Classification of OBDD Size for Monotone 2-CNFs

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—— Abstract ——

We introduce a new graph parameter called linear upper maximum induced matching width LU-MIM WIDTH, denoted for a graph G by $lu(G)$. We prove that the smallest size of the OBDD for φ, the monotone 2-CNF corresponding to G, is sandwiched between $2^{lu(G)}$ and $n^{O(lu(G))}$. The upper bound is based on a combinatorial statement that might be of an independent interest. We show that the bounds in terms of this parameter are best possible.

The new parameter is closely related to two existing parameters: linear maximum induced matching width (LMIM WIDTH) and linear special induced matching width (LSIM WIDTH). We prove that LU-MIM WIDTH lies strictly in between these two parameters, being dominated by LSIM WIDTH and dominating LMIM WIDTH. We conclude that neither of the two existing parameters can be used instead of LU-MIM WIDTH to characterize the size of OBDDs for monotone 2-CNFs and this justifies introduction of the new parameter.

1 Introduction

Statement of the results. Monotone 2-CNFs are CNFs with two positive literals per clause. They can be viewed as graphs without isolated vertices. In particular, for such a graph G, $\varphi = \varphi(G)$ is a CNF consisting of clauses $(u \vee v)$ for each $\{u, v\} \in E(G)$. We refer to G as the *underlying graph* of φ.

In this paper we introduce a new graph parameter called (Linear Upper Maximum Induced Matching Width) (LU-MIM WIDTH). This parameter is located 'in-between' of two existing parameters: Linear Maximum Induced Matching Width (LMIM WIDTH) [13] and Linear Special Induced Matching Width (LSIM WIDTH) [8]. We prove that LU-MIM WIDTH captures the size of Ordered Binary Decision Diagrams (OBDDs) for monotone 2-CNFs with a quasipolynomial gap. In particular, we show that $2^{lu(G)} \leq obdd(\varphi) \leq n^{O(lu(G))}$ where $obdd(\varphi)$ is the smallest number of nodes of an OBDD for a monotone 2-CNF φ and $lu(G)$ is the LU-MIM WIDTH of the underlying graph G of φ. The upper bound is based on a combinatorial statement that may be of independent interest. In particular, we exhibit a connection of this statement to the Sauer-Shelah lemma (e.g. Theorem 10.1 in [7]).

We show that the bounds are best possible by demonstrating classes of graphs G_1 and G_2 such that $obdd(\varphi(G_1)) \geq n^{\Omega(lu(G_1))}$ and $obdd(\varphi(G_2)) \leq 2^{O(lu(G_2))}$.

Finally, we prove that LU-MIM WIDTH is located strictly in between LSIM WIDTH and LMIM WIDTH. In particular, we demonstrate classes of graphs G_1 and G_2 such that
- LSIM WIDTH of G_1 is at most 3 while LU-MIM WIDTH is at least $\Omega(n^{1/3})$ and
- LU-MIM WIDTH of G_2 is at most 4 while LMIM WIDTH of G_2 is at least $\Omega(n^{1/2})$.

We conclude from the above dependencies that LSIM WIDTH cannot capture the upper bound of OBDDs for monotone 2-CNFs while LMIM WIDTH cannot capture the lower bound.

Motivation. Monotone CNFs are essentially hypergraphs while monotone 2-CNFs are essentially graphs. Therefore, it is natural to try to characterize the size of models of the corresponding Boolean functions by graph parameters. It is particularly neat if such a parameter can 'capture' the size of a model on a class of CNFs, that is to tightly characterize both upper and lower bounds. It is also desirable for the parameter to be well known as, in this case, existing techniques can be harnessed for determining the value of the parameter for the given class of graphs.

An example of such a neat capturing is characterization of the size of non-deterministic read-once branching programs (1-NBPs) representing monotone 2-CNFs $\varphi(G)$ where G has a bounded degree. In this case, considering the degree constant, the size of the smallest 1-NBP representing $\varphi(G)$ is $2^{\Theta(pw(G))}$ where $pw(G)$ is the pathwidth of G: the upper bound has been established in [3], the lower bound in terms of linear maximum matching width in [10] and it has been shown in [11] that the maximum matching width and pathwidth are linearly related. Since the upper bound of [3] holds for OBDDs, a special case of 1-NBPs, the capturing also works for OBDDs It is natural to ask whether such a capturing is possible for graphs of unbounded degree.

In this paper we address the above question *partially*. First, we obtain the result for OBDDs, a special case of 1-NBPs. Generalization to 1-NBPs is left as an open question. It is important to remark that although, for bounded degrees, the pathwidth captures the sizes of both models, for the case of unbounded degree another parameter might be needed for 1-NBPs. Second, there is a quasipolynomial gap between the upper and lower bounds. We believe that this is still reasonable because the value of the parameter provides a good indication of the size of the resulting OBDD. Besides, we show that for the considered parameter, no tighter capturing is possible. Third, we introduced a new parameter rather than using an existing one. However, this parameter is closely related to existing ones and, as mentioned above, we demonstrate that the related existing parameters cannot be used for the stated purpose.

An additional motivation of the proposed results is that they contribute to understanding the combinatorics of MIM WIDTH, a parameter becoming increasingly popular among graph algorithms researchers (see the related work part for the relevant references).

Related work. Here we overview related results that have not been mentioned in the earlier parts of the introduction.

The size of Decomposable Negation Normal Forms (DDNFs) for monotone 2-CNFs of bounded degree is captured by treewidth. In particular an FPT upper bound for CNFs of bounded (primal) treewidth is proved in Theorem 16 of [2]. A matching lower bound for CNFs of bounded arity and bounded number of variable occurrences follows from the combination of Theorem 8.3 and Lemma 8.4. [1] [1]

A lower bound for OBDDs for monotone CNFs is established in [1]. For 2-CNFs, the lower bound is $2^{\Omega(pw(G)/d^2)}$ where $pw(G)$ and d are the pathwidth and the max-degree of G. The lower bound provided in this paper is better because $lu(G) = \Omega(pw(G)/d)$ due to pathwidth and linear maximum matching width being linearly related [11]. The proof of the $n^{O(lu(G))}$ upper bound is similar in spirit to Lemma 1 of [12].

The MIM-WIDTH [13] has proven useful for design of efficient algorithms for intractable problems for restricted classes of graphs, see for example the recent series of papers [5], [6],[4]. Lower bounds of MIM-WIDTH for several graph classes have been established in [9].

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[1] I would like to thank Florent Capelli for pointing the result out to me.

Structure of the paper. Section 2 introduces the necessary background. Section 3 introduces the LU-MIM WIDTH parameter. Section 4 proves upper and lower bounds on the OBDD size. Section 5 proves that, in terms of the parameter, the bounds are essentially tight. Section 6 justifies the introduction of a new parameter by showing that neither LMIM-WIDTH nor LSIM can be used for the purpose of capturing the OBDD size for monotone 2-CNFs. Finally, Section 7 outlines directions of further research.

2 Preliminaries

A *literal* is a Boolean variable or its negation. Throughout this paper, when we refer to a set S of literals, we mean that S is a *proper* set of literals, that is a variable cannot occur in S along with its negation. The set of variables occurring in S is denoted by $Var(S)$. A variable $x \in Var(S)$ can occur in S either *positively*, if $x \in S$ or *negatively*, if $\neg x \in S$. We can also call S an assignment (to $Var(S)$ if the clarification is needed).

We view a *Conjunctive Normal Form* (CNF) as a set of *clauses* and each clause is just a proper set of literals. An assignment S *satisfies* a clause C if $S \cap C \neq \emptyset$. An assignment *satisfies* a CNF φ if $S \cap C \neq \emptyset$ for each $C \in \varphi$. For an assignment S, the CNF $\varphi|_S$ is obtained from φ by removal of all the clauses satisfied by S and removal the occurrences of $Var(S)$ from each remaining clause. We denote by $Var(\varphi)$ the set of all variables occurring in the clauses of φ. Customarily, $|Var(\varphi)|$ is denoted by n.

For a CNF φ, $U \subseteq Var(\varphi)$, let $\mathbf{A}(U) = \mathbf{A}_\varphi(U)$ be the set of all assignments S to U that can be extended to a satisfying assignment of φ. We denote by $\mathbf{BF}(U) = \mathbf{BF}_\varphi(U)$ the set of all Boolean functions represented by $\varphi|_A$ for $A \in \mathbf{A}(U)$.

▶ **Example 1.** Let $\varphi = (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_2 \vee x_4)$. Let $U = \{x_1, x_2\}$. Then $\mathbf{A}(U) = \{\{x_1, x_2\}, \{x_1, \neg x_2\}, \{\neg x_1, x_2\}\}$. Note that $\{\neg x_1, \neg x_2\}$ is not included in $\mathbf{A}(U)$ because the assignment falsifies the clause $(x_1 \vee x_2)$ and hence cannot be extended to a satisfying assignment of φ. Then $\mathbf{BF}(U)$ is the set of functions on x_3, x_4 represented by the following set of CNFs $\{(x_3 \vee x_4), (x_3), (x_4)\}$.

An *Ordered Binary Decision Diagram* (OBDD) is a popular model for representation of Boolean functions. For the purpose of this paper, we do not need a formal definition of OBDDs because the only fact about OBDDs we use is Proposition 3 but we provide a definition for the sake of completeness.

▶ **Definition 2.** *An OBDD Z is a directed acyclic graph (DAG) with one source and two sinks. Each non-sink vertex has exactly two outgoing neighbours. The vertices and edges of Z are labelled in the way specified below.*

Each non-sink vertex is labelled with a variable, one of the sinks is labelled with $true$, the other is labelled with $false$. Let u be a non-sink node of Z labelled with a variable x. Then one outgoing edge of u is labelled with the positive literal of x, that is x, the other is labelled with the negative literal of x, that is $\neg x$.

The labelling of non-sink nodes also needs to observe two principles: being read-once *and being* ordered*. The read-once property means that in any directed path P of Z the labels of all the non-sink nodes of P are distinct (no variable occurs twice). Being ordered means that there is a permutation $\pi(Z)$ of the variables labelling the nodes of Z so that for any path P from a non-sink node u to a non-sink node v the label of u precedes in $\pi(Z)$ the label of v.*

For a directed path P of Z, we denote by $A(P)$ the set of literals labelling the edges of P. Let $x_1, \ldots, x_n$ be the variables labelling the nodes of Z. The function f_Z represented by Z is defined as follows. Let S be a set of literals with $Var(S) = \{x_1, \ldots, x_n\}$. Then f_Z is true on S if and only if Z has a path P from the source to the true sink such that $A(P) \subseteq S$.

We refer the reader to [14] for an extensive study of OBDDs. For the results of this paper, we only need bounds on $obdd(\varphi)$, the smallest OBDD size (the number of vertices) for a CNF φ as in the next statement that follows from Theorem 3.1.4 of [14].

▶ **Proposition 3.**
1. *Suppose that for each permutation π of $Var(\varphi)$ there is a prefix π' of π [2] such that $|\mathbf{BF}(\pi')| \geq m$. [3] Then $obdd(\varphi) \geq m$.*
2. *Assume that there is a permutation π of $Var(\varphi)$ such that for every prefix π' of π, $|\mathbf{BF}(\pi')| \leq m$. Then $obdd(\varphi) = O(n * m)$.*

In case of OBDDs representing monotone 2-CNFs upper and lower bounds can be stated in graph theoretical terms as described below. We follow a standard graph-theoretical notation. In particular $G[U]$ denotes the subgraph induced by $U \subseteq V(G)$. $N(U)$ is the set of all neighbours of vertices of U excluding U, the considered graph may be added as a subscript if not clear from the context. The CNF $\{(u \vee v)|\{u,v\} \in E(G)\}$ is denoted by $\varphi(G)$.

▶ **Definition 4.** *Let $U \subseteq V(G)$. We denote by $\mathbf{ISET}(U)$ the family of all the independent subsets of U. Let $V = V(G) \setminus U$. We define $\mathbf{TRACES}(U) = \{N(S) \cap V|S \in \mathbf{ISET}(U)\}$. The subscript G can be used for $\mathbf{TRACES}(U)$ and $\mathbf{ISET}(U)$ if the graph in question is not clear from the context.*

▶ **Example 5.** Let G be a graph with vertices x_1, x_2, x_3, x_4 and edges $\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}$. (This is the graph corresponding to the CNF considered in Example 1.) Let $U = \{x_1, x_2\}$. Then $\mathbf{ISET}(U) = \{\emptyset, \{x_1\}, \{x_2\}\}$, $\mathbf{TRACES}(U) = \{\emptyset, \{x_3\}, \{x_4\}\}$.

Combination of Examples 1 and 5 illustrates that $\mathbf{TRACES}_G(U)$ and $\mathbf{BF}_\varphi(U)$ are of the same size where $\varphi = \varphi(G)$. The following lemma shows that this is not a coincidence.

▶ **Lemma 6.** *Let $\varphi = \varphi(G)$. Then $|\mathbf{BF}(U)| = |\mathbf{TRACES}(U)|$.*

Proof. It is not hard to see that $\mathbf{A}(U) = \{A(S)|S \in \mathbf{ISET}(U)\}$ where $A(S)$ is an assignment on U where all the elements of S occur negatively and the rest occur positively. Furthermore, it is not hard to see that $\varphi|_{A(S)}$ is a CNF of the form $\{(u)|u \in N(S) \cap V\} \cup \{(u \vee v)|\{u,v\} \in E(G[V])\}$ (which is equivalent to $\{(u)|u \in N(S) \cap V\} \cup \{(u \vee v)|\{u,v\} \in E(G[V \setminus N(S)])\}$). It follows that for $S_1, S_2 \in \mathbf{ISET}(U)$ and $N(S_1) \cap V = N(S_2) \cap V$, $\varphi|_{A(S_1)} = \varphi|_{A(S_2)}$. Conversely, we need to show that if $N(S_1) \cap V$ and $N(S_2) \cap V$ are distinct then so are the functions of $\varphi|_{A(S_1)}$ and $\varphi|_{A(S_2)}$. Assume w..l.o.g. the existence of $v \in (N(S_1) \cap V) \setminus (N(S_2) \cap V)$ This means that v occurs positively in all satisfying assignments of $\varphi_{A(S_1)}$ but can occur negatively in $\varphi_{A(S_2)}$: just assign positively the rest of the variables. ◀

Finally, we need one more definition.

▶ **Definition 7.** *Let $U, V \subseteq V(G)$. A (U, V)-matching is a matching of G consisting of edges with one end in U and the other in V. Let M be such a matching. We denote by $U(M)$ the set of ends of the edges of M that belong to U. Let S be an independent subset of U. We say that S enables an induced (U, V) matching if there is an induced (U, V)-matching M with $U(M) = S$.*

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[2] We think of the permutation as a linear order of $Var(\varphi)$ so a prefix is naturally defined.
[3] Here and in several other places we slightly abuse the notation by using a sequence as a set. The correct use will always be clear from the context.
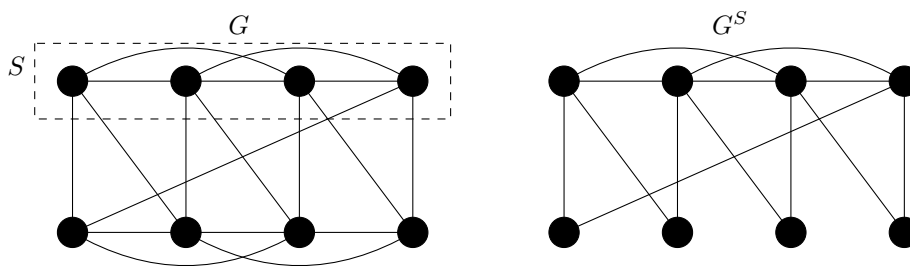
3 Linear upper induced matching width

In this section we introduce the parameter of *Linear Upper Maximum Induced Matching Width* (LU-MIM WIDTH). In order to present the parameter in the right context we compare it with two existing parameters: Linear Maximum Induced Matching Width (LMIM WIDTH) and Linear Special Induced Matching Width (LSIM WIDTH).

The definition of all three parameters follows the same pattern. First, we fix a permutation $\pi = (v_1, \ldots, v_n)$, denote each $\{v_1, \ldots, v_i\}$ by V_i and define the width of the prefix $(v_1, \ldots, v_i)$ as the largest size of an induced $(V_i, V(G) \setminus V_i)$-matching of some subgraph of G. The difference between the above three parameters is in the choice of the subgraph. The rest of the definition is identical for all the three parameters and also pretty standard: the width of π is the largest width among all the prefixes of π and the width of G is the smallest width among all the permutations.

To define the width of a permutation prefix for LU-MIM WIDTH, we need the notion of an *upper subgraph* introduced in the definition below.

▶ **Definition 8.** *Let $U \subseteq V(G)$ and $V = V(G) \setminus U$. The upper subgraph G^U of G w.r.t. U is a spanning subgraph of G with $E(G^U) = E(G) \setminus E(G[V])$.*

In words, G^U is obtained from G by removal of all the edges whose both ends are outside of U. See Figure 1 for an illustration of this notion.

■ **Figure 1** An example of an upper subgraph.

▶ **Definition 9** (LU-MIM WIDTH). *Let $\pi = (v_1, \ldots, v_n)$ be a permutation of $V(G)$ and denote $\{v_1, \ldots, v_i\}$ by V_i. Let r_i be the size of the largest induced $(V_i, V(G) \setminus V_i)$-matching of G^{V_i}. Let $r(\pi) = max_{i=1}^n r_i$. The Linear Upper Induced Matching Width (LU-MIM WIDTH) of G denoted by $lu(G)$ is the smallest $r(\pi)$ over all the permutations π of $V(G)$. We call a permutation π such that $r(\pi) = lu(G)$ a witnessing permutation for $lu(G)$.*

▶ **Example 10.** In the graph G of Figure 1, consider the permutation π first traversing all the top vertices from the left to the right and then all the bottom vertices from the left to the right. Let V_i be the set of all the top vertices (denoted by S in the picture). It is not hard to see that the largest induced $(V_i, V(G) \setminus V_i)$- matching of G^{V_i} is of size 1. The widths of the rest of the prefixes are also at most 1, So, $r(\pi) = 1$. Since the graph is connected, any permutation will have width at least one. So, we conclude that $lu(G) = 1$.

The parameter LU-MIM WIDTH can be considered as lying between existing parameters LMIM WIDTH and LSIM WIDTH. In particular, to compute the width of a prefix V_i for LMIM WIDTH, the edges having both ends in V_i are discarded along with the edges having both of their ends out of V_i. In Example 10, with $V_i = S$, only the edges between the top and the

bottom vertices remain, so the largest size of an induced $(V_i, V(G) \setminus V_i)$- matching of the resulting graph becomes 2. For LSIM WIDTH, no edges are discarded at all, so the width of V_i is the largest induced $(V_i, V(G) \setminus V_i)$-matching for the whole G.

It is clear that for any graph G, its LSIM WIDTH is smaller than or equal to its LU-MIM WIDTH which, in turn, is smaller than or equal to its LMIM WIDTH. For the latter two we can, in fact, demonstrate a class of graphs where LU-MIM WIDTH is bounded while LMIM WIDTH unbounded but we leave the exact relationship between the former two as an open question. We postpone to Section 6 a more detailed discussion of relationship between the parameters as well as justifying the need of the new parameter for bounding the size of OBDDs. The reason of this arrangement is that we need first to prove the main results of the paper so that we can refer to them for the purpose of the justification.

4   OBDD bounds in terms of LU-MIM width

In this section we establish upper and lower bounds on the size of OBDDs representing monotone two CNFs. The upper bound is the more interesting of these two because it is based on the following combinatorial statement.

▶ **Theorem 11.** *Let $U \subseteq V(G)$ such that $V = V(G) \setminus U$ is independent. Then $|\mathbf{TRACES}(U)| \leq n^{r+1}$ where $n = |V(G)|$ and r is the size of the largest induced (U, V)-matching.*

Before we provide a proof of Theorem 11, let us remark that if U is independent (that is G is a bipartite graph with U and V being its parts) then the statement follows from Sauer-Shelah lemma. This is just because, in this case, the size of the largest induced matching of G is exactly the VC-dimension of $\mathbf{TRACES}(U)$. Indeed, let $W = \{w_1, \dots, w_q\}$ be a set of the largest size shattered by $\mathbf{TRACES}(U)$. Then we can identify subsets $U_1, \dots, U_q$ such that $N(U_i) \cap W = \{w_i\}$ for $1 \leq i \leq q$. In particular, in each U_i we can identify a vertex u_i such that u_i is adjacent to w_i but not adjacent to any other vertex of W. Consequently, the edges $\{u_1, w_1\}, \dots, \{u_q, w_q\}$ constitute an induced matching. Conversely, let $\{u_1, w_1\}, \dots, \{u_q, w_q\}$ be an induced matching. Then the set $\{w_1, \dots, w_q\}$ is shattered by neighborhoods of all possible subsets of $\{u_1, \dots u_q\}$. Hence the VC dimension of $\mathbf{TRACES}(U)$ is at least q.

If U is not an independent set, the first part of the above reasoning does not work. Indeed, the vertices $u_1, \dots, u_q$ extracted from $U_1, \dots, U_q$ do not necessarily form an independent set and hence the resulting matching is not necessarily induced. We were unable to upgrade the above argument to prove Theorem 11 and hence we provide a self-contained proof.

Proof of Theorem 11.

▷ **Claim 12.**   Let $S \subseteq U$ be an independent subset of U. Let $u \in S$. Suppose that $S \setminus \{u\}$ enables an induced (U, V)-matching while S does not. Then there is a subset $S' \subset S$ enabling an induced (U, V)-matching such that $N(S') \cap V = N(S) \cap V$.

Proof. By induction on $|S|$. For $|S| = 1$ the statement holds in a vacuous way. For each $u' \in S$ let $T(u') = (N(u') \cap V) \setminus (N(S \setminus \{u'\}) \cap V)$ be called the *individual trace* of u'. Suppose all the individual traces are non-empty. For all u' fix an arbitrary $v' \in T(u')$. Then $\{\{u', v'\} | u' \in S\}$ is an induced matching (recall that V is independent) contradicting our assumption. It follows that there is $u' \in S$ such that $T(u') = \emptyset$. But then $N(S) \cap V \subseteq N(S \setminus \{u'\}) \cap V$ and hence $N(S) \cap V = N(S \setminus \{u'\}) \cap V$. If $S' = S \setminus \{u\}$ enables an induced (U, V)-matching, we are done. Otherwise, apply the induction assumption to S'.   ◁

▷ **Claim 13.** Let $S \subseteq U$ be an independent subset of U. Then there is $S' \subseteq S$ enabling an induced (U, V)-matching such that $N(S) \cap V = N(S') \cap V$.

Proof. Let q be the size of the largest subset of S enabling an induced (U, V)-matching. We proceed by induction on $|S| - q$. If it is zero then put $S' = S$. Otherwise, let S_0 be a subset of S of size q enabling an induced (U, V) matching and let $u \in S \setminus S_0$. By Claim 12, there is $S_1 \subset S_0 \cup \{u\}$ enabling an induced (U, V) matching such that $N(S_1) \cap V = N(S_0 \cup \{u\}) \cap V$.

Let $S_2 = S \setminus (S_0 \cup \{u\})$. Then

$$N(S_1 \cup S_2) \cap V = (N(S_1) \cap V) \cup (N(S_2) \cap V) =$$
$$(N(S_0 \cup \{u\}) \cap V) \cup (N(S_2) \cap V) = N(S_0 \cup \{u\} \cup S_2) \cap V = N(S) \cap V \quad (1)$$

Further on, $S_1 \cup S_2$ has a subset of size at least $|S_1|$ enabling an induced (U, V)-matching. But $|S_2 \cup S_1| - |S_1| = |S_2| = |S| - q - 1$. Apply the induction assumption to $S_1 \cup S_2$ to find a subset $S_3 \subseteq S_1 \cup S_2$ enabling an induced (U, V) matching such that $N(S_3) \cap V = N(S_1 \cup S_2) \cap V$. Since $S_1 \cup S_2 \subseteq S$, $S_3 \subseteq S$ and $N(S_3) \cap V = N(S) \cap V$ by (1), we put $S' = S_3$. □ ◁

By assumption an independent subset of U enabling an induced (U, V) matching is of size at most r. It follows from Claim 13 that $\mathbf{TRACES}(U) = \{N(S) \cap V | S \in \mathbf{ISET}(U), |S| \leq r\}$. Clearly the size of the right-hand set is upper bounded by the number of subsets of U of size at most r which is clearly upper bounded as claimed in the theorem. ◀

▶ **Theorem 14** (OBDD bounds). *For $\varphi = \varphi(G)$, $2^{lu(G)} \leq obdd(\varphi) \leq n^{O(lu(G))}$.*

Proof. Let $\pi = (v_1, \ldots, v_n)$ be a permutation of $V(G)$ witnessing $lu(G)$. Let V_i and r_i be as in Definition 9. By combination of Lemma 6 and Theorem 11, $|\mathbf{BF}(V_i)| \leq n^{r_i+1} \leq n^{lu(G)+1}$. The upper bound follows from the second statement of Proposition 3.

For the lower bound we assume now that $\pi = (v_1, \ldots, v_n)$ is an arbitrary permutation with the meaning of V_i and r_i retained. Furthermore, we assume that $(v_1, \ldots, v_i)$ is selected so that $r_i \geq lu(G)$ (such a prefix exists by definition of LU-MIM WIDTH). We are going to show that $|\mathbf{TRACES}(V_i)| \geq 2^{r_i}$. The lower bound will then follow from combination of Lemma 6 and the first statement of Proposition 3.

Let $U^* = \{u_1, \ldots, u_{r_i}\}$ be a subset of V_i enabling an induced $(V_i, V(G) \setminus V_i)$- matching of G^{V_i} of size r_i and let $M = \{\{u_1, v_1\}, \ldots, \{u_{r_i}, v_{r_i}\}\}$ be the edges of this matching. Let U_1, U_2 be two distinct subsets of U^*. We claim that $N(U_1) \cap (V(G) \setminus V_i) \neq N(U_2) \cap (V(G) \setminus V_i)$. Indeed, assume w.l.o.g. that there is $u_j \in U_1 \setminus U_2$. Then $v_j \in N(U_1) \cap (V(G) \setminus V_i)$ while $v_j \notin N(U_2) \cap (V(G) \setminus V_i)$. Thus 2^{r_i} subsets of U^* have pairwise distinct neighborhoods in V witnessing that $|\mathbf{TRACES}(V_i)| \geq 2^{r_i}$. ◀

5 No tighter bounds

We are now going to prove that the bounds in the statement of Theorem 14 are asymptotically best possible. This will imply that the quasypolynmial gap between the upper and lower bounds cannot be narrowed down. For the lower bound the proof will be straightforward. For the upper bound we will need a 'gadgeted' construction developed below.

▶ **Definition 15.** *Let $U = (u_1, \ldots, u_q)$, $V = (v_1, \ldots, v_q)$. The graph $SKEW(U, V)$ over vertices $\{u_1, \ldots, u_q, v_1, \ldots, v_q\}$ has the set of edges $\{\{u_i, v_j\} | i \leq j\}$.*

Let $U_1, \ldots, U_p$ be mutually disjoint sequences of q elements. We define a graph G over $U_1 \cup \cdots \cup U_p$ (here we interpret U_i as sets) as follows. For each $1 \leq i \leq p-1$, $G[U_i \cup U_{i+1}]$ is $SKEW(U_i, U_{i+1})$. We call G a p, q-path of skewed graphs. We call $U_1, \ldots, U_p$ the sequence of layers of G and give them numbers $1, \ldots, p$ in the order listed.

▶ **Definition 16.** *Let P be a path. The 1-subdivision of P is the graph obtained by introducing exactly one subdivision to each edge of P.*

▶ **Definition 17.** *Let $G^1, \ldots, G^r$ be p, q-paths of skewed graphs with respective sequences $(U_1^1 \ldots, U_p^1), \ldots, (U_1^r, \ldots, U_p^r)$ of layers. Connect the vertices of each $U_i^1 + \cdots + U_i^r$ into a path P in the order specified and 1-subdivide the resulting path. Let G be the resulting graph. We call G a p, q, r-grid of skewed graphs (we may omit the parameters if they are not relevant in the context).*

The vertices $V(G^1) \cup \cdots \cup V(G^r)$ are referred to as the main vertices *and the vertices introduced by the 1-subdivision are the* auxiliary vertices. *The subdivided paths are referred to as the* layers *of G with the i-th layer being the one containing $U_i^1, \ldots, U_i^r$ as the main vertices. Let us enumerate the main vertices of each layer i as in the sequence $U_i^1 + \ldots, +U_i^r$ starting from 1. The number each vertex receives is the* coordinate *of this vertex.*
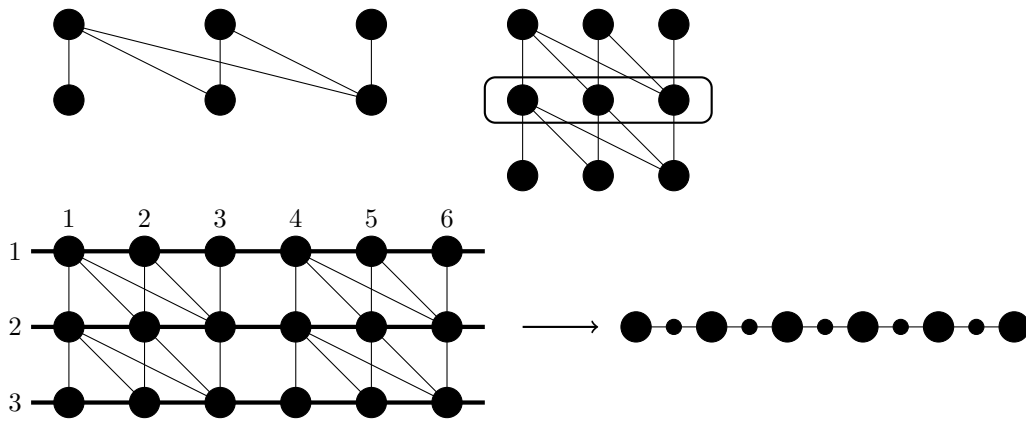
▶ **Figure 2** A grid of skewed graphs.

Figure 2 demonstrates a grid of skewed graphs. The top-left graph is $SKEW(U, V)$ where U is the sequence of three vertices on the top enumerated from the left to the right and V is the respective sequence of the bottom vertices. The graph on the top-right is a $3, 3$-path of skewed graphs. The graph has three layers enumerated from the top to the bottom. The vertices of the second layer are surrounded by the oval. The graph at the bottom-left is the $3, 3, 2$-grid of skewed graphs. For the sake of a better visualization, the auxiliary vertices are not shown and the layers are denoted by thick lines, the meaning of a thick line is specified on the bottom right of the picture. The grid has three layers and the coordinates of the main vertices range from 1 to 6 as specified in the picture.

▶ **Definition 18.** *Let G be an p, q, r-grid of skewed graphs. Let U be a set of main vertices one of each coordinate and none belonging to the last layer. For each vertex $u_1 \in U$ let u_2 be the vertex of the same coordinate lying at the next layer. Let V be the set of all vertices u_2. Let H be the subgraph of G induced by $U \cup V$. We call H a* horizontal *subgraph of G. We call U, V the* top and bottom forming sets *of H. Let M be the matching consisting of all the edges $\{u_1, u_2\}$ as above. We call M the* core matching *of H. $U \cup V$ is partitioned into r* main intervals $1, \ldots, r$ *where vertices of the i-th main interval are those having coordinates $(i-1) * q + 1, \ldots, i * q$.*

▶ **Lemma 19.** *With the notation as in Definition 18, both $|\mathbf{TRACES}_H(U)| \geq (q+1)^r$ and $|\mathbf{TRACES}_H(V)| \geq (q+1)^r$*

Proof. We prove only the first statement, the second is symmetric.

Let $H_1, \ldots, H_r$ be the subgraphs of H induced by the respective main intervals $1, \ldots, r$. For each H_i denote $V(H_i) \cap U$ and $V(H_i) \cap V$ by U_i and V_i, respectively.

It is not hard to see that H is the disjoint union of $H_1, \ldots, H_r$, hence $|\textbf{TRACES}_H(U)| = \prod_{i=1}^{r} |\textbf{TRACES}_{H_i}(U_i)|$. It is thus sufficient to prove that for each i, $|\textbf{TRACES}_{H_i}(U_i)| \geq q + 1$. W.l.o.g. we only prove that $|\textbf{TRACES}_{H_1}(U_1)| \geq q + 1$.

For $U' \subseteq U_1$, let $first(U')$ be the vertex $u' \in U'$ located at the layer having the largest number (among the vertices of U'), and, among those vertices of U' located at the layer, having the smallest coordinate. Let $u_1 = first(U_1)$ and for $2 \leq i \leq q$, $u_i = first(U_1 \setminus \{u_1, \ldots, u_{i-1}\})$. Let $v_1, \ldots, v_q$ be the other ends of the edges of M (the core matching of H) incident to $u_1, \ldots, u_q$, respectively. Observe that H has no edge $\{u_i, v_j\}$ such that $i > j$. Indeed, otherwise, either the layer of u_j is smaller than the layer of u_i or the coordinate of u_j is larger than the coordinate of u_i, both cases contradict the choice of vertices $u_1, \ldots, u_q$.

Consider the sets $W_1, \ldots W_{q+1}$ such that $W_{q+1} = \emptyset$ and $W_j = \{u_j\}$ for $1 \leq j \leq q$. It follows that for each $1 \leq j \leq q$, $v_j \in N(W_j) \cap V_1$ and $v_j \notin N(W_k) \cap V_1$ for $k > j$. It follows that the sets $N(W_1) \cap V_1, \ldots, N(W_{q+1}) \cap V_1$ are all distinct thus confirming that $|\textbf{TRACES}_{H_1}(U_1)| \geq q + 1$. ◀

▶ **Lemma 20.** *Let G be a p, q, r-grid of skewed graphs where $p > 1, q > 1, r \geq 1$, and $p = 2 * r \lceil \log q \rceil$. Let $n = V(G)$. Then for $\varphi = \varphi(G)$, $obdd(\varphi) \geq n^{r/2}$ for a fixed r and a sufficiently large n.*

Proof. We prove the q^r lower bound instead of $n^{r/2}$. Indeed $n^{r/2} \leq (2 * q * p * r)^{r/2} = q^{r/2} * (4r^2 \lceil \log q \rceil)^{r/2} \leq q^r$ for a fixed r and a sufficiently large q. We consider an arbitrary permutation π and show existence of a prefix π' such that $|\textbf{TRACES}(\pi')| \geq q^r$. The lemma will then follow from the combination of the first statement of Proposition 3 and Lemma 6.

Let π' be the shortest prefix of π containing all the vertices of some layer x. Assume existence of a layer y none of which vertices are contained in π'. Assume that $y > x$. For each coordinate i, specify main vertices u_i, v_i both having coordinate i with the layer of v_i being the next after the layer of u_i and such that $u_i \in \pi'$ while $v_i \notin \pi'$. To see that such vertices exist, start from the main vertex with coordinate i at layer x and iteratively move down. Since the respective vertex of coordinate i at y is not in π', the required vertices u_i, v_i will eventually be found. The set $\{u_1, \ldots, u_{qr}, v_1, \ldots, v_{qr}\}$ induce a horizontal subgraph H of G with $U = \{u_1, \ldots, u_{qr}\}$ being the top set. Then $|\textbf{TRACES}(\pi')| \geq |\textbf{TRACES}_H(U)| \geq (q + 1)^r$, the last inequality follows from Lemma 19.

If $y < x$, the reasoning is symmetric and we use the second statement of Lemma 19 rather than the first one. It remains to assume that at least one vertex of each layer of G is contained in π'. Remove from π' the last vertex and let π^* be the resulting prefix. By definition of π', in each layer of G there is at least one vertex inside π^* and at least one vertex outside π^*. Since layers induce connected subgraphs of G, we can identify edges $\{u_i, v_i\}$ of G for $1 \leq i \leq p$ with u_i, v_i located at layer i, u_i is contained in π^* while v_i is not. We notice that the edges $\{u_i, v_i\}$ with odd indices form an induced matching. Indeed, in G two vertices are adjacent only if they are in the same layer or in consecutive layers. Let U be the set of vertices u_i with i being odd. Applying the argument as in the lower bound proof for Theorem 14, we observe that the neighborhoods of the subsets of U in $V(G) \setminus \pi^*$ are pairwise distinct. Taking into account the definition of p and that $|U| = p/2$ by construction, we conclude that $\textbf{TRACES}(\pi^*) \geq 2^{|U|} = 2^{p/2} \geq q^r$. ◀

▶ **Lemma 21.** *With the notation as in Lemma 20, $r \leq lu(G) \leq r + 2$.*

Proof. For the lower bound we argue as in Lemma 20. Recall that for an arbitrary permutation π we considered two cases. In the first case we observed existence of a prefix π' such that there is a horizontal subgraph H of G with all vertices of the top forming set contained in π' and all vertices of the bottom forming set being outside of π' (or vice versa). For each main interval take one edge of the core matching whose vertex coordinates belong to this interval. These edges, taken together constitute an induced matching of G of size r.

If prefix π' as above does not exist then there is a prefix π^* such that for each layer $1 \leq i \leq p$ there is an edge $\{u_i, v_i\}$ with $u_i \in \pi^*$ and $v_i \notin \pi^*$. As we have observed the edges with odd indices comprise an induced matching of G of size at least $r \log q > r$.

For the upper bound, we consider a permutation π where vertices occur layer by layer: first layer 1, then layer 2 and so on. Within each layer the vertices occur along the path induced by the layer starting from the main vertex with coordinate 1.

Consider a prefix π' of π. Let x be the largest layer number (some of) whose vertices are contained in π'. By definition of π all the vertices whose layer numbers are smaller than x belong to π'. It follows that the edges between π' and $V(G) \setminus \pi'$ belong to one of the following categories.

1. Edges between layer x and layer $x+1$. Suppose that π' contains vertices of layer x laying in intervals $1, \ldots r'$. Then this category of edges can contribute at most r' edges to an induced matching of $G^{\pi'}$.
2. Edges between layer $x-1$ and layer x. This category of edges can contribute at most $r - r' + 1$ edges to the induced matching (the extra one is on the account that not all vertices of interval r' and layer x may be present in π') so there may be an edge of vertices of interval r' between layers $x-1$ and x contributing to the considered induced matching.
3. Edges with both ends in layer x. Since π' contains an initial fragment of the path of layer x, there may be at most one such an edge.

Summing up the above three items, we conclude that the size of induced matching of $G^{\pi'}$ cannot be greater than $r+2$.   ◀

▶ **Theorem 22** (best possible bounds). *For every fixed $r \geq 1$ there are infinite classes $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs of LU-MIM WIDTH $\Theta(r)$ and such that for each $G_1 \in \mathcal{G}_1$, $obdd(\varphi(G_1)) \leq 2^{O(r)}$ while for each $G_2 \in \mathcal{G}_2$, $obdd(\varphi(G_2)) \geq n^{\Omega(r)}$.*

Proof. Let $\mathcal{G}_1$ be the set of all $p \times r$ grids for a sufficiently large p. Each graph of this class has pathwidth of $\Theta(r)$ and hence the OBDD size is at most $2^{O(r)}$ by [3]. Because of the bounded degree, the pathwidth and the LU-MIM WIDTH of graphs in $\mathcal{G}_1$ are linearly related. Hence, we conclude that for each $G_1 \in \mathcal{G}_1$, $obdd(\varphi(G_1)) = 2^{O(lu(G))}$. Let $\mathcal{G}_2$ be the class of p, q, r grids for a sufficiently large q and $p = 2 * r\lceil \log q \rceil$. The required properties are immediate from the combination of Lemma 20 and Lemma 21.   ◀

6   Why is the new parameter needed

In this section we justify the need for the new parameter of LU-MIM WIDTH. In particular, we explain why we cannot use two existing closely related parameters: LMIM WIDTH and LSIM WIDTH. For the sake of completeness, let us define the latter two parameters.

▶ **Definition 23.** *Let $\pi = (v_1, \ldots, v_n)$ be a permutation of $V(G)$. Denote $\{v_1, \ldots, v_i\}$ by V_i. Let x_i be the largest size of an induced matching of $G[V_i, V(G) \setminus V_i]$ which is the graph induced by the edges between V_i and $V(G) \setminus V_i$. Let y_i be the largest size of an induced $(V_i, V(G) \setminus V_i)$-matching of G. Let $x(\pi)$ be the maximum of all x_i and let $y(\pi)$ be the maximum of all y_i. The Linear Maximum Induced Matching Width (LMIM WIDTH) of G*

denoted by $lmimw(G)$ is the minimum $x(\pi)$ over all permutations π of $V(G)$. The Linear Special Induced Matching Width (LSIM WIDTH) of G denoted by $lsimw(G)$ is the minimum $y(\pi)$ over all the permutations π of $V(G)$.

The parameter LMIM WIDTH cannot be used for our purposes because it does not capture the lower bound for OBDDs representing monotone 2-CNFs. In particular, below we demonstrate an infinite class of graphs having LMIM WIDTH of order of the square root of the number of vertices whose corresponding CNFs can be represented by polynomial size OBDDs.
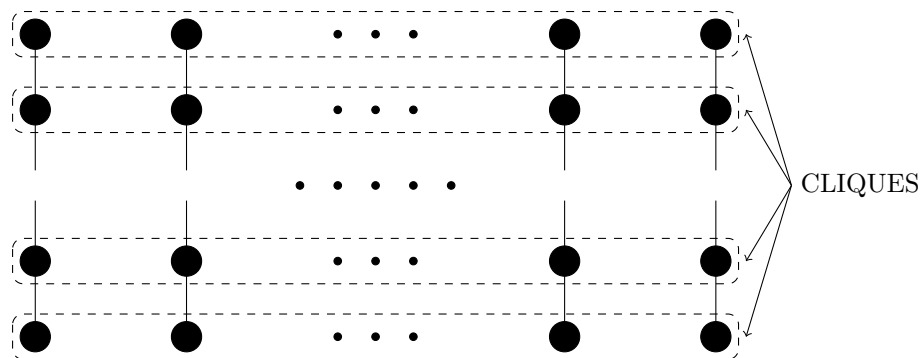

Figure 3 Schematic illustration of graphs H_n.

▶ **Theorem 24.** *For each integer $r \geq 2$, there is an infinite class of graphs H_r of $n = r^2$ vertices such that $lu(H_r) = 2$ (and hence $\varphi(H_r)$ can be represented by an OBDD of size at most $n^{O(1)}$ by Theorem 14) while $lmimw(H_r) \geq (r-1)/2$.*

Proof. $V(H_r)$ consists of disjoint union of sets $V_1, \ldots, V_r$ of r vertices each. Each V_i is a clique in H_r. Denote the vertices of each V_i by $v_{i,1}, \ldots, v_{i,r}$. The graph H_r has paths $v_{1,i}, \ldots, v_{r,i}$ for each $1 \leq i \leq r$. $E(H_r)$ contains no other edges besides those specified above. Figure 3 schematically illustrates the graphs H_r.

To demonstrate that $lu(H_r)$ is small, consider the permutation of $V(H_r)$ by the alphabetic ordering of their indices, that is, $v_{1,1}, \ldots, v_{1,r}, \ldots, v_{r,1}, \ldots, v_{r,r}$. Let V' be the set of vertices of a prefix of this permutation. Let $1 \leq q \leq r$ be such that $V_q \cap V' \neq \emptyset$ while for each $q < i \leq r$, $V_i \cap V' = \emptyset$. It follows that for each $1 \leq i < q$, $V_i \subseteq V'$. Hence, by construction, any edge connecting V' to $V(H_r) \setminus V'$ has an end either in V_q or in V_{q-1}. Thus for any three edges between V' and $V(H_r) \setminus V'$ either two of them have an end in V_q or two of them have an end in V_{q-1}. In both cases these ends are connected by an edge with both ends lying in V' and hence such edges cannot constitute an induced matching of $H_r^{V'}$. We conclude that the largest possible size of the such an induced matching is 2. It follows from Theorem 14 that $\varphi(H_r)$ can be represented by an OBDD of size upper bounded by $n^{O(1)}$.

Let us know establish an $\Omega(r) = \Omega(n^{1/2})$ lower bound on $lmimw(H_r)$. For vertices $v_{i,j}$ of H_r let us call their first coordinates *rows* and their second coordinates *columns*. Let π be an arbitrary permutation of $V(H_r)$. Let π_0 be the longest prefix of π that does not contain vertices with all the row coordinates and does not contain vertices with all the column coordinates. Since this is already not the case for the immediate successor of π_0, it is either that π_0 contains vertices with $r-1$ row coordinates or vertices with $r-1$ column coordinates. Assume the former. Then there is a set I of $r-1$ rows i such that π_0 contains some $v_{i,j}$. By assumption, for each $i \in I$, there is some q such that $v_{i,q} \notin \pi_0$. Since each V_i is connected we can identify, for each $i \in I$ an edge $\{v_{i,j_1}, v_{i,j_2}\}$ such that one end of this edge is in π_0 while

the other end is outside. At least half of such edges have the same parity of the row. Let M be a such a subset of edges. By definition of H_r vertices with the same row parity are not adjacent hence this matching is induced in H_r and of size at least $(r-1)/2$ by definition.

It remains to assume that there is a set I of $r-1$ columns j such that there is a vertex $v_{i,j} \in \pi_0$. By assumption, at least one vertex of $v_{1,j}, \ldots, v_{r,j}$ does not belong to π_0. As $v_{1,j}, \ldots, v_{r,j}$ induce a path, there is an edge $\{v_{i,j}, v_{i+1,j}\}$ such that one end belong to π_0 while the other end is outside. Let E' be a set of such edges one per column of I. For an edge $\{v_{i,j}, v_{i+1,j}\}$ of E' we call its end that belongs to π_0 the *inner end* and the other one the *outer end*. We call the edge *even* if the row of the inner end is even and *odd* otherwise. Clearly at least $(r-1)/2$ edges of E' have the same parity. Assume without loss of generality that these are even edges. It remains to demonstrate that there are no distinct columns j_1 and j_2. such that the inner end of the edge of E' of column j_1 is adjacent to the outer end of the edge corresponding to j_2. Since the columns are different the adjacency may be only because the adjacent ends belong to the same clique V_i. But this means that the row of the inner end of j_2 is odd, a contradiction. ◀

Regarding LSIM WIDTH, the situation is opposite: LSIM WIDTH cannot represent the OBDD upper bound. In particular, we present below a class of graphs whose LSIM WIDTH is at most 3 while the LU-MIM WIDTH is lower bounded the number of vertices to the power of $1/3$. Hence, the size of the corresponding OBDDs is exponential in the number of vertices (in a positive power). We conclude that LSIM WIDTH cannot be used for representation of the OBDD upper bound for monotone 2-CNFs.

▶ **Definition 25.** *For each integer $r \geq 2$, we define the graph X_r of $n = 2r^3$ vertices as follows. Let $L_1, \ldots, L_{2r}$ be mutually disjoint sets of r^2 vertices in each and call the sets* layers. $V(X_r) = L_1 \cup \cdots \cup L_{2r}$.

For the purpose of introducing edges, each L_i is arbitrarily partitioned into sets $L_{i,1}, \ldots, L_{i,r}$ of r vertices in each. These sets are called sublayers *of layer i. The vertices of each $L_{i,j}$ are arbitrarily enumerated.*

The edges of X_r are the following.

1. *For each $1 \leq i \leq 2r-1$, for each $1 \leq j \leq r$ and for each $1 \leq k \leq r$, introduce an edge between vertex number k of $L_{i,j}$ and vertex number k of $L_{i+1,j}$. We call these edges* inter-layer *ones.*
2. *For each odd i for every $1 \leq j < k \leq r$ introduce an edge between each vertex of $L_{i,j}$ and each vertex of $L_{i,k}$.*
3. *For each even i and each $1 \leq j \leq r$, make $L_{i,j}$ into a clique.*

▶ **Lemma 26.** *$lsimw(X_r) \leq 3$.*

Proof. Consider the following permutation π of X_r. The vertices are traversed layer by layer, first L_1 then L_2 then L_3 and so on. Within each L_i first vertices of $L_{i,1}$ are traversed then of $L_{i,2}$, and so on. Within each $L_{i,j}$ vertices are traversed by the increasing order of the numbers assigned to them.

Consider an arbitrary prefix V' of this permutation. Let q be the largest number such that $L_q \cap V' \neq \emptyset$. It follows that for each $1 \leq i \leq q-1$ $L_i \subseteq V'$. Consequently, by construction, the edges between V' and $V(X_r) \setminus V'$ may be divided into the following three categories.

1. Edges between L_q and L_{q+1}. Then any two such edges will have an edge between their ends. Indeed, any such an edge connects an odd layer with an even layer. Let us call the end in the odd layer the *odd end* and the end in the even layer the *even end*. Note that both odd ends belong to the same sublayer if and only if both even ends do. If both even

ends belong to the same sublayer then they are adjacent by construction. Otherwise, both odd ends belong to different sublayers and, again, are adjacent by construction.

2. Edges between L_{q-1} and L_q. The same principle applies that any two edges must have adjacent ends.

3. Edges inside L_q. If q is odd then the ends of any edge e belong to different sublayers. By construction $N(e) \cap L_q = L_q \setminus e$ hence any two edges have adjacent ends. In case q is even, there are two cases two consider. The first is when each sublayer of L_q is either a subset of V' or a subset of $V(X_r) \setminus V'$. In this case there are no edges between V' and $V(X_r) \setminus V'$ with both ends in L_q. Otherwise, there is precisely one sublayer of L_q that is in part in V' and in part outside of V'. But since this sublayer induced a clique of X_r clearly any two edges of this category wil have adjacent ends.

If we take arbitrary four edges between V' and $V(X_r) \setminus V'$ then two of them will get to the same category (by the pigeonhole principle because there are three categories) and, as specified above there is an edge between their ends. Hence the width of the prefix is at most 3. ◄

▶ **Lemma 27.** $lu(X_r) \geq r$.

Proof. Let π be an arbitrary permutation of $V(X_r)$.

▷ **Claim 28.** Suppose that π has a prefix π' such that for each odd layer L_i, $L_i \cap \pi' \neq \emptyset$ and $L_i \setminus \pi' \neq \emptyset$. Then $X_r^{\pi'}$ has an induced matching of size r.

Proof. As each odd L_i induces a connected subgraph we can identity an edge in $X_r[L_i]$ with one end in π' the other edge out of π'. Let E' be the set of such edges of all r odd layers. By construction they form an induced matching. ◁

▷ **Claim 29.** Assume that π has a prefix π' such that for each $1 \leq i \leq r$ there is an interlayer edge whose ends belong to sublayer i with the even end in π' and the odd end outside π'. Let E' be a set of such n edges. Then they form an induced matching of $X_r^{\pi'}$.

Proof. Indeed, let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ be two such edges with u_1 and u_2 being the even ends. There is no edge between u_1 and u_2 as vertices of even layers belonging to different sublayers are not adjacent by construction. As vertices outside of π' form an independent set by definition of $H_n^{\pi'}$ v_1 and v_2 are not adjacent. Now u_1 and v_2 may be adjacent only if they belong to the same sublayer which is not the case, likewise for u_2 and v_1. ◁

It remains to assume that the cases as in the above two claims do not hold. Let π_0 be the shortest prefix such that there is an odd layer L_i with $L_i \subseteq \pi_0$ (*the full layer*). Then there is another odd layer L_j such that $L_j \cap \pi' = \emptyset$ (*the empty layer*). Indeed, otherwise, let π_1 be the immediate predecessor of π_0. The only difference of π_1 from π_0 is that one vertex of L_i is outside of π_1 simply due to the minimality of π_0. By assumption about π_0 each odd layer has a vertex inside π_1. By minimality of π_0 each odd layer has a vertex outside π_1. This is exactly the situation as in the Claim 28 in contradiction to our assumption.

Next, to avoid the premises of Claim 29 to apply, we identify $1 \leq k \leq r$ such that there is no inter-layer edge between π' and $V(G) \setminus \pi'$ with the both ends in sublayer k, the end in π' being the even one. We can assume w.l.o.g. that $k = 1$.

For each $1 \leq x \leq r$ there is an interlayer edge whose nodes have number x in layer 1, one end inside π' the other end outside π'. Indeed, assume w.l.o.g. that $j > i$ (recall that i and j are the numbers of the full and empty layers respectively). Let P be the path formed of interlayer edges of sublayer 1 whose ends have number x consisting of vertices at layers

$i, i+1, \ldots, j$. The vertex at layer i is in π', the vertex at layer j is outside π', hence one of the edges of P ought to be as desired. Let E' be the set of r such edges. By our assumption each edge of E' has its odd end inside π' and the even end outside π'. We are going to show that the edges of E' form an induced matching of $X_r^{\pi'}$.

Let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ be two edges of E', u_1 and u_2 being the ends inside π', v_1 and v_2 being the ends outside π'. Now v_1 and v_2 are not adjacent by definition of an upper graph. The vertices u_1 and u_2 are not adjacent because, by construction two different vertices of the same sublayer of odd layers are not adjacent. Finally u_1 and v_2 as well as u_2 and v_1 are vertices of layers of different parity lying in the same sublayers but having different numbers. Again, by construction, such vertices cannot be adjacent.   ◀

▶ **Theorem 30.** *There is no function f such that for any graph G, $obdd(\varphi(G)) \leq n^{f(lsimw(G))}$.*

Proof. Consider the graphs X_r. By Lemma 27, $lu(X_r) \geq r = (n/2)^{1/3}$ and hence, by Theorem 14, $obdd(\varphi(X_r)) \geq 2^{n^{\Omega(1)}}$.

Clearly $obdd(\varphi(X_r))$ cannot be upper-bounded by any polynomial function of n. On the other hand, by Lemma 26, whatever function f we take, $n^{f(lsimw(X_n))}$ is upper-bounded by $n^{O(c)}$ where c is the maximum of $f(1)$, $f(2)$, and $f(3)$.   ◀

7   Future research

In this section we discuss several interesting open questions related to representation of monotone 2-CNFs by circuit models more powerful than OBDDs. A natural question in this direction is to consider Nondeterministic Read-Once Branching Programs (1-NBPs) instead of OBDDs, For example, is it true that the size of 1-NBP representing a monotone 2-CNF $\varphi = \varphi(G)$ is lower bounded by $2^{\Omega(lu(G))}$?

Similarly to MIM WIDTH and SIM WIDTH, it is possible to formulate the 'non-linear' version of LU-MIM WIDTH in terms of the branch decompositions rather than permutations. It is interesting to investigate whether the size of Decomposable Negation Normal Forms (DNNFs) or its restricted classes such as Deterministic DNNFs representing monotone 2-CNFs can be captured by this non-linear parameter. We conjecture that the resulting non-linear parameter 'captures' the size of Structural Deterministic DNNFs but for more general models the situation is unclear and is likely to depend on the situation with 1-NBPs. Our belief relies on a plausible analogy with the bounded degree case where pathwidth captures the size of 1-NBPs while its non-linear counterpart (that is treewidth) captures the size of DNNFs [1].

Finally, a natural question arising from the results of this paper is to capture the size of OBDDs on monotone CNFs of higher arity. One possibility to achieve this might be through a concise generalization of LU-MIM WIDTH to hypergraphs.

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