number of $D[T_1 \setminus \{c_1, t\}]$ is at most $\alpha - 1$ as any independent set can be extended using $s$.

The previous argument does not apply to $C_{1,2}$, because the vertices of $C_{1,2}$ can be adjacent to $s$ or $t$ (some vertices of $C_{1,2}$ can be adjacent to $s$ and some can be adjacent to $t$). This is the case that requires a stronger and more technical definition for a $k$-cut preserving set. In particular, we need to understand what happens to the vertices of $D$ that are on a path from $s$ to $t$ but do not belong to a $k$-cut preserving set for this pair.

### 1.3 Deriving Polynomial Kernels for DFAS

Let us now briefly explain how to derive a polynomial kernel for DFAS when the input digraph belongs to $\mathcal{D}_\alpha$, from our result on fault-tolerant subgraphs. First note that if $D \in \mathcal{D}_\alpha$ then every induced cycle in $D$ has length at most $2\alpha + 1$. Let $(D, k)$ be an instance of DFAS, and consider a maximal set of arc disjoint induced cycles in $D$. If this set consists of more than $k$ cycles, then any solution to $(D, k)$ has to pick one arc per cycle, and $(D, k)$ is a NO instance. If not, let $S$ be the union of these cycles. $S$ is a set of less than $(2\alpha + 1) \cdot k$ vertices such that $D - S$ is acyclic. Therefore, we can apply our result to find a solution $H$ to the problem of Fault-Tolerance $(S, S)$-Reachability of size at most $|S|^2 k^{\mathcal{O}(4^\alpha)}$ . We claim that $H$ is the desired kernel. Indeed, suppose that $A$ is a set of arcs such that $H - A$ is acyclic, but $D - A$ contains a cycle. By construction of $S$, this cycle must use vertices of $S$. However, we know that if a path exists between two vertices of $S$ in $D - A$, then such a path also exists in $H - A$. This implies the existence of a closed walk in $H - A$, a contradiction.

## 2 Preliminaries

For standard notations and terminology that is not defined here, we refer to [25].

**Sets.** For positive integer $i, j$, $[i]$ denotes the set $\{1, \ldots, i\}$ and $[i, j]$ denote the set $\{i, i + 1, \ldots, j\}$. For a set $S$, $S^2$ denotes the set of ordered pairs of $S$, that is $S^2 = \{(u, v) \mid u \in S, v \in S\}$.

**Digraphs.** For a digraph $D$, $V(D)$ denotes the vertex set of $D$ and $E(D)$ denotes the arc set of $D$. For any $X \subseteq V(D)$ (resp. $X \subseteq E(D)$), $D - X$ denotes the digraph obtained by deleting the vertices (resp. edges) of $X$. For any $v \in V(D)$, $N^+_D(v)$ (resp. $N^-_D(v)$) denotes the set of out-neighbours (resp.in-neighbours) of $v$ in $D$, that is $N^+_D(v) = \{u \in V(D) \mid (v, u) \in E(D)\}$ (resp. $N^-_D(v) = \{u \in V(D) \mid (u, v) \in E(D)\}$). Whenever the digraph $D$ is clear from the context, we drop the subscript $D$ in $N^+_D(v)$ (resp. $N^-_D(v)$). For any $X, Y \subseteq V(D)$, $E(X, Y)$ denotes the set of arcs of $D$ with tail in $X$ and head in $Y$, that is, $E(X, Y) = \{(u, v) \in E(D) \mid u \in X, v \in Y\}$. A digraph $D$ is called *strongly connected* if for each $u, v \in V(D)$ there is a path from $u$ to $v$ and, a path from $v$ to $u$ in $D$. A set $X \subseteq V(D)$ is called a *strongly connected component* of $D$ if $D[X]$ is a strongly connected digraph and for each $X' \supseteq X$, $D[X']$ is not a strongly connected digraph. A *tournament* is a digraph where there is exactly one arc between each pair of vertices. A digraph with no cycles is called a *directed acyclic graph (dag)*. A tournament with no cycles is called a *transitive tournament*.

**Paths.** A *path* $P$ is a graph such that there exists an ordering $(v_1, \ldots, v_q)$ of its vertex set $V(P)$ such that $E(P) = \{(v_i, v_{i+1}) \mid i \in [q-1]\}$. Such a path is called a $(v_1, v_q)$*-path*, $v_1, v_q$ are called the *end-points of* $P$ and $v_2, \ldots, v_{q-1}$ are called the *internal vertices of* $P$. A path $P$ is *even* (resp. odd) if the number of arcs/edges in it is even (resp. odd). We say that $P$ is a path in the digraph $D$ if $P$ is a subgraph of $D$. We say that $P$ is an *induced path* in $D$ if $P$ is an induced subgraph of $D$. For paths $P$ and $P'$, by $P \circ P'$ we denote the composition of $P$ and $P'$, that is, the path obtained by appending $P'$ after $P$. For paths $P, P_1, P_2, \ldots, P_q$ such that $P = P_1 \circ P_2 \circ \ldots \circ P_q$, we say that $P_1 \circ P_2 \circ \ldots \circ P_q$ is a *partition* of $P$. For a digraph $D$ and $X \subseteq V(D)$, we say that a $(u, v)$-path $P$ in $D$ is $X$*-free* if none of the internal vertices of $P$ are from $X$. The $X$*-based partition of* $P$ in $D$ is the partition $P = P_1 \circ \ldots \circ P_q$ such the union of the end-points of $P_i$, $i \in [q]$, is exactly the set $(X \cap V(P)) \cup \{u, v\}$. A *semi-*$X$*-based partition* of $P$, $P = P_1 \circ \ldots \circ P_q$, is such that the end-points of the paths $P_i$, $i \in [q]$, are a subset of $(X \cap V(P)) \cup \{u, v\}$. Paths $\{P_1, \ldots, P_q\}$ are *internally vertex-disjoint* if for all distinct $i, j \in [q]$, the sets of internal vertices of $P_i$ and $P_j$ are disjoint.

**Vertex and Arc Cuts.** For a digraph $D$ and $u, v \in V(D)$, a $(u, v)$*-arc cut* is a set of arcs of $D$, say $X$, such that $D - X$ has no $(u, v)$-path. A $(u, v)$*-vertex cut* is a set of vertices of $D$, say $Y$, such that $D - Y$ has no $(u, v)$-path and $u, v \notin Y$ if $(u, v) \notin E(D)$.

▶ **Observation 2.1.** *Let $D \in \mathcal{D}_\alpha$. The length of the shortest cycle in $D$ is at most $2\alpha + 1$. Also, the length of any induced path in $D$ is at most $2\alpha + 1$.*

In this article, we focus of the proof of Lemma 1.1 alone. The proofs of Lemma 1.2 and Theorems 1.6, 1.8, 1.5, 1.6, 1.7, 1.8 will be made available later in the full version of the paper.

## 3 Finding Small $k$-Cut Preserving Sets

We give the precise definition of a $k$-cut preserving set here.

▶ **Definition 3.1** ($k$-Cut Preserving Set). *For digraph $D$, an ordered pair $(u, v)$ of vertices of $D$ and a positive integer $k$, $\{u, v\} \subseteq \mathcal{Z} \subseteq V(D)$ is a* $k$-cut preserving set for $(u, v)$ in $D$ *if the following holds. For any $(u, v)$-path $P$ in $D$, there exists a semi-$\mathcal{Z}$-based partition $P_1 \circ \ldots \circ P_d$ of $P$ with the following two properties. For each $i \in [d]$, $P_i$ is an $(s_i, t_i)$-path in $D$ with $s_i, t_i \in \mathcal{Z}$. Moreover, either $V(P_i) \subseteq \mathcal{Z}$ or there exists a list $\mathcal{L}_i$ of $k+1$ internally vertex-disjoint $(V(D) \setminus \mathcal{Z})$-free $(s_i, t_i)$-paths. A list $\mathcal{L}_i$ with the above property is called a* replacement kit *for $P_i$ in $\mathcal{Z}$. Such a semi-$\mathcal{Z}$-based partition of $P$ is called a* $\mathcal{Z}$-replacement witness *for $P$.*

Before moving to the computational aspects of a $k$-cut preserving set, we give the following lemma that can be considered as the main utility of $k$-cut preserving sets, and relate to the intuition we gave in the previous section.

▶ **Lemma 3.2.** *Let $D$ be a digraph, $u, v \in V(D)$ and $\mathcal{Z}$ be a $k$-cut preserving set for $(u, v)$ in $D$. For any set $A \subseteq E(D)$ of at most $k$ arcs, if there exists a $(u, v)$-path in $D - A$, then there also exists one in $D[\mathcal{Z}] - A$.*

**Proof.** Consider some $A \subseteq E(D)$ such that $|A| \leq k$. Suppose there exists a $(u,v)$-path $P$ in $D - A$ . Since $\mathcal{Z}$ is a $k$-cut preserving set for the pair $(u,v)$, there exists a semi-$\mathcal{Z}$-based partition $P = P_1 \circ \ldots \circ P_d$ such that for each $j \in [d]$, $P_j$ is an $(s_j,t_j)$-path, $s_j, t_j \in \mathcal{Z}$ and, either $V(P_j) \subseteq \mathcal{Z}$, in which case $P_j$ is a path in $D[\mathcal{Z}] - A$, or there exist $k+1$ internally vertex- disjoint $(s_j,t_j)$-paths in $D[\mathcal{Z}]$. In the later case, at least one of the $k+1$ paths is in $D[\mathcal{Z}] - A$ (because $|A| \leq k$). This implies the existence of a walk from $u$ to $v$ (and hence also a $(u,v)$-path) in $D[\mathcal{Z}] - A$. This concludes the proof. ◀

The main goal of this section is to prove the following lemma.

▶ **Lemma 3.3** ($k$-Cut Preserving Lemma). *Let $D$ be an acyclic digraph, and $u,v \in V(D)$ be such that $N^-(u) = N^+(v) = \emptyset$. Additionally, let $D - \{u,v\} \in \mathcal{D}_\alpha$. Then there exists a $k$-cut preserving set for $(u,v)$ in $D$ of size at most $f(\alpha)$, where $f(1) = k^3 + 5k^2 + 3k$ and for $\alpha > 1$, $f(\alpha) = k^2 g(\alpha) + 2kh(\alpha)$, $g(\alpha) = (2k + (k + kf(\alpha-1))^2)f(\alpha-1)$ and $h(\alpha) = (k^2+k)g(\alpha) + kf(\alpha-1)$. Moreover, such a set can be found in time $n^{\mathcal{O}(1)}$, where $n = |V(D)|$.*

Note that $V(D)$ is always a $k$-cut preserving set for any pair of vertices $(u,v)$ in $D$, for any $k$. We now define a notation, for the sake of convenience, that will be used throughout this section. For any digraph $D$, $u,v \in V(D)$ and $X \subseteq V(D)$, let $ver_D(u,v;X)$ denote the union of the sets of vertices of all $X$-free $(u,v)$-paths in $D$. Observe that $ver_D(u,v;X) \cap X \subseteq \{u,v\}$. We begin by making an observation that forms the base line for computing small sized $k$-cut preserving sets using an appropriate induction.

▶ **Observation 3.4.** *Let $D$ be a digraph, $u,v \in V(D)$, $\mathcal{Z} \subseteq V(D)$ and $k$ be a positive integer. Let $P$ be a $(u,v)$-path in $D$, and $P = P_1 \circ \ldots \circ P_d$ be a semi-$\mathcal{Z}$-based partition of $P$. If for each $i \in [d]$, there is a $\mathcal{Z}_i$-replacement witness for $P_i$ in $D_i$, for some $\mathcal{Z}_i \subseteq \mathcal{Z}$ and $D_i$ subgraph of $D$, then there is a $\mathcal{Z}$-replacement witness for $P$.*

**Proof.** For each $i \in [d]$, let $P_I = P_{i,1} \circ \ldots \circ P_{i,c_i}$ be a $\mathcal{Z}_i$-replacement witness for $\mathcal{Z}_i$ in $D_i$. Then, consider the semi-$\mathcal{Z}$-based partition $P = P_{1,1} \circ \ldots \circ P_{1,c_1} \circ P_{2,1} \circ \ldots \circ P_{2,c_2} \circ \ldots \circ P_{d,1} \circ \ldots \circ P_{d,c_d}$. Then, for each $i \in [d]$ and $j \in [c_i]$, either $V(P_{i,j}) \subseteq \mathcal{Z}_i \subseteq \mathcal{Z}$, or there exists a list $\mathcal{Z}_{i,j}$ containing $k+1$ internally vertex-disjoint $(V(D_i) \setminus \mathcal{Z}_i)$-free $(x_{i,j}, y_{i,j})$-paths in $D_i$ such that $P_{i,j}$ is a $(x_{i,j}, y_{i,j})$-path. Since $\mathcal{Z}_i \subseteq \mathcal{Z}$ and $D_i$ is a subgraph of $D$, the paths in $\mathcal{L}_{i,j}$ are $(V(D) \setminus \mathcal{Z})$-free and exist in $D$. ◀

Next, we give two lemmas (Lemmas 3.5 and 3.6) that basically use Observation 3.4 in a more concrete setting required to prove the $k$-Cut Preserving Lemma by induction on the size of the maximum independent set in the digraph.

▶ **Lemma 3.5.** *Let $D$ be a digraph, $u,v \in V(D)$ and $k$ be a positive integer. Let $C$ be some $(u,v)$-vertex cut in $D$. For each $c \in C$, let $\mathcal{Z}(u,c)$ (resp. $\mathcal{Z}(c,v)$) be a $k$-cut preserving set for $(u,c)$ (resp. $(c,v)$) in $D[ver_D(u,c;C)]$ (resp. $D[ver_D(c,v;C)]$). For each $(c,c') \in C^2$, $c \neq c'$, let $\mathcal{Z}(c,c')$ be a $k$-cut preserving set for $(c,c')$ in $D[ver_D(c,c';C)]$. Then, $\mathcal{Z} := \bigcup_{c \in C}(\mathcal{Z}(u,c) \cup \mathcal{Z}(c,v)) \cup \bigcup_{(c,c') \in C^2, c \neq c'} \mathcal{Z}(c,c')$ is a $k$-cut preserving set for $(u,v)$ in $D$.*

**Proof.** First observe, from the definition of a $k$-cut preserving set and the construction of $\mathcal{Z}$, that $C \subseteq \mathcal{Z}$. Consider any $(u,v)$-path $P$ in $D$. Let $P = P_1 \circ \ldots \circ P_q$ be the $C$-based partition of $P$. Since $C \subseteq \mathcal{Z}$, $P_1 \circ \ldots \circ P_q$ is a semi-$\mathcal{Z}$-based partition of $P$. Then $P_1$ is a $C$-free $(u,c_1)$-path in $D$ for some $c_1 \in C$, $P_q$ is a $C$-free $(c_2,v)$-path in $D$ for some $c_2 \in C$, and for each $i \in [2,q-1]$, $P_i$ is a $C$-free $(c_{j_i}, c_{j_i'})$-path in $D$, for some

$c_{j_i}, c_{j_i'} \in C$, $j_i \neq j_i'$. Thus, $P_1$ is a $(u, c_1)$-path in $D[ver_D(u, c_1; C)]$, $P_q$ is a $(c_2, v)$-path in $D[ver_D(c_2, v; C)]$, and for each $i \in [2, q-1]$, $P_i$ is a $(c_{j_i}, c_{j_i'})$-path in $D[ver_D(c_{j_i}, c_{j_i'}; C)]$. Since $\mathcal{Z}(u, c_1), \mathcal{Z}(c_2, v), \cup_{i \in [2,q-1]} \mathcal{Z}(c_{j_i}, c_{j_i'}) \subseteq \mathcal{Z}$, we are done by Observation 3.4. ◀

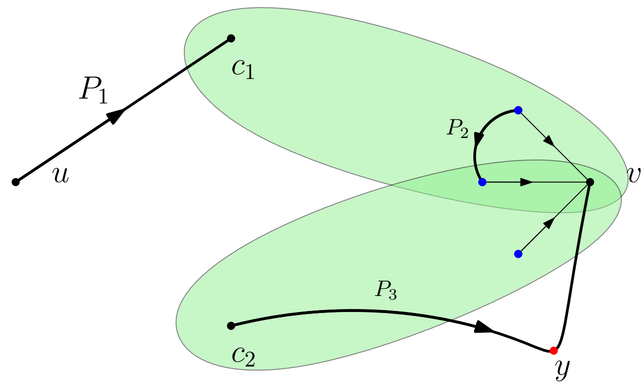


**Figure 2** $(c_1, c_2)$ is a $(u, v)$ vertex-cut, the green parts correspond to the $\mathcal{Z}(c_i, v)$ and the blue vertices are the vertices of $X$. $P_1$ is a path of Type $(u, \square)$, $P_2$ is a path of Type $(\boxtimes, \boxtimes)$ and $P_3$ is a path of Type $(\square, \boxminus, v)$ with $y \in Y$.

▶ **Lemma 3.6.** *Let $D$ be a digraph, $u, v \in V(D)$, and $k$ be a positive integer. Let $C$ be some $(u, v)$-vertex cut in $D$. For each $c \in C$, let $\mathcal{Z}(u, c)$ (resp. $\mathcal{Z}(c, v)$) be a $k$-cut preserving set for $(u, c)$ (resp. $(c, v)$) in $D[ver_D(u, c; C)]$ (resp. $D[ver_D(c, v; C)]$). Let $X = N_D^-(v) \cap \bigcup_{c \in C} \mathcal{Z}(c, v)$. For each $(\mathtt{a}, \mathtt{b}) \in (C \cup X)^2$, $\mathtt{a} \neq \mathtt{b}$, let $\mathcal{Z}(\mathtt{a}, \mathtt{b})$ be a $k$-cut preserving set for $(\mathtt{a}, \mathtt{b})$ in $D[ver_D(\mathtt{a}, \mathtt{b}; C \cup N_D^-(v))]$. Then, $\mathcal{Z} := \bigcup_{c \in C}(\mathcal{Z}(u, c) \cup \mathcal{Z}(c, v)) \cup \bigcup_{(\mathtt{a}, \mathtt{b}) \in (C \cup X)^2, \mathtt{a} \neq \mathtt{b}} \mathcal{Z}(\mathtt{a}, \mathtt{b})$ is a $k$-cut preserving set for $(u, v)$ in $D$.*

**Proof.** First observe that $\{u, v\} \cup C \cup X \subseteq \mathcal{Z}$. Let $Y = N_D^-(v) \setminus X$. We begin by defining some special types of paths (see Figure 2).

1. A path $P$ *is of Type* $(u, \square)$ *(resp.* $(\square, v)$*)* if it is a $C$-free $(u, c)$-path (respectively $(c, v)$-path) in $D$ for some $c \in C$.
2. A path $P$ *is of Type* $(\boxtimes, \boxtimes)$ if it is a $(C \cup N_D^-(v))$-free $(\mathtt{a}, \mathtt{b})$-path in $D$ for some $(\mathtt{a}, \mathtt{b}) \in (C \cup X)^2$.
3. A path $P$ *is of Type* $(\square, \boxminus, v)$ if it is a $(c, v)$-path in $D$ for some $c \in C$ and there exists $y \in V(P) \cap Y$ such that the $(c, y)$-subpath of $P$ is $C$-free.[3]

We now begin with the proof of the lemma. Let $P$ be some $(u, v)$-path. We need to show that there is a $\mathcal{Z}$-replacement witness for $P$. Let $P = P'_1 \circ \ldots \circ P'_q$ be the $(C \cup X)$-based partition of $P$. If $P$ is not $Y$-free, that is, $V(P) \cap Y \neq \emptyset$, let $s' \in [q]$ be the least integer such that $V(P'_{s'}] \cap Y \neq \emptyset$. If $P$ is $Y$-free, let $s' = q$. Let $s \leq s'$ be the largest integer such that $P_s$ is an $(\mathtt{a}, \mathtt{b})$-path, where $\mathtt{a} \in C$ and $\mathtt{b} \in C \cup X \cup \{v\}$. WE first show that such a $s$ always exists. From the definition of $s'$, either there exists some $y \in Y$ in $V(P'_{s'})$ or $v \in V(P'_{s'})$. In the later case, since $C$ is a $(u, v)$-vertex cut, there exists $c \in C$ such that $c$ appears on $P$. Since $P = P'_1 \circ \ldots \circ P'_q$ is a $C \cup X$-based partition of $P$, there exists $s \leq s'$ such that $P_s$ is a $(\mathtt{a}, \mathtt{b})$-path where $\mathtt{a} \in C$. In the former case again, since $y \in Y \subseteq N_D^-(v)$ and $C$ is a $(u, v)$-vertex cut using previous arguments the existence of the desired $s$ is guaranteed.

[3] Specifically, if there exists $y \in V(P) \cap Y$ with this property, then the first vertex of $P$ that belongs to $Y$ also has that property.

Consider the partition $P = P_1 \circ \ldots \circ P_s$, such that $P_i = P'_i$, if $i < s$ and $P_s = P'_s \circ P'_{s+1} \circ \ldots \circ P'_q$. Observe that, since $C \cup X \subseteq \mathcal{Z}$, $P = P_1 \circ \ldots \circ P_s$ is a semi-$\mathcal{Z}$-based partition of $P$.

▷ Claim 3.7. $P_1$ is a Type $(u, \square)$ path, for each $i \in [2, s-1]$, $P_i$ is a Type $(\boxtimes, \boxtimes)$ path and, $P_s$ is either a Type $(\square, v)$ or Type $(\square, \boxminus, v)$ path.

Proof. Recall that $P = P'_1 \circ \ldots \circ P'_q$ is the $(C \cup X)$-based partition of $P$. Thus, we have the following.

1. For each $i \in [q]$, $P'_i$ is $(C \cup X)$-free path.
2. For each $i \in [2, q-1]$, $P'_i$ is a $(\mathtt{a}, \mathtt{b})$-path, where $(\mathtt{a}, \mathtt{b}) \in (C \cup X)^2$.
3. Since $C$ is a $(u, v)$-vertex cut in $D$ and $X \subseteq N^-_D(v)$, $P'_1$ is a $(u, c)$-path for some $c \in C$.
4. From the choice of $s$, for each $i \in [s-1]$, $V(P'_i) \cap Y = \emptyset$. Since for $i \in [s-1]$, $P_i = P'_i$ and $X \cup Y = N^-_D(v)$, $P_i$ is $(C \cup N^-_D(v))$-free.

Thus, from Points 2 and 4, for each $i \in [s-1]$, $P_i$ is of Type $(\boxtimes, \boxtimes)$. Also, from Points 3 and 4, $P_1$ is of Type $(u, \square)$. We now show that $P_s$ is of Type $(\square, v)$ or $(\square, \boxminus, v)$. From the choice of $s$ and the construction of $P_s$, $P_s$ is a $(c, v)$-path for some $c \in C$. If $P$ is $Y$-free, then $P_s$ is of Type $(\square, v)$, otherwise, $P_s$ is of Type $(\square, \boxminus, v)$. ◁

For each $i \in [s]$, define $\mathcal{Z}_i$ and $D_i$ as follows.

$$\mathcal{Z}_i = \begin{cases} \mathcal{Z}(u, c) & \text{if } i = 1,\ P_1 \text{ is a } (u, c)\text{-path}, c \in C \\ \mathcal{Z}(\mathtt{a}, \mathtt{b}) & \text{if } i \in [2, s-1],\ P_i \text{ is a } (\mathtt{a}, \mathtt{b})\text{-path}, (\mathtt{a}, \mathtt{b}) \in (C \cup X)^2 \\ \mathcal{Z}(c, v) & \text{if } i = s,\ P_s \text{ is a } (c, v)\text{-path}, c \in C \end{cases}$$

$$D_i = \begin{cases} D[ver_D(u, c; C)] & \text{if } i = 1, P_1 \text{ is a } (u, c)\text{-path}, c \in C \\ D[ver_D(\mathtt{a}, \mathtt{b}; (C \cup N^-_D(v))] & \text{if } i \in [2, s-1],\ P_i \text{ is a } (\mathtt{a}, \mathtt{b})\text{-path}, (\mathtt{a}, \mathtt{b}) \in (C \cup X)^2 \\ D[ver_D(c, v)] & \text{if } i = s,\ P_s \text{ is a } (c, v)\text{-path}, c \in C \end{cases}$$

Recall the construction of $\mathcal{Z}$ from the lemma statement. Observe that for each $i \in [s]$, $\mathcal{Z}_i \subseteq \mathcal{Z}$. From Observation 3.4, to give a $\mathcal{Z}$-replacement witness for $P$, it is enough to give a $\mathcal{Z}_i$-replacement witness for each $P_i$, in $D_i$, $i \in [s]$. Thus, the following claim will finish the proof of the lemma.

▷ Claim 3.8. For each $i \in [s]$, $P_i$ has a $\mathcal{Z}_i$-replacement witness in $D_i$.

Proof. We prove the claim using the following cases.

- **Case $i = 1$:** From Claim 3.7, $P_1$ is a $C$-free $(u, c)$-path in $D$ for some $c \in C$. Thus, $P_1$ is a $(u, c)$-path in $D_1$. Since $Z_1$ is a $k$-cut preserving set for $(u, c)$ in $D_i$, there exists a $\mathcal{Z}_1$-replacement witness for $P_1$ in $D_1$.
- **Case $i \in [2, s-1]$:** From Claim 3.7, when $i \in [2, s-1]$, then $P_i$ is a $(C \cup N^-_D(v))$-free $(\mathtt{a}, \mathtt{b})$-path in $D$ for some $(\mathtt{a}, \mathtt{b}) \in (C \cup X)^2$. Thus, $P_i$ is an $(\mathtt{a}, \mathtt{b})$-path in $D_i$. Since $Z_i$ is a $k$-cut preserving set for $(\mathtt{a}, \mathtt{b})$ in $D_i$, there exists a $\mathcal{Z}_i$-replacement witness for $P_i$ in $D_i$.
- **Case $i = s$:** From Claim 3.7, $P_s$ is of either Type $(\square, v)$ or Type $(\square, \boxminus, v)$.
  - $P_s$ **is of Type** $(\square, v)$**:** From the definition of Type $(\square, v)$, $P_s$ is a $C$-free $(c, v)$-path in $D$, for some $c \in C$. Thus, $P_s$ is a $(c, v)$-path in $D_s$. Since $\mathcal{Z}_s$ is a $k$-cut preserving set for $(c, v)$ in $D_s$, there exists a $\mathcal{Z}_s$-replacement witness for $P_s$ in $D_s$.

- $P_s$ **is of Type** $(\square, \boxminus, v)$**:** From the definition of Type $(\square, \boxminus, v)$, $P_s$ is a $(c, v)$-path in $D$, for some $c \in C$, and there exists $y \in V(P) \cap Y$ such that the $(c, y)$-subpath of $P$ is $C$-free. Let $P_s^\dagger$ be the $(c, y)$-subpath of $P$. Recall that $Y = N_D^-(v) \setminus X$. Consider the $(c, v)$-path in $D$, denoted by $\widetilde{P_s}$, obtained by appending the arc $(y, v)$ at the end of $P_s^\dagger$. That is, $\widetilde{P_s} = P_s^\dagger \circ (y, v)$. Since $P_s^\dagger$ is a $C$-free path, so is $\widetilde{P_s}$. Thus $\widetilde{P_s}$ is a $(c, v)$-path in $D_s$. Since $\mathcal{Z}_s$ is a $k$-cut preserving set for $(c, v)$ in $D_s$, there exists a semi-$\mathbb{Z}_s$-based partition of $\widetilde{P_s}$ which is a $\mathcal{Z}_s$-replacement witness for $\widetilde{P_s}$ in $D_s$. Let $\widetilde{P_s} = \widetilde{P_{s,1}} \circ \ldots \circ \widetilde{P_{s,r}}$ be one such partition. Since $y \in Y = N_D^-(v) \setminus X$ and $\mathcal{Z}_s \subseteq X$, $y \notin \mathcal{Z}_s$. Thus, $y$ is an internal vertex of $\widetilde{P_{s,r}}$. Let $\widetilde{P_{s,r}}$ be an $(x, v)$-path. Clearly, $x \in \mathcal{Z}_s$ because $\widetilde{P_s} = \widetilde{P_{s,1}} \circ \ldots \circ \widetilde{P_{s,r}}$ is a semi-$\mathcal{Z}_s$-based partition. Let $P_{s,r}^\dagger$ be the $(x, v)$-subpath of $P_{s,r}$. We claim that $P_s = \widetilde{P_{s,1}} \circ \ldots \circ \widetilde{P_{s,r-1}} \circ P_{s,r}^\dagger$ is a semi-$\mathcal{Z}_s$-based partition of $P_s$ and is also a $\mathcal{Z}_s$-replacement witness for $P_s$ in $D_s$. It is clear from the discussion above that $P_s = \widetilde{P_{s,1}} \circ \ldots \circ \widetilde{P_{s,r-1}} \circ P_{s,r}^\dagger$ is a semi $\mathcal{Z}_s$-based partition of $P_s$. We will now show that it is a $\mathcal{Z}_s$-replacement witness for $P_s$ in $D_s$.
  Since $\widetilde{P_s} = \widetilde{P_{s,1}} \circ \ldots \circ \widetilde{P_{s,r}}$ is a $\mathcal{Z}_s$-replacement witness for $\widetilde{P_s}$, we have that for each $j \in [r]$, either $V(\widetilde{P_{s,j}}) \subseteq \mathcal{Z}_s$ or there exists a list $\mathcal{L}_j$ containing $k+1$ vertex disjoint paths from the start vertex of $\widetilde{P_{s,j}}$ to its end vertex. Also, since $y \notin \mathcal{Z}_s$ and $y$ is an internal vertex of $\widetilde{P_{s,r}}$, $V(\widetilde{P_{s,r}}) \not\subseteq \mathcal{Z}_s$. Thus, there is a list $\mathcal{L}_r$ containing $k+1$ vertex disjoint $(x, v)$-paths (recall $x$ and $v$ are the start and end vertices, respectively, of $\widetilde{P_{s,r}}$). Since $P_s = \widetilde{P_{s,1}} \circ \ldots \circ \widetilde{P_{s,r-1}} \circ P_{s,r}^\dagger$, and $P_{s,r}^\dagger$ is an $(x, v)$-path, from the above discussion for each $j \in [r-1]$, either $V(\widetilde{P_{s,j}}) \subseteq \mathcal{Z}_s$ or there exists a list $\mathcal{L}_j$ containing $k+1$ vertex disjoint paths from the start vertex of $\widetilde{P_{s,j}}$ to its end vertex. Also, there exists a list, $\mathcal{L}_r$, containing $k+1$ vertex disjoint paths from the start vertex of $P_{s,r}^\dagger$ to its end vertex. This completes the proof of the claim. ◁

As argued earlier, this completes the proof of the lemma. ◀

## 3.1 Finding a Small $k$-Cut Preserving Set for a Pair with Large Flow

As explained in Section 1.2, the proof of Lemma 3.3 will distinguish whether there is a $k$ vertex-cut for $(s, t)$ or not. The case where there is a no $k$ vertex-cut is the easiest one, and will be dealt with the following lemma by simply keeping $k+1$ vertex disjoint paths.

▶ **Lemma 3.9.** *Let $D \in \mathcal{D}_\alpha$ be an acyclic digraph and $u, v \in V(D)$ be such that each $(u, v)$-vertex cut in $D$ has size at least $k+1$. Then, a $k$-cut preserving set for $(u, v)$ in $D$ of size at most $(2\alpha - 1)(k+1) + 2$ exists and is computable in $n^{\mathcal{O}(1)}$ time, where $n = |V(D)|$.*

**Proof.** Since every $(u, v)$-vertex cut in $D$ has size at least $k+1$, from Menger's Theorem, there are at least $k+1$ vertex-disjoint $(u, v)$-paths in $D$. Let $Q_1', \ldots, Q_{k+1}'$ be a collection of some $k+1$ of these paths. We will now obtain a collection of $Q_1, \ldots Q_{k+1}$ vertex disjoint paths where the length of each $Q_i$ is at most $2\alpha + 1$. To this end, we define each $Q_i$ as some shortest $(u, v)$-path using the vertices of $V(Q_i')$. We first claim that the length of $Q_i$ is at most $2\alpha + 1$. For the sake of contradiction, suppose not. Then, from Observation 2.1, there exist $x, y \in V(Q_i)$ such that $(x, y) \in E(D)$. Since $D$ is acyclic, $x$ appears before $y$ in the path $Q_i$. This contradicts that $Q_i$ is a shortest $(u, v)$-path in $V(Q_i')$. Let $\mathcal{Z} = \bigcup_{i \in [k+1]} V(Q_i)$. Clearly, $\{u, v\} \subseteq \mathcal{Z}$ and $|\mathcal{Z}| \leq (2\alpha - 1)(k+1) + 2$. The size bound follows because the length of each $Q_i$ is at most $2\alpha + 1$, and $u, v$ are the vertices common in each $Q_i$. To show that $\mathcal{Z}$ is a $k$-cut preserving set for $(u, v)$ in $D$, consider the semi-$\mathcal{Z}$-based partition of $P$ that is $P$ itself. Then, $\{Q_1, \ldots, Q_{k+1}\}$ is the list for $P$ containing $k+1$ internally vertex-disjoint $(V(D) \setminus \mathcal{Z})$-free $(u, v)$-paths. ◀

## 3.2 Finding a Small $k$-Cut Preserving Set of a Pair in a Tournament

As explained before, the proof of Lemma 3.3 will use induction on $\alpha$. The next lemma handles the base case where $\alpha = 1$. It is somewhat more complicated compared to the arguments in Section 1.2; the reason for the complication is that we consider the digraph $D$ such that the $D - \{u, v\} \in \mathcal{D}_\alpha$. Thus $D$ is not "exactly" a tournament. This is required in the inductive case for the proof of Lemma 3.3.

▶ **Lemma 3.10.** *Let $D$ be an acyclic digraph. Let $u, v \in V(D)$ be such that $N^+(u) = N^-(v) = \emptyset$ and $D - \{u, v\}$ is a tournament. Then, a $k$-cut preserving set for $(u, v)$ in $D$ of size at most $k^3 + 5k^2 + 3k$ exists and is computable in polynomial time.*

**Proof.** If all $(u, v)$-vertex cuts in $D$ have size at least $k + 1$, then the correctness follows from Lemma 3.9. Thus, for the rest of the proof assume that there is a $(u, v)$-vertex-cut in $D$ of size at most $k$. Let $C = \{c_1, \ldots, c_\ell\}$ be a minimal $(u, v)$-vertex cut in $D$ of size $\ell \leq k$.

▷ Claim 3.11. $C \subseteq N_D^+(u) \cup N_D^-(v)$.

Proof. Suppose not. Then, there exists $c_i \in C$ such that $c_i \notin N^+(u) \cup N^-(v)$. Since $C$ is a minimal $(u, v)$-vertex cut in $D$, there exists a path, say $P$, from $u$ to $v$ in $D - (C \setminus \{c_i\})$. Let $u'$ be the first vertex on $P$ after $u$ and $v'$ be the last vertex of $P$ before $v$. Since $D - \{u, v\}$ is an acyclic tournament, $(u', v') \in E(D)$. Since $u', v' \notin C$, we get a $(u, v)$-path in $D - C$, contradicting that $C$ is a $(u, v)$-vertex cut in $D$. ◁

Let $I = \{i \in [\ell] \mid c_i \in N_D^-(v)\}$ and $J = \{j \in [\ell] \mid c_j \in N_D^+(u)\}$. For all $i \in I$, let $U_i = ver_D(u, c_i; C)$ and $D_i = D[U_i]$. For all $j \in J$, let $V_j = ver_D(c_j, v; C)$ and $D_j = D[V_j]$. For all $(i, j) \in [\ell]^2$, $i \neq j$, let $Q_{i,j} = ver_D(c_i, c_j; \emptyset)$ and $D_{i,j} = D[Q_{i,j}]$.

For each $i \in I$ (resp. $j \in J$, resp. $(i, j) \in [\ell]^2$, $i \neq j$), we will compute a $k$-cut preserving set $\mathcal{Z}_i$ (resp. $\mathcal{Z}_j$, resp. $\mathcal{Z}_{i,j}$) of $(u, c_i)$ (resp. $(c_j, v)$, resp. $(c_i, c_j)$) in $D_i$ (resp. $D_j$, resp. $D_{i,j}$) of size at most $2k + 3$ (resp. $2k + 3$, resp. $k + 3$). The procedure to do so is as follows.

- **Computing $\mathcal{Z}_i$, $i \in I$:** First observe that $U_i$ is a candidate for $\mathcal{Z}_i$. Thus, if $|U_i| \leq 2(k+1)$, set $\mathcal{Z}_i = U_i$. Otherwise, we have that $|U_i| \geq 2k + 3$. Since $D - \{u, v\}$ is an acyclic tournament, let $\pi$ be the unique topological ordering of $D - \{u, v\}$. We divide this case further into two cases.
  **Case 1:** $|N^+(u) \cap U_i| \leq k$**:** Let $\widetilde{U_i}$ be the last $k + 1$ vertices of $U_i$ in $\pi$. Observe that $\widetilde{U_i} \subseteq N^-(c_i) \cap U_i$. Define $\mathcal{Z}_i = (N^+(u) \cap U_i) \cup \widetilde{U_i} \cup \{u, c_i\}$. Clearly, $|\mathcal{Z}_i| \leq 2k + 3$. To prove that $\mathcal{Z}_i$ is a $k$-cut preserving set for $(u, c_i)$ in $D_i$, consider some $(u, c_i)$-path $P$ in $D_i$, such that $V(P) \not\subseteq \mathcal{Z}_i$. We will show a $\mathcal{Z}_i$-replacement witness for $P$ in $D_i$. Consider the semi-$\mathcal{Z}_i$-based partition of $P$, $P = P_1 \uplus P_2$, where $P_1$ is the arc $(u, x) \in E(P)$, for some $x \in N^+(u) \cap U_i$ and $P_2$ is the $(x, c_i)$-subpath of $P$. Clearly, $V(P_1) \subseteq \mathcal{Z}_i$. We claim that there are $k + 1$ vertex-disjoint $(x, c_i)$-paths in $\mathcal{Z}_i$. To see this, consider the following argument. Since $V(P) \not\subseteq \mathcal{Z}_i$, there exists a vertex $y \in V(P)$ such that $y \notin \mathcal{Z}_i$. Then, $y \in V(P_2)$. Since $y \notin \mathcal{Z}_i$, it in particular holds that $y \notin \widetilde{U_i}$. Thus, all the vertices of $\widetilde{U_i}$ appear after $y$ in $\pi$. Since there is a $(x, y)$-path in $D_i$, $x$ appears before $y$ in $\pi$. Thus, $x$ appears before all the vertices of $\widetilde{U_i}$ in $\pi$. Thus, because $D - \{u, v\}$ is a tournament, $\widetilde{U_i} \subseteq N^+(x) \cap U_i$. Since $\widetilde{U_i} \subseteq N^-(c_i) \cap U_i$, there are $|\widetilde{U_i}|$ many vertex disjoint $(x, c_i)$-paths in $\mathcal{Z}$. This completes the proof.
  **Case 2:** $|N^+(u) \cap U_i| > k$**:** First observe that all the vertices of $N^+(u) \cap U_i$ appear before $c_i$ in $\pi$. Since $\pi$ is a topological ordering of $D - \{u, v\}$, there are $|N^+(u) \cap U_i| > k$ vertex-disjoint $(u, c_i)$-paths in $\mathcal{Z}_i$. Thus, each $(u, c_i)$-vertex-cut in $D_i$ has size at least

$k+1$. In this case, let $\mathcal{Z}_i$ be the $k$-cut preserving set for $(u, c_i)$ in $D_i$ obtained from Lemma 3.9. Observe that $|\mathcal{Z}_i| \leq k+3$.

- **Computing $\mathcal{Z}_j$, $j \in J$:** $\mathcal{Z}_j$ can be computed using arguments symmetric to the previous case.
- **Computing $\mathcal{Z}_{i,j}$, $(i,j) \in [\ell]^2$, $i \neq j$:** First observe that all the vertices of $Q_{i,j} \setminus \{c_i, c_j\}$ appear after $c_i$ and before $c_j$ in $\pi$. Thus, there are $|Q_{i,j} \setminus \{c_i, c_j\}|$ many vertex-disjoint $(c_i, c_j)$-paths in $D_{i,j}$. If $|Q_{i,j}| \leq k-2$, then set $\mathcal{Z}_i = Q_{i,j}$, otherwise let $\mathcal{Z}_i$ be the $k$-cut preserving set for $(c_i, c_j)$ in $D_{i,j}$ obtained from Lemma 3.9. In either case, $|\mathcal{Z}_i| \leq k+3$.

Let $\mathcal{Z} := \bigcup_{i \in I} \mathcal{Z}_i \cup \bigcup_{j \in J} \mathcal{Z}_j \cup \bigcup_{(i,j) \in [\ell]^2, i \neq j} \mathcal{Z}_{i,j}$. Observe that $C \subseteq \mathcal{Z}$. First note that $|\mathcal{Z}| \leq |I|(2k+3) + |J|(2k+3) + \ell^2(k+3) \leq k^3 + 5k^2 + 3k^2$ (the last inequality holds because $|I| + |J| = \ell$ and $\ell \leq k$). We will now show that $\mathcal{Z}$ is a $k$-cut preserving set for $(u,v)$ in $D$. To see this, consider some$(u,v)$-path $P$, in $D$. Since $C$ is a $(u,v)$-vertex-cut in $D$ there exists a vertex of $C$ on $P$. Let $c_i$ be the first vertex of $C$ on $P$ and $c_j$ be the last vertex of $C$ on $P$ ($c_i$ could be the same as $c_j$). Let $P_1$ be the $(u, c_i)$-subpath of $P$, $P_2$ be the $(c_i, c_j)$-subpath of $P$ and $P_3$ be the $(c_j, v)$-subpath of $P$ (if $c_i$ is the same as $c_j$, then $P_2$ is empty). Thus, $P = P_1 \circ P_2 \circ P_3$ is a semi-$\mathcal{Z}$-based partition of $P$ (as $C \subseteq \mathcal{Z}$). Since $\mathcal{Z}_i$ is a $k$-cut preserving set for $(u, c_i)$ in $D_i$, $\mathcal{Z}_i$ is a $k$-cut preserving set for $(c_j, v)$ in $D_j$ and $\mathcal{Z}_{i,j}$ is a $k$-cut preserving set for $(c_i, c_j)$ in $D_{i,j}$, and $\mathcal{Z}_i, \mathcal{Z}_j, \mathcal{Z}_{i,j} \subseteq \mathcal{Z}$, from Observation 3.4, $\mathcal{Z}$ is a $k$-cut preserving set for $(u,v)$ in $D$. ◀

## 3.3 Finding a small $k$-cut preserving set for a pair in a $D \in \mathcal{D}_\alpha$

We are now ready to prove Lemma 3.3.

▶ **Lemma 3.3** ($k$-Cut Preserving Lemma). *Let $D$ be an acyclic digraph, and $u, v \in V(D)$ be such that $N^-(u) = N^+(v) = \emptyset$. Additionally, let $D - \{u,v\} \in \mathcal{D}_\alpha$. Then there exists a $k$-cut preserving set for $(u,v)$ in $D$ of size at most $f(\alpha)$, where $f(1) = k^3 + 5k^2 + 3k$ and for $\alpha > 1$, $f(\alpha) = k^2 g(\alpha) + 2kh(\alpha)$, $g(\alpha) = (2k + (k + kf(\alpha-1))^2)f(\alpha-1)$ and $h(\alpha) = (k^2+k)g(\alpha) + kf(\alpha-1)$. Moreover, such a set can be found in time $n^{\mathcal{O}(1)}$, where $n = |V(D)|$.*

**Proof.** We prove this lemma using induction on $\alpha$. When $\alpha = 1$, the proof follows from Lemma 3.10.

▷ Claim 3.12. Let $x, y \in V(D) \setminus \{x, y\}$. Then, a $k$-cut preserving set for $(x,y)$ of size $g(\alpha)$ in any digraph $D'$ that is a subgraph of $D$ where $u, v \notin V(D')$, can be found in polynomial time.

*Proof.* Let $W$ be a minimum $(x,y)$-vertex-cut in $D'$. If $|W| > k$, then the claim follows from Lemma 3.9. Thus, we are now in the case where $|W| \leq k$. For each $w \in W$, let $\mathcal{Z}(x,w)$ (resp. $\mathcal{Z}(w,y)$) be a $k$-cut preserving set for $(x,w)$ (resp. $(w,y)$) in $D'[ver_{D'}(x,w;W)]$ (resp. $D'[ver_{D'}(w,y;W)]$). Let $B = N^-_{D'}(y) \cap \bigcup_{w \in W} \mathcal{Z}(w,y)$. For each $(\mathtt{a},\mathtt{b}) \in (W \cup B)^2$, let $\mathcal{Z}(\mathtt{a},\mathtt{b})$ be a $k$-cut preserving set for $(\mathtt{a},\mathtt{b})$ in $D'[ver'_D(\mathtt{a},\mathtt{b}; W \cup N^-(y))]$. Then, from Lemma 3.6, $\mathcal{Z}(x,y) := \bigcup_{w \in W}(\mathcal{Z}(x,w) \cup \mathcal{Z}(w,y)) \cup \bigcup_{(\mathtt{a},\mathtt{b}) \in (W \cup B)^2} \mathcal{Z}(\mathtt{a},\mathtt{b})$ is a $k$-cut preserving set for $(x,y)$ in $D'$.

We will now show that for any $w \in W$ and $(\mathtt{a},\mathtt{b}) \in (W \cup B)^2$, each digraph among $D'[ver_{D'}(x,w;W)]$, $D'[ver_{D'}(w,y;W)]$ and $D'[ver_{D'}(\mathtt{a},\mathtt{b}; W \cup N^-_{D'}(y))]$ has independence number strictly smaller than $\alpha$. Then, from induction hypothesis and the expression for $\mathcal{Z}(x,y)$ written above, we will conclude that a $k$-cut preserving set for $(x,y)$ in $D'$ of size $g(\alpha)$ can be found in polynomial time. To see that the independence number of $D'[ver_{D'}(x,w;W)]$

is strictly less than $\alpha$, observe that $y$ is not adjacent to any vertex in $ver_{D'}(x, w; W)$, as $W$ is an $(x, y)$-vertex cut in $D'$. Thus, any independent set of $D'[ver_{D'}(x, w; W)]$ together with $y$ is an independent set of $D'$ and hence of $D$. Since $y \notin \{u, v\}$, $u, v \notin V(D')$ and the independence number of $D - \{u, v\}$ is $\alpha$, we have that the independence number of $D'[ver_{D'}(x, w; W)]$ is strictly smaller than $\alpha$. A similar argument holds for $D'[ver_{D'}(w, y; W)]$ as in this case $x$ is not adjacent to any vertex of $ver_{D'}(w, y; W)$. For $D'[ver_{D'}(\mathtt{a}, \mathtt{b}; W \cup N^-_{D'}(y))]$, since $ver_{D'}(\mathtt{a}, \mathtt{b}; W \cup N^-_{D'}(y)) \cap N^-_{D'}(y) = \emptyset$, $u, v \notin V(D')$ and $N^+_{D'}(y) = \emptyset$, any independent set of $D'[ver_{D'}(\mathtt{a}, \mathtt{b}; W \cup N^-_{D'}(y))]$ together with $y$ is an independent set in $D - \{x, y\}$. Since $D - \{x, y\}$ has independence number $\alpha$, $D'[ver_{D'}(\mathtt{a}, \mathtt{b}; W \cup N^-_{D'}(y))]$ has independence number strictly smaller than $\alpha$. ◁

Let $C$ be a minimum $(u, v)$-vertex-cut in $D$. If $|C| > k$, then the lemma follows from Lemma 3.9. Thus, for the remainder of the proof we assume that $|C| \leq k$. For each $c \in C$, let $U_c = ver_D(u, c; C)$, $V_c = ver_D(c, v; C)$, $\mathcal{Z}(u, c)$ be a $(u, c)$ $k$-cut preserving set in $D[U_c]$, and $\mathcal{Z}(c, v)$ be a $(c, v)$ $k$-cut preserving set in $D[V_c]$. For each $(c, c') \in C^2$, $c \neq c'$, let $Q_{c,c'} = ver_D(c, c'; C)$, and $\mathcal{Z}(c, c')$ be a $k$-cut preserving set in $D[Q_{c,c'}]$. Then from Lemma 3.5, $\mathcal{Z} := \bigcup_{c \in C} \mathcal{Z}(u, c) \cup \mathcal{Z}(c, v) \cup \bigcup_{(c,c') \in C^2, c \neq c'} \mathcal{Z}(c, c')$ is a $k$-cut preserving set for $(u, v)$ in $D$. Since $C \cap \{u, v\} = \emptyset$, from Claim 3.12, for each $(c, c') \in C^2$, $c \neq c'$, $\mathcal{Z}(c, c')$ of size $g(\alpha)$ can be computed in polynomial time. In the remainder of the proof, we will show how to compute $\mathcal{Z}(u, c)$ and $\mathcal{Z}(c, v)$, for any $c \in C$, of the desired size. We will only give the proof of construction of $\mathcal{Z}(u, c)$ as the proof for $\mathcal{Z}(c, v)$ is symmetrical.

▷ Claim 3.13. For any $c \in C$, $\mathcal{Z}(u, c)$ of size $h(\alpha)$ can be computed in polynomial time.

Proof. For ease of notation, let $\widehat{D} = D[U_c]$. Let $A$ be a minimum $(u, c)$-vertex-cut in $\widehat{D}$. First note that $A \cap \{u, v\} = \emptyset$. If $|A| > k$, then the claim follows from Lemma 3.9. Thus, for the remainder of the proof, assume that $|A| \leq k$.

For each $a \in A$, let $\widehat{U_a} = ver_{\widehat{D}}(u, a; A)$, $\widehat{V_a} = ver_{\widehat{D}}(a, c; A)$, $\widehat{\mathcal{Z}}(u, a)$ be a $(u, a)$ $k$-cut preserving set in $\widehat{D}[\widehat{U_a}]$ and $\widehat{\mathcal{Z}}(a, c)$ be a $(a, c)$ $k$-cut preserving set in $\widehat{D}[\widehat{V_a}]$. For each $(a, a') \in A^2$, $a \neq a'$, let $R_{a,a'} = ver_{\widehat{D}}(a, a'; A)$ and $\widehat{\mathcal{Z}}(a, a')$ be a $k$-cut preserving set in $\widehat{D}[R_{a,a'}]$. Then from Lemma 3.5, $\mathcal{Z}(u, c) := \bigcup_{a \in A} (\widehat{\mathcal{Z}}(u, a) \cup \widehat{\mathcal{Z}}(a, c)) \cup \bigcup_{(a,a') \in A^2, a \neq a'} \widehat{\mathcal{Z}}(a, a')$ is a $k$-cut preserving set for $(u, c)$ in $D$. Since $A \cap \{u, v\} = \emptyset$ and $c \in \{u, v\}$, from Claim 3.12, for each $a \in A$, $(a, a') \in A^2$, $a \neq a'$, $\widehat{\mathcal{Z}}(a, c)$ and $\widehat{\mathcal{Z}}(a, a')$ of size $g(\alpha)$ can be computed in polynomial time. Moreover, the independence number of $\widehat{D}[\widehat{U_a}] - \{u, a\}$ is strictly smaller than $\alpha$ because $c (\neq v)$ is not adjacent to any vertex in $\widehat{U_a}$, besides possibly $u$ and $a$. Thus, for each $a \in A$, a set $\widehat{\mathcal{Z}}(u, a)$ of size $f(\alpha - 1)$ can be computed in polynomial time by the induction hypothesis. This finishes the proof of the claim. ◁

Thus, from the previous arguments and Claim 3.13, we have that $\mathcal{Z}$ is a $k$-cut preserving set for $(u, v)$ in $D$ of size at most $k^2 g(\alpha) + 2kh(\alpha)$. ◀

A rough computation gives that, for any $k$, $g(\alpha) \leq 6k^2 f(\alpha - 1)$ and $h(\alpha) \leq 8k^4 f(\alpha - 1)$. This imply that $f(\alpha) \leq 22k^5 f(\alpha - 1)^3$. By noting that $f(1) \leq 22k^5$, we can show the following observation.

► **Observation 3.14.** *For any $\alpha$ and $k$, there exists a $k$-cut preserving set of size smaller than $f(k, \alpha) = (22k^5)^{4^\alpha}$.*

### 3.4 $k$-Cut Preserving Sets for a Set of Vertices

Below we also define a notion of $k$-cut preserving sets for a set of vertices. Such a notion will come handy in our applications. Given a digraph $D$ and $X \subseteq V(D)$, for each $(u,v) \in X^2$, we define the digraph $D^X_{(u,v)}$ as follows (note that $u$ could be equal to $v$). Let $R = V(D) - X$. Then, $D^X_{(u,v)}$ is the supergraph of $D[R]$ obtained by adding two new vertices $u^+$ and $v^-$ together with the following set of additional arcs: $\{(u^+, x) : x \in R, (u,x) \in E(D)\} \cup \{(x, v^-) : x \in R, (x,v) \in E(D)\}$ .

▶ **Definition 3.15** ($k$-Cut Preserving Set for a Set of Vertices)**.** *For any digraph $D$, a positive integer $k$ and $X \subseteq V(D)$, we say that $X \subseteq \mathcal{Z} \subseteq V(D)$ is a $k$-*cut preserving set for $X$*, if for all $(u,v) \in X^2$, $\mathcal{Z}$ is a $k$-cut preserving set for $(u,v)$ in $D^X_{(u,v)}$.*

▶ **Lemma 3.16.** *For any digraph $D \in \mathcal{D}_\alpha$, a positive integer $k$, and $S \subseteq V(D)$ such that $D - S$ is a acyclic, a $k$-cut preserving set for $S$ of size at most $|S|^2 f(k,\alpha)$ can be found in polynomial time, where $f(k,\alpha) \le (22k^5)^{4^\alpha}$.*

**Proof.** For each pair $(u,v) \in S^2$ ($u$ and $v$ could be equal), let $\mathcal{Z}_{(u,v)}$ be the a $k$-cut preserving set for $(u^+, v^-)$ in $D^S_{(u,v)}$ obtained from Lemma 3.3. From the definition of $k$-cut preserving set for $S$, $\mathcal{Z} = \bigcup_{(u,v)\in S^2} \mathcal{Z}_{(u,v)}$ is a $k$-cut preserving set for $S$. From Observation 3.14, for any $(u,v) \in S^2$, $|\mathcal{Z}_{(u,v)}| \le f(k,\alpha)$. Thus, we conclude the correctness of the lemma. ◀

## 4 Fault-Tolerant $(S,S)$-Reachability

In this section, we prove Lemma 1.1. Recall that $(D,S,\ell,k)$ is an instance of $\mathrm{FTR}(S,S)$ where $D \in \mathcal{D}_\alpha$, $S \subseteq V(D)$ and $\ell, k$ are positive integers such that each strongly connected component of $D - S$ has size at most $\ell$. The goal is to compute a subgraph $H$ of $D$ of size $k^{2^{\mathcal{O}(\alpha)}}$ such that, for any $A \subseteq E(D)$ of size at most $k$, for any $s,t \in S$, if $D - A$ has an $(s,t)$-path, then so does $H - A$. It is not difficult to see from Lemma 3.2 that if $\mathcal{Z}$ is a $k$-cut preserving set for $S$ in $D$, then $H = D[\mathcal{Z}]$ is a solution for $(D,S,\ell,k)$ (for any $\ell$). When $\ell = 1$, $D - S$ is acyclic and hence a $k$-cut preserving set for $S$ can be computed using Lemma 3.16. When $\ell > 1$, in order to use Lemma 3.16 we modify the digraph $D$ to turn $D - S$ acyclic. We now describe the operation, which we call dagify, that is used to turn $D - S$ acyclic. Informally, for each strongly connected component $SC$ of $D$ we turn it into an independent set while preserving the paths in $D$ that use the vertices of $SC$. This is achieved by creating a new vertex for every ordered pair of vertices (say, $(u,v)$) in $SC$. Such a vertex represents the existence of a $(u,v)$-path in the strongly connected component $SC$. In fact, in the path in the modified graph, each new vertex corresponding to some pair $(u,v)$ can be replaced by some $(u,v)$-path from the strongly connected component $SC$ to yield a path in the original graph. Then, arcs between two vertices in this newly constructed vertex set are put in such a way that the concatenation of the paths corresponding to these new vertices gives a path in $D$. This idea is formalized below.

▶ **Definition 4.1** (dagify$(D,R)$)**.** *Let $D$ be a digraph, $R \subseteq V(D)$ and $S = V(D) \setminus R$. Let $SC_1, \ldots, SC_d$ be the strongly connected components of $D[R]$. For $a \in [d]$, let $V(SC_a) = \{v^a_1, \ldots, v^a_{n_a}\}$, where $n_a = |V(SC_a)|$. Then, $D^\dagger_R :=$ dagify$(D,R)$ is the digraph defined as:*

***Vertex set of $D^\dagger_R$:*** *For each $a \in [d]$, let $SC^\dagger_a = \{\mathbf{v}^a_{ij} \mid (v^a_i, v^a_j) \in \{SC_a\}^2, i,j \in [n_a]\}$. Let $R^\dagger = \cup_{a\in[d]} SC^\dagger_a$ and $V(D^\dagger_R) = R^\dagger \cup S$.*

***Arc set of $D^\dagger_R$:*** *It contains all the arcs of $D$ with both end-points in $S$. For each $a \in [d]$, ${SC^\dagger}_a$ is an independent set in $D^\dagger_R$. For any $a \in [d]$, $s \in S$ and $i,j \in [n_a]$, $(s, \mathbf{v}^a_{ij}) \in E(D^\dagger_R)$ if and only if $(s, v^a_i) \in E(D)$. Similarly, $(\mathbf{v}^a_{ij}, s) \in E(D^\dagger_R)$ if and only if $(v^a_j, s) \in E(D)$. We put the arcs between $SC^\dagger_a$ and $SC^\dagger_b$, for distinct $a,b \in [d]$ as follows. For any $i,j \in [n_a]$ and $i',j' \in [n_b]$, $(\mathbf{v}^a_{ij}, \mathbf{v}^b_{i'j'}) \in E(D^\dagger_R)$ if and only if $(v^a_j, v^b_{i'}) \in E(D)$.*

For a set of vertices of $X^\dagger \subseteq D_R^\dagger$, $\mathit{full\text{-}comp}(X^\dagger)$ denotes the set of vertices of $V(D)$ such that, for each $\mathbf{v}_{i,j}^a \in X^\dagger$, all the vertices of $SC_a$ belong to full-comp$(X^\dagger)$. Also all the vertices of $S$ that belong to $X^\dagger$, belong to $\mathit{full\text{-}comp}(X^\dagger)$. Observe that $|\mathit{full\text{-}comp}(X^\dagger)| \leq \ell^2 \cdot |X^\dagger|$, where $\ell$ is the upper bound on the size of each $SC_a$. Note from the construction above that, for any $s, t \in S$ and an $(s,t)$-path $P^\dagger$ in $D_R^\dagger$, there exists an $(s,t)$-path $P$ in $D$ such that $V(P) \subseteq \mathit{full\text{-}comp}(P^\dagger)$. The following observations state a few properties of the digraph $D_R^\dagger$ that would be useful when we want to find a $k$-cut preserving set for $D_R^\dagger$ using Lemma 3.16.

▶ **Observation 4.2.** *$D_R^\dagger[R^\dagger]$ is acyclic.*

**Proof.** Recall, from the construction of $D_R^\dagger$, that $R^\dagger = \bigcup_{a \in [d]} SC_a^\dagger$ and each $SC_a^\dagger$ is an independent set in $D_R^\dagger$. Without loss of generality, let $SC_1, \ldots, SC_d$ be the strongly connected components of $D[R]$ ordered as in their topological ordering. Then, there is no arc from a vertex of $SC_b$ to a vertex of $SC_a$, for any $b > a$, in $D$. Thus, from the construction of $D_R^\dagger$, there is no arc from any $\mathbf{v}_{ij}^b$ to any $\mathbf{v}_{i'j'}^a$ $(b > a)$. This shows that $D_R^\dagger[R^\dagger]$ is acyclic. ◀

▶ **Observation 4.3.** *If $D \in \mathcal{D}_\alpha$ and every strongly connected component of $D[R]$ has size at most $\ell$, then $D_R^\dagger \in \mathcal{D}_{\ell^2\alpha}$.*

**Proof.** Recall that $R^\dagger = \bigcup_{a \in [d]} SC^\dagger{}_a$ and $D_R^\dagger[SC_a^\dagger]$ has no arc. From the construction of $D_R^\dagger$, for each $a \in [d]$, $|SC_a^\dagger| \leq \ell^2$. Finally, since $D \in \mathcal{D}_\alpha$, from the construction of $D_R^\dagger$, the size of any maximum independent set in $D_R^\dagger$ is at most $\max_{a \in [d]} |SC_a^\dagger| \cdot \alpha \leq \ell^2\alpha$. ◀

We define some terminology that would come handy later. For any $A \subseteq E(D)$, we say that a vertex $v \in V(D)$ *is affected by $A$* if there exists some arc of $A$ that is incident on $v$. The *set affected by $A$ in $D_R^\dagger$* is the set of vertices of $D_R^\dagger$ containing the union of the vertices in $SC_a^\dagger$, for each $a \in [d]$ such that a vertex in $SC_a$ is affected by $A$ in $D$.

▶ **Observation 4.4.** *Let $D$ be a digraph, $R \subseteq V(D)$ and $S = V(D) \setminus R$. Let $A \subseteq E(D)$ of size at most $k$. Let $\mathcal{A}^\dagger$ be the set affected by $A$ in $D_R^\dagger$. Recall the construction of $D_R^\dagger$ from Definition 4.1. For some $\mathbf{v}_{ij}^a, \mathbf{v}_{i'j'}^b \in R^\dagger$, let $P^\dagger$ be an $\mathcal{A}^\dagger$-free $(\mathbf{v}_{ij}^a, \mathbf{v}_{i'j'}^b)$-path in $D_R^\dagger$. Then there exists a $(v_i^a, v_{j'}^b)$-path $P$ in $D$ such that: $V(P) \subseteq \mathit{full\text{-}comp}(P^\dagger)$ and, $P$ does not use any arc of $A$.*

**Proof.** Recall the construction of $\mathsf{dagify}(D,R)$. Consider any path $P$ obtained from $P^\dagger$ by replacing all the vertices of $R^\dagger$ as follows. If for any $c \in [d]$, $i^*, j^* \in [n_c]$, $\mathbf{v}_{i^*j^*}^c \in V(P^\dagger)$, then replace $\mathbf{v}_{i^*j^*}^c$ in $P^\dagger$ by any $(v_{i^*}^c, v_{j^*}^c)$-path in the strongly connected component $SC_c$. Clearly, the path $P$ obtained is a $(v_i^a, v_{j'}^b)$-path in $D$ and $V(P) \subseteq \mathit{full\text{-}comp}(P^\dagger)$. Also from the definition of $\mathcal{A}^\dagger$ and the fact that $P^\dagger$ is $\mathcal{A}^\dagger$-free, we get that $P$ cannot use an arc of $A$. ◀

From the construction in Definition 4.1, for any $s, t \in S$, for an $(s,t)$-path $P$ in $D$, we can associate a unique $(s,t)$-path $P^\dagger$ in $D_R^\dagger$. This is elaborated below. Consider the digraph $D_R^\dagger$ obtained by $\mathsf{dagify}(D,R)$. $(v_i^a, v_j^a) \in SC_a^2$ for some component $SC_a$ of $D[R]$. Let $s, t \in S$. Let $P$ be an $S$-free $(s,t)$-path in $D$. For any such path $P$, we define the notion of a *reduced path* of $P$ in $D_R^\dagger$ as follows. Consider the unique partition $P = P_s \circ P_{i_1} \circ \ldots \circ P_{i_q} \circ P_t$ such that $P_s$ is an arc $(s,u)$ where $u \in V(SC_{i_1})$, $P_t$ is an arc $(v,t)$ where $v \in V(SC_{i_q})$ and for each $j \in [q]$, $V(P_{i_j}) \subseteq V(SC_{i_j})$, where $i_1, \ldots, i_j \in [d]$ and $i_1 < \ldots < i_q$. For each $j \in [q]$, let $P_{i_j}$ be a $(v_{p_j}^{i_j}, v_{r_j}^{i_j})$-path. Consider the vertex $\mathbf{v}_{p_j,r_j}^{i_j}$ in $V_{i_j} \subseteq R^\dagger \subseteq V(D_R^\dagger)$. From the construction of $D_R^\dagger$, we get the $(s,t)$-path $P^\dagger = s \circ \mathbf{v}_{p_1,r_1}^{i_1} \circ \mathbf{v}_{p_2,r_2}^{i_2} \circ \ldots \circ \mathbf{v}_{p_q,r_q}^{i_q} \circ t$ in $D_R^\dagger$. This $(s,t)$-path $P^\dagger$ in $D_R^\dagger$ is called the *reduced path of $P$ in $D_R^\dagger$*.

**Proof of Lemma 1.1.** Recall $(D, S, \ell, k)$ is an instance of FTR$(S, S)$. Let $R = V(D) \setminus S$. Let $D_R^\dagger$ be obtained by dagify$(D, R)$. From Observations 4.2 and 4.3, Lemma 3.16 can be used to compute a $(2k\ell^2 + 1)$-cut preserving set for $S$ in $D_R^\dagger$. Let $\mathcal{Z}^\dagger$ be such a set. Let $\mathcal{Z} = \mathit{full\text{-}comp}(\mathcal{Z}^\dagger)$. We claim that $H = D[\mathcal{Z}]$ is a solution to the instance $(D, S, \ell, k)$. (First note that the size bound on $H$ follows from Lemma 3.16 and the fact that each strongly connected component of $R$ has size at most $\ell$.)

Towards this let $A \subseteq E(D)$ of size at most $k$, $s, t \in S$ and $P$ be an $(s,t)$-path in $D - A$. We need to show that there is some $(s,t)$-path in $H - A$ too. Let $P = P_1 \circ \ldots \circ P_q$ be the $S$-based partition of $P$ such that each $P_i$ is an $(s_i, t_i)$-path. Then it suffices to show that for each fixed $i \in [q]$, there is some $(s_i, t_i)$ path in $H - A$ (these paths would yield a closed walk from $s$ to $t$ in $H - A$ and hence an $(s,t)$-path in $H - A$). In the remaining part of the proof, we focus on proving this. Note that each $P_i$ is $S$-free. Fix any $i \in [q]$. For the ease of notation, let us call the path $P_i$ as $P$, vertices $s_i, t_i$ as $s, t$ respectively.

Let $P^\dagger$ be the reduced path corresponding of $P$ in $D_R^\dagger$. Since $\mathcal{Z}^\dagger$ is a $(2k\ell^2 + 1)$-cut preserving set for $P^\dagger$ in $D_R^\dagger$, consider a $\mathcal{Z}^\dagger$-witnessing replacement $P^\dagger = P_1^\dagger \circ \ldots \circ P_r^\dagger$. Recall the notation from the construction in Definition 4.1.

For an arbitrary $c \in [r]$, let $P_c^\dagger$ be a $(\mathbf{v}_{ij}^a, \mathbf{v}_{i',j'}^b)$-path (or $(s, \mathbf{v}_{ij}^a)$-path or $(\mathbf{v}_{ij}^a, s)$-path). Observe that, since $P^\dagger$ is the reduced path of $P$, to finish the proof of the lemma, it is enough to show a $(v_i^a, v_j^b)$-path (or $(s, v_i^a)$-path or $(v_i^a, s)$-path) exists in $H - A$. Without loss of generality, let $P_c^\dagger$ be a $(\mathbf{v}_{ij}^a, \mathbf{v}_{i',j'}^b)$-path, the other cases hold due to similar arguments.

As $P^\dagger = P_1^\dagger \circ \ldots \circ P_d^\dagger$ is a $\mathcal{Z}^\dagger$-witnessing replacement, one of the following cases arises.

1. $V(P_c^\dagger) \subseteq \mathcal{Z}^\dagger$. Since $P^\dagger$ is the reduced path of $P$, consider the $(v_i^a, v_j^b)$-subpath, say $P'_c$, of $P$. Then, $V(P'_c) \subseteq \mathit{full\text{-}comp}(P_c^\dagger) \subseteq \mathcal{Z}$ (because $V(P_c^\dagger) \subseteq \mathcal{Z}^\dagger$). Also since $P$ does not have an arc in $A$, so does $P'_c$. Thus, by the construction of $H$, $P'_c$ is a path in $H - A$.
2. There is a list $\mathcal{L}_i$ of $2k\ell^2 + 1$ internally vertex-disjoint $(\mathbf{v}_{ij}^a, \mathbf{v}_{i'j'}^b)$-paths in $D_R^\dagger[\mathcal{Z}^\dagger]$. Let $\mathcal{A}^\dagger$ be the set of affected vertices of $A$ in $D_R^\dagger$. Clearly, $|\mathcal{A}^\dagger| \leq 2k\ell^2$. Then there exists a path in $\mathcal{L}_i$ that is $\mathcal{A}^\dagger$-free. Then from Observation 4.4, there exists a $(v_i^a, v_j^b)$-path, say $P'_c$, such that $V(P'_C) \subseteq \mathit{full\text{-}comp}(P_c^\dagger) \subseteq \mathcal{Z}$ and, that does not use an arc of $A$. From the construction of $H$, $P'_c$ is a path in $H - A$.

This finishes the proof of the lemma. ◀

## 5 Conclusion

In this paper, we presented a sparsification procedure for the class of acyclic digraphs (or more generally, "almost" acyclic) of bounded independence, to preserve the (both normal and parity-based) reachability from a given terminal set $S$ to a given terminal set $T$ under the failure of any set of at most $k$ arcs. In particular, it outputs a digraph whose size is completely *independent* of $n$ and polynomial in $k$, while even the simple classes of directed paths and tournaments admit no sparsifier whose output is a digraph of less than $n - 1$ arcs already when $k = 1$. Apart from being interesting on its own from the perspective of fault tolerance, we also showed that our sparsification procedure finds applications in Kernelization. Specifically, we proved that the classic Directed Feedback Arc Set problem as well as Directed Edge Odd Cycle Transversal (which, in sharp contract, is W[1]-hard on general digraphs) admit polynomial kernels on bounded independence number digraphs. In fact, for any $p \in \mathbb{N}$, we designed a polynomial kernel for hitting all cycles of length $\ell$ where ($\ell$ mod $p = 1$). Additionally, we derived complementary results that assert the NP-hardness of DEOCT on tournaments, as well as its admittance of a sub-exponential time parameterized algorithm on digraphs of bounded independence.

We conclude the paper with a few directions for further research. Our result, currently, holds when the input digraph $D$ is "almost acyclic" and has bounded independence number. From the example of the tournament described in the introduction (the one that is obtained by taking a transitive tournament and reversing the arcs along the Hamiltonian path defined by its topological ordering), it seems that some notion of "almost acyclic" might be necessary to have fault tolerant subgraphs whose size avoid the dependence on $n$. On the other hand, it might be possible to ask for something weaker than bounded independence number. For example, forbidding the existence of an induced $P_\alpha$, the directed path on $\alpha$ vertices.

**Question 1.** Does $\text{FTR}(S, S)$ admit a subgraph of size independent of $n$ on digraphs that are "almost acyclic" and have no induced $P_\alpha$, for some fixed positive integer $\alpha$?

It is not very difficult to observe that our results (Lemmas 1.1 and 1.2) also hold when the input graph is undirected and has bounded independence number. It would be interesting (because of the arguments discussed earlier) if one could obtain similar results when the input undirected graph has no induced $P_\alpha$.

**Question 2.** Does $\text{FTR}(S, S)$ admit a subgraph of size independent of $n$ when the input graph is undirected and has no induced $P_\alpha$, for some fixed positive integer $\alpha$?

It would also be interesting to discover other (di)graph classes where the dependence on $n$ of the size of the output subgraph can be sublinear, for example, $\log n$, for $\text{FTR}(S, S)$ and also for other fault tolerant graph properties.

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