Local-To-Global Agreement Expansion via the Variance Method

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Abstract

Agreement expansion is concerned with set systems for which local assignments to the sets with almost perfect pairwise consistency (i.e., most overlapping pairs of sets agree on their intersections) implies the existence of a global assignment to the ground set (from which the sets are defined) that agrees with most of the local assignments.

It is currently known that if a set system forms a *two-sided* or a *partite* high dimensional expander then agreement expansion is implied. However, it was not known whether agreement expansion can be implied for *one-sided* high dimensional expanders.

In this work we show that agreement expansion can be deduced for one-sided high dimensional expanders assuming that all the vertices' links (i.e., the neighborhoods of the vertices) are agreement expanders. Thus, for one-sided high dimensional expander, an agreement expansion of the large complicated complex can be deduced from agreement expansion of its small simple links.

Using our result, we settle the open question whether the well studied Ramanujan complexes are agreement expanders. These complexes are neither partite nor two-sided high dimensional expanders. However, they are one-sided high dimensional expanders for which their links are partite and hence are agreement expanders. Thus, our result implies that Ramanujan complexes are agreement expanders, answering affirmatively the aforementioned open question.

The local-to-global agreement expansion that we prove is based on the *variance method* that we develop. We show that for a high dimensional expander, if we define a function on its top faces and consider its local averages over the links then the variance of these local averages is much smaller than the global variance of the original function. This decreasing in the variance enables us to construct one global agreement function that ties together all local agreement functions.

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1 Introduction

Agreement expansion has been extensively studied. It plays an important role in nearly all PCP constructions, and has found various applications in many recent works (see e.g., [8, 5, 1, 9] and [3]). Previous works could only prove agreement expansion for complexes which are two-sided local spectral expanders or partite one-sided local spectral expanders. However, the question for the general case of one-sided local spectral expanders remained open. In this work we show that all one-sided local spectral expanders are agreement expanders, given that

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the local views of their vertices are agreement expanders. In particular, this solves an open question regarding the well studied Ramanujan complexes, which are not two-sided local spectral expanders nor partite, but the local views of their vertices are agreement expanders.

Our main theorem is a local-to-global agreement expansion. We show that agreement expansion of the entire complex can be deduced from agreement expansion of its vertices, i.e., if locally the complex is an agreement expander, then it is also globally expanding. In many cases of high dimensional expanders, the entire complex is a complicated object which is hard to understand, but it is composed of many simple local objects. We show that it is enough to argue regarding the simple local views of the complex, which then implies that the whole complicated complex is also an agreement expander.

The main theorem that we prove is based on the *variance method* that we develop. We show that for one-sided local spectral expanders, the variance of any function decreases as we go down the dimensions. Namely, if we define a function on the top faces of the complex and consider its local averages over the links then the variance of these local averages is much smaller than the global variance of the original function. This decreasing in variance enables us to construct a single global function that stitches together all the local agreement functions.

Agreement tests

Let S a collection of subsets over some ground set V. A *local assignment* is a collection of functions $f = \{f_\sigma\}_{\sigma \in S}$, where each f_σ is a function that specifies a $0, 1$ value for each $u \in \sigma$. Any f_σ is local in the sense that it gives values only to the elements in σ, independently of the rest of the functions and the ground set. A *global assignment* is a single function $g : V \to \{0, 1\}$ which specifices a value for every vertex in the ground set. We say that a local assignment $f = \{f_\sigma\}$ comes from a global assignment g, if for every $\sigma \in S$ it holds that $f_\sigma = g\big|_\sigma$, i.e., f_σ is just the restriction of g to σ.

Denote by $\mathcal{G}$ the set of all global assignments. An agreement test is specified by a distribution $\mathcal{D}$ on tuples $(\tau, \sigma_1, \sigma_2)$ such that $\sigma_1, \sigma_2 \in S$ and $\tau \subseteq \sigma_1 \cap \sigma_2$. The test picks τ, σ_1, σ_2 according to $\mathcal{D}$ and accepts if f_{σ_1} and f_{σ_2} agree on τ, i.e., if for every $u \in \tau$, $f_{\sigma_1}(\tau) = f_{\sigma_2}(\tau)$.[1] We denote the acceptance probability of the test by

$$\mathrm{agree}_{\mathcal{D}}(f) = \Pr_{\tau, \sigma_1, \sigma_2 \sim \mathcal{D}}[f_{\sigma_1}\big|_\tau = f_{\sigma_2}\big|_\tau].$$

It is easy to see that if a local assignment comes from a global assignment then the test accepts with probability 1. Denote by $\mathrm{dist}(f, g)$ the fraction of $\sigma \in S$ for which $f_\sigma \neq g\big|_\sigma$. An agreement theorem states that if $\mathrm{agree}_{\mathcal{D}}(f) \geq 1 - \varepsilon$ then f is $1 - O(\varepsilon)$ close to a global assignment, i.e., there exists a global assignment $g \in \mathcal{G}$ such that $\mathrm{dist}(f, g) \leq O(\varepsilon)$. In other words, an agreement theorem guarantees that the agreement test provides a good approximation for the distance of a local assignment from the global assignments.

High dimensional expanders

A d-dimensional simplicial complex X is a $(d+1)$-hypergraph which is closed under containment: For any $(d+1)$-hyperedge σ in X, all of its subsets $\tau \subset \sigma$ also belong to X. A hyperedge σ is also called a face of the complex, and its dimension is $|\sigma| - 1$. The set of all k-dimensional faces of the complex is denoted by $X(k)$.

[1] A weaker notion of an agreement test, which we do not discuss in this paper, is concerned with an *approximate* global consistency, i.e., that f_σ agrees with $g\big|_\sigma$ on most of the elements in σ.

For any face $\sigma \in X$, its *link* is the subcomplex obtained by all the faces in X which contain σ after removing σ from all of them, and denoted by $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in X\}$. Note that if X is of dimension d then X_σ is of dimension $d - |\sigma|$.

▶ **Definition 1.1** (Partite complex). *A d-dimensional complex X is called* partite *if its vertices can be partitioned into $d+1$ sets $X(0) = V_1 \cup V_2 \cup \cdots \cup V_{d+1}$ such that any d-dimensional face $\sigma \in X(d)$ contains a vertex from each V_i, i.e., $|\sigma \cap V_i| = 1$.*

In recent years, several distinct notions of high dimensional expansion have been studied. For a detailed survey regarding high dimensional expanders we refer the reader to [10]. In this work we focus on the spectral expansion of the links of the complex.

▶ **Definition 1.2** (Local spectral expansion). *A d-dimensional complex X is called a* λ-one-sided local spectral expander *(or* λ-two-sided local spectral expander*) if for every $-1 \leq k \leq d-2$ and every $\sigma \in X(k)$, the underlying graph*[2] *of X_σ is a λ-one-sided spectral expander (or λ-two-sided spectral expander, respectively).*

High dimensional expanders and agreement expansion

The work of [4] initiated the relation of high dimensional expanders to agreement tests. We follow their definition of *agreement expansion*.

▶ **Definition 1.3** (Agreement expansion). *A d-dimensional complex X is called a* c-agreement expander for dimension k *if there exists a distribution $\mathcal{D}$ such that for every local assignment $f = \{f_\sigma\}_{\sigma \in X(k)}$ there exists a global agreement function $g \in \mathcal{G}$ such that*

$$\mathrm{dist}(f, g) \leq c \cdot \mathrm{disagree}_{\mathcal{D}}(f),$$

where $\mathrm{disagree}_{\mathcal{D}}(f) = 1 - \mathrm{agree}_{\mathcal{D}}(f)$ *is the rejection probability of the test.*

The name "agreement expansion" comes from its similarity to other expansion measures, since X is an agreement expander if and only if

$$\min_f \frac{\mathrm{disagree}_{\mathcal{D}}(f)}{\mathrm{dist}(f, \mathcal{G})} \geq \frac{1}{c},$$

where the minimum is taken over all local assignments that do not come from a global assignment.

In [6], the authors prove that there exists a constant $c > 0$ such that the d-dimensional *complete complex*, which is the complex that contains all possible sets of size $\leq d+1$, is a c-agreement expander for its top dimension. Building on [6], [4] show that there exists a constant $c > 0$ such that a d-dimensional two-sided local spectral expander is a c-agreement expander *for dimension* $k = O(\sqrt{d})$. Their proof goes by a reduction to the complete complex, and therefore they could not prove agreement expansion for the top dimension of the complex, but only for some lower dimension.

In a recent work, Dikstein and Dinur [2] prove that there exists a constant $c > 0$ such that a d-dimensional two-sided local spectral expander or a partite one-sided local spectral expander are c-agreement expanders *for their top dimension*. However, the general question for one-sided local spectral expanders remained open.

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[2] The graph whose vertices are $X_\sigma(0)$ and its edges are $X_\sigma(1)$.

Our main theorem in this work is that agreement expansion of a complex can be deduced from the agreement expansion of the links of its vertices. In particular, for any constant $c > 0$ there exists a constant $c' > 0$ *which is dependent only on* c such that a d-dimensional one-sided local spectral expander is a c'-agreement expander *for its top dimension*, given that the links of its vertices are c-agreement expanders for their top dimension.

▶ **Theorem 1.4** (Main Theorem, informal). *For any constant $c > 0$ there exists a constant $c' = c'(c)$ such that if the links of the vertices of a one-sided local spectral expander are c-agreement expanders for their top dimension, then the entire complex is a c'-agreement expander for its top dimension. Moreover, the global agreement function that agrees with most of the local functions is defined by majority decoding.*

Our result extends [2] to general one-sided local expanders, which are not necessarily partite. Moreover, the result of [2] ensures that if the agreement test accepts with probability $1 - \varepsilon$ then there exists *some global function* that agrees with $1 - c\varepsilon$ of the local functions. The global function that [2] construct is not necessarily the majority function, but rather some conditional majority. We show here that the *majority function* agrees with $1 - c'\varepsilon$ of the local functions, *regardless* of which functions agree with the local functions on each vertex. Our proof technique can be used for two-sided local spectral expanders and for partite one-sided local spectral expanders as well: As a first step use [2] to construct a local agreement function for each vertex, and then by our work conclude that the majority function agrees with most of these local agreement functions.

▶ **Corollary 1.5.** *There exists a constant $c > 0$ such that any d-dimensional two-sided local spectral expander and any d-dimensional partite one-sided local spectral expander is a c-agreement expander for its top dimension, where the global agreement function is defined by majority decoding.*

Ramanujan complexes

Much of the motivation for the study of high dimensional expanders comes from the existence of Ramanujan complexes, whose properties are optimal in almost every measure. Ramanujan complexes are the high dimensional analogs of the celebrated LPS Ramanujan graphs [11], which arise from number theory. In [12] the authors describe an explicit construction of a family of bounded degree Ramanujan complexes, i.e., every vertex is contained in a bounded number of faces. Thus, the number of d-dimensional faces in these complexes is linear in the number of vertices.

Ramanujan complexes are known to be one-sided local spectral expanders [7], and their links are also partite. However, Ramanujan complexes are not two-sided local spectral expanders nor partite, so previous works could not show that they are agreement expanders. As a corollary of our theorem, we settle this open question.

▶ **Corollary 1.6.** *There exists a constant $c > 0$ such that Ramanujan complexes are c-agreement expanders for their top dimension.*

The variance method

Our local-to-global agreement theorem is based on the *variance method*. The decreasing in variance idea has first appeared in [4], where the authors proved that in high dimensional expanders, the difference of variances of a function on successive dimensions is decreasing. The authors in [4] proved it *only for two-sided* local spectral expanders, and used it just for the single purpose of showing that random walks on high dimensional expanders have an optimal convergence rate. We extend their work and show that the same holds for *one-sided* local spectral expanders as well. In addition, we show the general statement that the variance of any function on the top faces of a high dimensional expander decreases by a factor of d when considered on the vertices.

▶ **Theorem 1.7** (The variance method, informal). *Let X be a d-dimensional one-sided local spectral expander. For any function $h : X(d) \to \mathbb{R}$,*

$$\operatorname*{Var}_{u\in X(0)} \mathop{\mathbb{E}}_{\sigma\in X_u(d-1)} h(u\sigma) \leq O\left(\frac{1}{d}\right) \operatorname*{Var}_{\sigma\in X(d)} h(\sigma).$$

The small variance is important in order to get a global function that agrees with $1 - O(\varepsilon)$ of the local functions. Recall that we are given a local assignment $f = \{f_\sigma\}$ and a probability distribution $\mathcal{D}$ such that $\text{disagree}_{\mathcal{D}}(f) \leq \varepsilon$ and our goal is to construct a global function g such that $\text{dist}(f, g) \leq O(\varepsilon)$. By a simple union bound argument, it is easy to get $\text{dist}(f, g) \leq O(d\varepsilon)$. In order to get an agreement which is not dependent on d, we must bound the probability of the bad events by $O(\varepsilon/d)$ and then by a union bound to get an agreement of $O(\varepsilon)$. Using the variance method, we can show that the variance on the vertices decreases by a factor of d, and thus the bad events happen with a probability bounded by $O(\varepsilon/d)$. We believe that this method will find many more applications in the future.

Organization

In Section 2 we provide some required preliminaries regarding expander graphs and high dimensional expanders. In Section 3 we introduce the variance method. In Section 4 we prove our main theorem.

2 Preliminaries

2.1 Expander graphs

Let $G = (V, E)$ be a graph with positive weights on the edges $w : E \to \mathbb{R}_{>0}$. Assume without loss of generality that the sum of the weights is 1 (otherwise just normalize), i.e., w is a probability distribution on the edges. These weights induce a probability distribution on the vertices as well, where the probability of a vertex $u \in V$ is given by $\frac{1}{2}\sum_{e\ni u} w(e)$. All the following probabilities are taken according to these distributions.

For any two functions $h, h' : E \to \mathbb{R}$, we define their inner product by

$$\langle h, h'\rangle = \sum_{e\in E} w(e)h(e)h'(e) = \mathop{\mathbb{E}}_{e\in E} h(e)h'(e).$$

Similarly, for any two functions $g, g' : V \to \mathbb{R}$, their inner product is defined by

$$\langle g, g'\rangle = \mathop{\mathbb{E}}_{u\in V} g(u)g'(u).$$

The adjacency operator $A = A(G) : \mathbb{R}^V \to \mathbb{R}^V$ is defined by $Ag(u) = \mathbb{E}_{uv|u}\, g(v)$, where $uv|u$ denotes the event that the edge $\{u, v\}$ was chosen given that the vertex u was chosen.

Denote the eigenvalues of the adjacency operator of G by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|}$. It is easy to see that $\lambda_1 = 1$ (with corresponding eigenfunction $\mathbf{1}$) and that $\lambda_{|V|} \geq -1$. Let $0 < \lambda < 1$ be a positive constant. The graph G is said to be a λ-*one-sided spectral expander* if $\lambda_2 \leq \lambda$, and G is said to be a λ-*two-sided spectral expander* if both $\lambda_2 \leq \lambda$ and $\lambda_{|V|} \geq -\lambda$.

2.2 High dimensional expanders

Recall that a d-dimensional simplicial complex X is a $(d+1)$-hypergraph which is closed under containment. Throughout the paper we refer to the probability of choosing a face of the complex, which is given by the following probabilistic process. Consider the random sequence of faces $\sigma_d \supset \sigma_{d-1} \supset \cdots \supset \sigma_0 \supset \emptyset$, where $\sigma_d \in X(d)$ is chosen uniformly at random, and then for each $k = d-1, \cdots, 0$, σ_k is chosen by removing a uniformly random vertex from σ_{k+1}. For any $\sigma \in X(k)$, we denote the probability of choosing σ by

$$\Pr[\sigma] = \Pr[\sigma_d \supset \sigma_{d-1} \supset \cdots \supset \sigma_0 \supset \emptyset \mid \sigma_k = \sigma].$$

For any $-1 \leq k < d$ and a face $\sigma \in X(k)$, its *link* is the $(d-k-1)$-dimensional complex defined by $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in X\}$. The probability of choosing a face $\tau \in X_\sigma$ is given by $\Pr[\tau \in X_\sigma] = \Pr[\tau \mathbin{\dot\cup} \sigma \mid \sigma_k = \sigma]$.

For the agreement test, we use the $\mathcal{D}_{d,\ell}$ distribution as defined in [2].

▶ **Definition 2.1** (The $\mathcal{D}_{d,\ell}$ distribution). *Let X be a d-dimensional simplicial complex and $\ell < d$ be a positive integer. The $\mathcal{D}_{d,\ell}$ distribution is defined by the following random process:*

1. *Choose $\tau \in X(\ell)$ at random.*
2. *Choose $\sigma_1, \sigma_2 \in X_\tau(d-\ell-1)$ independently at random.*

The tuple that is then returned is $(\tau, \tau \mathbin{\dot\cup} \sigma_1, \tau \mathbin{\dot\cup} \sigma_2)$.

3 The Variance Method

We show in this section that for one-sided local spectral expanders, the variance of any function decreases as we go down the dimensions of the complex. We first bound the variance in one-sided spectral expander graphs, and then show how to use this bound in order to bound the variance on functions of a complex.

3.1 Bounding the variance in expander graphs

Let $G = (V, E)$ be a weighted graph. Recall that the adjacency operator $A : \mathbb{R}^V \to \mathbb{R}^V$ is defined by $Ag(u) = \mathbb{E}_{uv|u}\, g(v)$. We define the following two additional averaging operators:

- $A^\downarrow : \mathbb{R}^E \to \mathbb{R}^V$ defined by $A^\downarrow h(u) = \mathbb{E}_{e|u}\, h(e)$.
- $A^\uparrow : \mathbb{R}^V \to \mathbb{R}^E$ defined by $A^\uparrow g(e) = \mathbb{E}_{u|e}\, g(u)$.

The following properties are pretty standard.

(1) $A^\downarrow$ and $A^\uparrow$ are adjoint to each other, i.e., for any two vectors $h \in \mathbb{R}^E, g \in \mathbb{R}^V$,

$$\langle A^\downarrow h, g\rangle = \mathop{\mathbb{E}}_{u} \mathop{\mathbb{E}}_{e|u} h(e)g(u) = \mathop{\mathbb{E}}_{e} \mathop{\mathbb{E}}_{u|e} h(e)g(u) = \langle h, A^\uparrow g\rangle.$$

(2) Denote by $\lambda_2(A^\downarrow A^\uparrow)$ and $\lambda_2(A)$ the second largest eigenvalues of $A^\downarrow A^\uparrow$ and A correspondingly. It is easy to check that $A^\downarrow A^\uparrow = (A+I)/2$, where I is the identity operator. Thus,

$$\lambda_2(A^\downarrow A^\uparrow) = \frac{1 + \lambda_2(A)}{2}.$$

(3) $A^\uparrow A^\downarrow$ and $A^\downarrow A^\uparrow$ have the same non-zero eigenvalues. In particular, $\lambda_2(A^\uparrow A^\downarrow) = \lambda_2(A^\downarrow A^\uparrow)$.

(4) By Rayleigh quotient

$$\lambda_2(A^{\uparrow}A^{\downarrow}) = \max_{\substack{h:E\to\mathbb{R} \\ h\perp\mathbf{1}}} \frac{\langle A^{\uparrow}A^{\downarrow}h, h\rangle}{\|h\|^2}.$$

We will use the following lemma, which is immediate from the aforementioned properties.

▶ **Lemma 3.1.** *Let $G=(V,E)$ be a λ-one-sided spectral expander graph. For any function $h:E\to\mathbb{R}$ such that $h\perp\mathbf{1}$ it holds that*

$$\left\|A^{\downarrow}h\right\|^2 \le \frac{1+\lambda}{2}\|h\|^2.$$

Proof. It follows immediately from the properties mentioned above, since

$$\left\|A^{\downarrow}h\right\|^2 = \langle A^{\downarrow}h, A^{\downarrow}h\rangle = \langle A^{\uparrow}A^{\downarrow}h, h\rangle \le \lambda_2(A^{\uparrow}A^{\downarrow})\|h\|^2 = \lambda_2(A^{\downarrow}A^{\uparrow})\|h\|^2 = \frac{1+\lambda_2(A)}{2}\|h\|^2,$$

where we used properties (1), (4), (3) and (2) in that order. ◀

We can now bound the variance of any function on the edges of the graph.

▶ **Lemma 3.2.** *Let $G=(V,E)$ be a λ-one-sided spectral expander graph. For any function $h:E\to\mathbb{R}$ it holds that*

$$\mathop{\mathbb{E}}_{u}\mathop{\mathrm{Var}}_{e|u} h(e) \ge \frac{1-\lambda}{1+\lambda}\mathop{\mathrm{Var}}_{u}\mathop{\mathbb{E}}_{e|u} h(e).$$

Proof. Assume without loss of generality that $\mathbb{E}_e h(e)=0$ (otherwise, define $h' = h - \mathbb{E}_e h(e)$ and continue with h'). Now, note that

$$\mathop{\mathrm{Var}}_{u}\mathop{\mathbb{E}}_{e|u} h(e) = \mathop{\mathbb{E}}_{u}\left(\mathop{\mathbb{E}}_{e|u} h(e)\right)^2 = \left\|A^{\downarrow}h\right\|^2. \tag{1}$$

Note also that

$$\mathop{\mathbb{E}}_{u}\mathop{\mathrm{Var}}_{e|u} h(e) = \mathop{\mathbb{E}}_{u}\left(\mathop{\mathbb{E}}_{e|u} h(e)^2 - \left(\mathop{\mathbb{E}}_{e|u} h(e)\right)^2\right) = \mathop{\mathbb{E}}_{e} h(e)^2 - \mathop{\mathbb{E}}_{u}\left(\mathop{\mathbb{E}}_{e|u} h(e)\right)^2 = \|h\|^2 - \left\|A^{\downarrow}h\right\|^2. \tag{2}$$

Since $\mathbb{E}_e h(e)=0$, by lemma 3.1 we have that

$$\|h\|^2 \ge \frac{2}{1+\lambda}\left\|A^{\downarrow}h\right\|^2. \tag{3}$$

Combining (1), (2) in (3) finishes the proof. ◀

3.2 Bounding the variance in one-sided local spectral expanders

The following lemma follows immediately from lemma 3.2.

▶ **Lemma 3.3.** *Let X be a d-dimensional λ-one-sided local spectral expander. For any $1\le k\le d$ and a function $h:X(k)\to\mathbb{R}$ it holds that*

$$\mathop{\mathbb{E}}_{\sigma\in X(k-1)}\mathop{\mathrm{Var}}_{u\in X_\sigma(0)} h(\sigma u) \ge \frac{1-\lambda}{1+\lambda}\mathop{\mathbb{E}}_{\sigma\in X(k-2)}\mathop{\mathrm{Var}}_{u\in X_\sigma(0)}\mathop{\mathbb{E}}_{v\in X_{\sigma u}(0)} h(\sigma uv).$$

Proof. By definition, for every $\sigma \in X(k-2)$, the underlying graph of X_σ is a λ-one-sided spectral expander. Thus, by lemma 3.2,

$$\mathop{\mathbb{E}}_{u \in X_\sigma(0)} \mathop{\mathrm{Var}}_{v \in X_{\sigma u}(0)} h(\sigma u v) \geq \frac{1-\lambda}{1+\lambda} \mathop{\mathrm{Var}}_{u \in X_\sigma(0)} \mathop{\mathbb{E}}_{v \in X_{\sigma u}(0)} h(\sigma u v).$$

Averaging over all $\sigma \in X(k-2)$ finishes the proof. ◀

We can now prove the following two lemmas for one-sided local spectral expanders.

▶ **Lemma 3.4.** *Let X be a $(d-1)$-dimensional λ-one-sided local spectral expander, where $d \geq k \cdot \ell$ for $k \geq 2$, $\ell \geq 1$. If $\lambda \leq 1/2d$ then for any function $h : X(d-1) \to \mathbb{R}$,*

$$\mathop{\mathrm{Var}}_{\sigma \in X(d-1)} h(\sigma) \leq \frac{k+2}{k-1} \mathop{\mathbb{E}}_{\tau \in X(\ell-1)} \mathop{\mathrm{Var}}_{\sigma \in X_\tau(d-\ell-1)} h(\tau\sigma).$$

Proof. Note that

$$\begin{aligned}
\mathop{\mathrm{Var}}_{\sigma \in X(d-1)} h(\sigma) &= \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d)^2 - \Big(\mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \Big)^2 \\
&= \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d)^2 - \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{d-1}} \Big(\mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \Big)^2 + \\
&\quad \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{d-1}} \Big(\mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \Big)^2 - \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{d-2}} \Big(\mathop{\mathbb{E}}_{u_{d-1}} \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \Big)^2 + \cdots \\
&\quad \cdots + \mathop{\mathbb{E}}_{u_1} \Big(\mathop{\mathbb{E}}_{u_2} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \Big)^2 - \Big(\mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \Big)^2 \qquad (4) \\
&= \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{d-1}} \mathop{\mathrm{Var}}_{u_d} h(u_1 \cdots u_d) + \\
&\quad \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{d-2}} \mathop{\mathrm{Var}}_{u_{d-1}} \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) + \cdots \\
&\quad \cdots + \mathop{\mathrm{Var}}_{u_1} \mathop{\mathbb{E}}_{u_2} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d),
\end{aligned}$$

where the second equality follows by a telescoping sum argument. Similarly,

$$\begin{aligned}
\mathop{\mathbb{E}}_{\tau \in X(\ell-1)} \mathop{\mathrm{Var}}_{\sigma \in X_\tau(d-\ell-1)} h(\tau\sigma) &= \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{d-1}} \mathop{\mathrm{Var}}_{u_d} h(u_1 \cdots u_d) + \\
&\quad \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{d-2}} \mathop{\mathrm{Var}}_{u_{d-1}} \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) + \cdots \\
&\quad \cdots + \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_\ell} \mathop{\mathrm{Var}}_{u_{\ell+1}} \mathop{\mathbb{E}}_{u_{\ell+2}} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d).
\end{aligned}$$

For any $1 \leq i \leq k-1$, invoking lemma 3.3 $i \cdot \ell$ times on each of the last ℓ summands of (4) yields

$$\begin{aligned}
&\mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{\ell-1}} \mathop{\mathrm{Var}}_{u_\ell} \mathop{\mathbb{E}}_{u_{\ell+1}} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) + \cdots + \mathop{\mathrm{Var}}_{u_1} \mathop{\mathbb{E}}_{u_2} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \\
&\quad \leq \Big(\frac{1+\lambda}{1-\lambda} \Big)^{i\ell} \Big(\mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{(i+1)\ell-1}} \mathop{\mathrm{Var}}_{u_{(i+1)\ell}} \mathop{\mathbb{E}}_{u_{(i+1)\ell+1}} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) + \cdots \\
&\qquad \cdots + \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{i\ell}} \mathop{\mathrm{Var}}_{u_{i\ell+1}} \mathop{\mathbb{E}}_{u_{i\ell+2}} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \Big) \\
&\quad \leq e \Big(\mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{(i+1)\ell-1}} \mathop{\mathrm{Var}}_{u_{(i+1)\ell}} \mathop{\mathbb{E}}_{u_{(i+1)\ell+1}} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) + \cdots \\
&\qquad \cdots + \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{i\ell}} \mathop{\mathrm{Var}}_{u_{i\ell+1}} \mathop{\mathbb{E}}_{u_{i\ell+2}} \cdots \mathop{\mathbb{E}}_{u_d} h(u_1 \cdots u_d) \Big),
\end{aligned}$$

where the last inequality follows since $i \cdot \ell \leq d-1$ and $\lambda \leq 1/2d$. Thus, by invoking lemma 3.3 $i \cdot \ell$ times on the last ℓ summands of (4) for $i = 1, \ldots, k-1$ we get

$$(k-1) \operatorname*{Var}_{\sigma \in X(d-1)} h(\sigma) \leq (k-1+e) \mathop{\mathbb{E}}_{\tau \in X(\ell-1)} \operatorname*{Var}_{\sigma \in X_\tau(d-\ell-1)} h(\tau\sigma),$$

which finishes the proof. ◀

▶ **Lemma 3.5.** *Let X be a d-dimensional λ-one-sided local spectral expander. If $\lambda \leq 1/(2d+1)$ then for any function $h : X(\ell) \to \mathbb{R}$, $\ell \leq d$,*

$$\operatorname*{Var}_{u \in X(0)} \mathop{\mathbb{E}}_{\sigma \in X_u(\ell-1)} h(u\sigma) \leq \frac{8}{5\ell} \operatorname*{Var}_{\sigma \in X(\ell)} h(\sigma).$$

Proof. Note that

$$\begin{aligned}\operatorname*{Var}_{\sigma \in X(\ell)} h(\sigma) = & \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_\ell} \operatorname*{Var}_{u_{\ell+1}} h(u_1 \cdots u_{\ell+1}) + \\ & \mathop{\mathbb{E}}_{u_1} \cdots \mathop{\mathbb{E}}_{u_{\ell-1}} \operatorname*{Var}_{u_\ell} \mathop{\mathbb{E}}_{u_{\ell+1}} h(u_1 \cdots u_{\ell+1}) + \cdots \\ & \cdots + \operatorname*{Var}_{u_1} \mathop{\mathbb{E}}_{u_2} \cdots \mathop{\mathbb{E}}_{u_{\ell+1}} h(u_1 \cdots u_{\ell+1}).\end{aligned} \tag{5}$$

By invoking lemma 3.3 on each summand of (5), where on the ith summand we invoke it $i-1$ times, we get

$$\operatorname*{Var}_{\sigma \in X(\ell)} h(\sigma) \geq \sum_{i=0}^{\ell} \left(\frac{1-\lambda}{1+\lambda}\right)^i \operatorname*{Var}_{u \in X(0)} \mathop{\mathbb{E}}_{\sigma \in X_u(\ell-1)} h(u\sigma) \geq \ell(1-e^{-1}) \operatorname*{Var}_{u \in X(0)} \mathop{\mathbb{E}}_{\sigma \in X_u(\ell-1)} h(u\sigma),$$

where the last inequality follows since $\lambda \leq 1/(2d+1)$. ◀

▶ **Corollary 3.6.** *By the assumptions of lemma 3.5, if the range of h is $[0,1]$ then*

$$\operatorname*{Var}_{u \in X(0)} \mathop{\mathbb{E}}_{\sigma \in X_u(\ell-1)} h(u\sigma) \leq \frac{8}{5\ell} \mathop{\mathbb{E}}_{\sigma \in X(\ell)} h(\sigma).$$

Proof. It follows immediately since for h with range in $[0,1]$, $\operatorname{Var}_{\sigma \in X(\ell)} h(\sigma) \leq \mathbb{E}_{\sigma \in X(\ell)} h(\sigma)$. ◀

4 Local-to-Global Agreement Expansion

In this section we state and prove our main theorem.

▶ **Theorem 4.1** (Main Theorem). *For any constant $c > 0$ and two natural numbers $d > \ell \geq 2$ such that $\ell = \Theta(d)$, there exists a constant $c' = c'(c, d/\ell)$ such that the following holds. Let X be a d-dimensional λ-one-sided local spectral expander, $\lambda \leq 1/(2d+1)$. If for every vertex $v \in X(0)$, the link X_v is a c-agreement expander with regard to the $\mathcal{D}_{d-1,\ell-1}$ distribution, then X is a c'-agreement expander with regard to the $\mathcal{D}_{d,\ell}$ distribution. Moreover, the global agreement function is defined by majority decoding.*

The general idea will be to decompose the global agreement probability to local agreements in the links, and to show that with high probability the local functions agree with the global majority function.

Let $f = \{f_\sigma\}_{\sigma \in X(d)}$. Define the majority function $\operatorname{maj} : X(0) \to \{0,1\}$ by

$$\operatorname{maj}(v) = \operatorname*{arg\,max}_{\alpha \in \{0,1\}} \Big\{ \Pr_{\sigma \in X_v(d-1)} [f_{v\sigma}(v) = \alpha] \Big\}.$$

Denote by $\varepsilon = \operatorname{disagree}(f)$. The proof will follow from the following claims.

▷ Claim 4.2. $\mathbb{E}_{v\in X(0)} \Pr_{\sigma\in X_v(d-1)}[f_{v\sigma}(v) \neq \mathrm{maj}(v)] \leq 2\Big(1+\frac{3}{d/\ell-1}\Big)\varepsilon.$

▷ Claim 4.3. $\mathbb{E}_{v\in X(0)} \Pr_{\sigma\in X_v(d-1)}\left[g_v\big|_\sigma \neq \mathrm{maj}\big|_\sigma\right] \leq 120\Big(\frac{2d}{\ell}c^2+2c+1+\frac{3}{d/\ell-1}\Big)\varepsilon,$

where $g_v : X_v(0) \to \{0,1\}$ is the function promised by the c-agreement expansion of X_v.

Proof of Main Theorem. Consider a vertex $v \in X(0)$ and a top face in its link $\sigma \in X_v(d-1)$. Note that if the following three events happen then $f_{v\sigma}$ agrees with $\mathrm{maj}\big|_{v\sigma}$:

(1) $f_{v\sigma}(v) = \mathrm{maj}(v)$,
(2) $f_{v\sigma}\big|_\sigma = g_v\big|_\sigma$,
(3) $g_v\big|_\sigma = \mathrm{maj}\big|_\sigma$.

Therefore,

$$\begin{aligned}\Pr_{\sigma\in X(d)}[f_\sigma \neq \mathrm{maj}\big|_\sigma] &= \mathbb{E}_{v\in X(0)} \Pr_{\sigma\in X_v(d-1)}[f_{v\sigma}\big|_{v\sigma} \neq \mathrm{maj}\big|_{v\sigma}]\\ &\leq \mathbb{E}_{v\in X(0)}\left(\Pr_{\sigma\in X_v(d-1)}[f_{v\sigma}(v)\neq \mathrm{maj}(v)] + \right.\\ &\qquad \Pr_{\sigma\in X_v(d-1)}[f_{v\sigma}\big|_\sigma \neq g_v\big|_\sigma] + \\ &\qquad \left.\Pr_{\sigma\in X_v(d-1)}[g_v\big|_\sigma \neq \mathrm{maj}\big|_\sigma]\right)\\ &\leq 2\Big(1+\frac{3}{d/\ell-1}\Big)\varepsilon + c\varepsilon + 120\Big(\frac{2d}{\ell}c^2+2c+1+\frac{3}{d/\ell-1}\Big)\varepsilon\\ &= \Big(\frac{240d}{\ell}c^2+241c+122+\frac{366}{d/\ell-1}\Big)\varepsilon,\end{aligned}$$

where the second inequality follows by claims 4.2 and 4.3, and by the c-agreement expansion of the link of every vertex in the complex. Finally, the theorem follows since $d/l = \Theta(1)$. ◀

4.1 Proofs of Claims 4.2 and 4.3

Proof of Claim 4.2. The idea of the proof is by the variance method, as follows. For any vertex $v \in X(0)$ we consider the indicator function on the d-faces whether a local function f_σ agrees with the majority on v. We note that the global variance of this function indicates the disagreement probability of two d-faces on v, and the average over $\tau \in X_v(\ell-1)$ of local variances indicates the disagreement probability of two d-faces on v *with intersection of size at least* ℓ. Since $\ell = \Theta(d)$ we conclude by the variance method that these two distributions are approximately equal. Details follow.

Let $v \in X(0)$. Define the indicator function $h_v : X_v(d-1) \to \{0,1\}$ by

$$h_v(\sigma) = \begin{cases} 1 & f_{v\sigma}(v) \neq \mathrm{maj}(v),\\ 0 & f_{v\sigma}(v) = \mathrm{maj}(v).\end{cases}$$

By definition, $\mathbb{E}_{\sigma\in X_v(d-1)} h_v(\sigma) \leq 1/2$. Thus,

$$\mathbb{E}_{\sigma\in X_v(d-1)} h_v(\sigma) \leq 2 \mathbb{E}_{\sigma\in X_v(d-1)} h_v(\sigma)\left(1-\mathbb{E}_{\sigma\in X_v(d-1)} h_v(\sigma)\right). \tag{6}$$

Since h_v is an indicator function

$$\mathbb{E}_{\sigma\in X_v(d-1)} h_v(\sigma)\left(1-\mathbb{E}_{\sigma\in X_v(d-1)} h_v(\sigma)\right) = \mathrm{Var}_{\sigma\in X_v(d-1)} h_v(\sigma). \tag{7}$$

Since X is a λ-one-sided local spectral expander, by lemma 3.4

$$\operatorname*{Var}_{\sigma\in X_v(d-1)} h_v(\sigma) \le \frac{d+2\ell}{d-\ell} \mathop{\mathbb{E}}_{\tau\in X_v(\ell-1)} \operatorname*{Var}_{\sigma\in X_{v\tau}(d-\ell-1)} h_v(\tau\sigma). \tag{8}$$

Again, since h_v is an indicator function

$$\operatorname*{Var}_{\sigma\in X_{v\tau}(d-\ell-1)} h_v(\tau\sigma) = \mathop{\mathbb{E}}_{\sigma\in X_{v\tau}(d-\ell-1)} h_v(\tau\sigma)\left(1 - \mathop{\mathbb{E}}_{\sigma\in X_{v\tau}(d-\ell-1)} h_v(\tau\sigma)\right). \tag{9}$$

Combining (6), (7), (8) and (9) yields

$$\Pr_{\sigma\in X_v(d-1)}[f_{v\sigma}(v)\neq \mathrm{maj}(v)] \le 2\frac{d+2\ell}{d-\ell} \mathop{\mathbb{E}}_{\tau\in X_v(\ell-1)} \Pr_{\sigma_1,\sigma_2\in X_{v\tau}(d-\ell-1)}[f_{v\tau\sigma_1}(v)\neq \mathrm{maj}(v)\wedge f_{v\tau\sigma_2}(v)=\mathrm{maj}(v)].$$

Finally, since $f_{v\tau\sigma_1}(v)\neq \mathrm{maj}(v)\wedge f_{v\tau\sigma_2}(v)=\mathrm{maj}(v)$ implies that $f_{v\tau\sigma_1}\big|_{v\tau}\neq f_{v\tau\sigma_2}\big|_{v\tau}$, we can conclude that

$$\begin{aligned}\mathop{\mathbb{E}}_{v\in X(0)} \Pr_{\sigma\in X_v(d-1)}[f_{v\sigma}(v)\neq \mathrm{maj}(v)] &\le 2\frac{d+2\ell}{d-\ell} \mathop{\mathbb{E}}_{\tau\in X(\ell)} \Pr_{\sigma_1,\sigma_2\in X_\tau(d-\ell-1)}[f_{\tau\sigma_1}\big|_\tau \neq f_{\tau\sigma_2}\big|_\tau] \\ &\le 2\Big(1+\frac{3}{d/\ell-1}\Big)\varepsilon.\end{aligned}$$

◁

Before proving Claim 4.3, let us define the following set of “bad neighbors”. For any vertex $u\in X(0)$, by the agreement expansion of X_u, there exists a function $g_u : X_u(0)\to\{0,1\}$ that agrees with most of the top faces on u. We say that $v\in X_u(0)$ is a bad neighbor of u if g_u disagrees with many top faces that contain v:

$$B_u = \left\{ v\in X_u(0) \;\middle|\; \Pr_{\sigma\in X_{uv}(d-2)}[f_{uv\sigma}\big|_{v\sigma}\neq g_u\big|_{v\sigma}] > \frac{3}{10}\right\}.$$

We will use the following two claims, which guarantee that on average almost all of the neighbors are not bad.

▷ Claim 4.4. $\displaystyle \mathop{\mathbb{E}}_{u\in X(0)} \Pr_{v\in X_u(0)}[v\in B_u] \le 160\Big(\frac{d}{\ell}c^2 + c\Big)\frac{\varepsilon}{d-1}.$

▷ Claim 4.5. $\displaystyle \mathop{\mathbb{E}}_{v\in X(0)} \Pr_{u\in X_v(0)}[g_u(v)\neq \mathrm{maj}(v)\ \wedge\ v\notin B_u] \le 80\Big(1+\frac{3}{d/\ell-1}\Big)\frac{\varepsilon}{d-1}.$

Proof of Claim 4.3. Note that

$$\begin{aligned}\mathop{\mathbb{E}}_{u\in X(0)} \Pr_{\sigma\in X_u(d-1)}[g_u\big|_\sigma\neq \mathrm{maj}\big|_\sigma] &= \mathop{\mathbb{E}}_{u\in X(0)} \Pr_{\sigma\in X_u(d-1)}[\exists v\in\sigma \text{ s.t. } g_u(v)\neq \mathrm{maj}(v)] \\ &\le d \mathop{\mathbb{E}}_{u\in X(0)} \Pr_{v\in X_u(0)}[g_u(v)\neq \mathrm{maj}(v)].\end{aligned}$$

By the law of total probability

$$\begin{aligned}\mathop{\mathbb{E}}_{u\in X(0)} \Pr_{v\in X_u(0)}[g_u(v)\neq \mathrm{maj}(v)] &= \mathop{\mathbb{E}}_{u\in X(0)} \Pr_{v\in X_u(0)}[g_u(v)\neq \mathrm{maj}(v)\ \wedge\ v\in B_u] + \\ &\quad \mathop{\mathbb{E}}_{v\in X(0)} \Pr_{u\in X_u(0)}[g_u(v)\neq \mathrm{maj}(v)\ \wedge\ v\notin B_u] \\ &\le 160\Big(\frac{d}{\ell}c^2+c\Big)\frac{\varepsilon}{d-1} + 80\Big(1+\frac{3}{d/\ell-1}\Big)\frac{\varepsilon}{d-1} \\ &= 80\Big(\frac{2d}{\ell}c^2 + 2c + 1 + \frac{3}{d/\ell-1}\Big)\frac{\varepsilon}{d-1},\end{aligned}$$

where the inequality follows by claims 4.4 and 4.5. Therefore,

$$\mathop{\mathbb{E}}_{u\in X(0)}\Pr_{\sigma\in X_u(d-1)}[g_u\big|_\sigma \neq \mathrm{maj}\big|_\sigma] \le 120\Big(\frac{2d}{\ell}c^2+2c+1+\frac{3}{d/\ell-1}\Big)\varepsilon,$$

where the inequality follows by the assumption that $d \ge 3$. ◁

4.2 Proofs of Claims 4.4 and 4.5

For any $u \in X(0)$, denote by ε_u the disagreement probability conditioned on u:

$$\varepsilon_u = \mathop{\mathbb{E}}_{\tau\in X_u(\ell-1)}\Pr_{\sigma_1,\sigma_2\in X_{u\tau}(d-\ell-1)}[f_{u\tau\sigma_1}\big|_{u\tau} \neq f_{u\tau\sigma_2}\big|_{u\tau}],$$

and define the following set of vertices:

$$S = \left\{ u \in X(0) \;\middle|\; \varepsilon_u \le \frac{1}{5c} \right\}.$$

The proofs will follow from the following claims.

▷ Claim 4.6. $\displaystyle\Pr_{u\in X(0)}[u \notin S] \le 160c^2\frac{\varepsilon}{\ell}$.

▷ Claim 4.7. For any vertex $u \in S$,

$$\Pr_{v\in X_u(0)}[v \in B_u \mid u \in S] \le 160c\frac{\varepsilon_u}{d-1}.$$

Proof of Claim 4.4. Assuming claims 4.6 and 4.7 the proof follows immediately since

$$\begin{aligned}\mathop{\mathbb{E}}_{u\in X(0)}\Pr_{v\in X_u(0)}[v\in B_u] &\le \Pr_{u\in X(0)}[u\notin S] + \mathop{\mathbb{E}}_{u\in S}\Pr_{v\in X_u(0)}[v\in B_u]\\ &\le 160c^2\frac{\varepsilon}{\ell} + 160c\frac{1}{d-1}\mathop{\mathbb{E}}_{u\in S}\varepsilon_u\\ &\le 160\Big(\frac{d}{\ell}c^2+c\Big)\frac{\varepsilon}{d-1},\end{aligned}$$

where the last inequality follows since $\mathbb{E}_{u\in X(0)}\,\varepsilon_u \le \varepsilon$. ◁

Proof of Claim 4.5. Consider a vertex $v \in X(0)$ and the sets of top faces which disagree with the majority on v. By the variance method we know that almost all the vertices in the link of v see the right amount of top faces that disagree with the majority on v. Thus, if for a vertex $u \in X_v(0)$, where v is not bad neighbor of u, the function g_u disagrees with the majority on v, it must see a lot of top faces that disagree with the majority. By the variance method we know that this happens with a small probability. Details follows.

For any vertex $v \in X(0)$ define the function $h_v : X_v(d-1) \to \{0,1\}$ by

$$h_v(\sigma) = \begin{cases} 1 & f_{v\sigma}(v) \neq \mathrm{maj}(v),\\ 0 & f_{v\sigma}(v) = \mathrm{maj}(v).\end{cases}$$

Note that $\mathbb{E}_{\sigma\in X_v(d-1)}\,h_v(\sigma) \le 1/2$. Note also that for any vertex $u \in X_v(0)$,

$$g_u(v) \neq \mathrm{maj}(v) \quad \wedge \quad v \notin B_u \quad \Rightarrow \quad \mathop{\mathbb{E}}_{\sigma\in X_{uv}(d-2)} h_v(u\sigma) > \frac{7}{10}.$$

Therefore,

$$\Pr_{u \in X_v(0)}[g_u(v) \neq \mathrm{maj}(v) \ \wedge \ v \notin B_u] \leq \Pr_{u \in X_v(0)}\left[\mathop{\mathbb{E}}_{\sigma \in X_{uv}(d-2)} h_v(u\sigma) - \mathop{\mathbb{E}}_{\sigma \in X_v(d-1)} h_v(\sigma) > \frac{1}{5}\right].$$

By corollary 3.6

$$\mathop{\mathrm{Var}}_{u \in X_v(0)} \mathop{\mathbb{E}}_{\sigma \in X_{uv}(d-2)} h_v(u\sigma) \leq \frac{8}{5(d-1)} \mathop{\mathbb{E}}_{\sigma \in X_v(d-1)} h_v(\sigma).$$

Thus, by Chebyshev inequality

$$\Pr_{u \in X_v(0)}\left[\mathop{\mathbb{E}}_{\sigma \in X_{uv}(d-2)} h_v(u\sigma) - \mathop{\mathbb{E}}_{\sigma \in X_v(d-1)} h_v(\sigma) > \frac{1}{5}\right] \leq \frac{40}{d-1} \mathop{\mathbb{E}}_{\sigma \in X_v(d-1)} h_v(\sigma).$$

Averaging over all $v \in X(0)$ yields

$$\begin{aligned}\mathop{\mathbb{E}}_{v \in X(0)} \Pr_{u \in X_v(0)}[g_u(v) \neq \mathrm{maj}(v) \ \wedge \ v \notin B_u] &\leq \frac{40}{d-1} \mathop{\mathbb{E}}_{v \in X(0)} \mathop{\mathbb{E}}_{\sigma \in X_v(d-1)} h_v(\sigma) \\ &\leq 80\left(1 + \frac{3}{d/\ell - 1}\right)\frac{\varepsilon}{d-1},\end{aligned}$$

where the last inequality follows by claim 4.2. ◁

4.3 Proofs of Claims 4.6 and 4.7

Both of these claims follow immediately by the variance method. In claim 4.6 we use the small variance of the function that measures the disagreement probability on a face $\tau \in X(\ell)$. Since the variance of the average of this function on the vertices is small, almost all the vertices have disagreement close to the expectation.

In claim 4.7 we use the small variance of the indicator function in the link of a vertex $u \in X(0)$, which indicates whether a top face disagrees with the function g_u. Since the variance of the average of this function on the vertices is small, almost all the vertices in the link of u see a small amount of top faces which disagree with g_u, i.e., almost all the vertices in the link of u are not bad neighbors. Details follow.

Proof of Claim 4.6. Note that $\mathbb{E}_{u \in X(0)} \varepsilon_u \leq \varepsilon$ and assume that $\varepsilon \leq 1/10c$. Thus, for a vertex $u \in X(0)$ to not be in S, it has to deviate from its mean by more than $1/10c$. By corollary 3.6

$$\mathop{\mathrm{Var}}_{u \in X(0)} \varepsilon_u \leq \frac{8}{5\ell} \mathop{\mathbb{E}}_{\tau \in X(\ell)} \Pr_{\sigma_1, \sigma_2 \in X_\tau(d-\ell-1)}[f_{\tau\sigma_1}|_\tau \neq f_{\tau\sigma_2}|_\tau] \leq \frac{8\varepsilon}{5\ell}.$$

Thus, by Chebyshev inequality

$$\Pr_{u \in X(0)}[u \notin S] \leq \Pr_{u \in X(0)}\left[\varepsilon_u - \mathop{\mathbb{E}}_{u' \in X(0)} \varepsilon_{u'} > \frac{1}{10c}\right] \leq \frac{160c^2\varepsilon}{\ell}.$$ ◁

Proof of Claim 4.7. Consider a vertex $u \in S$. Since X_u is a c-agreement expander, there exists a local function $g_u : X_u(0) \to \{0,1\}$ such that

$$\Pr_{\sigma \in X_u(d-1)}[f_{u\sigma}|_\sigma \neq g_u|_\sigma] \leq c\varepsilon_u.$$

Define the indicator function $h_u : X_u(d-1) \to \{0,1\}$ by

$$h_u(\sigma) = \begin{cases} 1 & f_{u\sigma}|_\sigma \neq g_u|_\sigma, \\ 0 & f_{u\sigma}|_\sigma = g_u|_\sigma. \end{cases}$$

By the definition of S, $\mathbb{E}_{\sigma\in X_u(d-1)} h_u(\sigma) \leq c\varepsilon_u \leq 1/5$. Thus, in order for a vertex $v \in X_u(0)$ to be in B_u, it has to deviate from its mean by more than $1/10$. By corollary 3.6

$$\operatorname*{Var}_{v\in X_u(0)} \mathop{\mathbb{E}}_{\sigma\in X_{uv}(d-2)} h_u(v\sigma) \leq \frac{8}{5(d-1)} \mathop{\mathbb{E}}_{\sigma\in X_u(d-1)} h_u(\sigma) \leq \frac{8c\varepsilon_u}{5(d-1)}.$$

Thus, by Chebyshev inequality

$$\Pr_{v\in X_u(0)}[v\in B_u \mid u\in S] \leq \Pr_{v\in X_u(0)}\left[\mathop{\mathbb{E}}_{\sigma\in X_{uv}(d-2)} h_u(v\sigma) - \mathop{\mathbb{E}}_{\sigma\in X_u(d-1)} h_u(\sigma) > \frac{1}{10}\right] \leq \frac{160c\varepsilon_u}{d-1}.$$

$\triangleleft$

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