Expressive Quantale-Valued Logics for Coalgebras: An Adjunction-Based Approach

Harsh Beohar
University of Sheffield, UK

Sebastian Gurke
Universität Duisburg-Essen, Germany

Barbara König
Universität Duisburg-Essen, Germany

Karla Messing
Universität Duisburg-Essen, Germany

Jonas Forster
Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Lutz Schröder
Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Paul Wild
Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Abstract

We address the task of deriving fixpoint equations from modal logics characterizing behavioural equivalences and metrics (summarized under the term *conformances*). We rely on an earlier work that obtains Hennessy-Milner theorems as corollaries to a fixpoint preservation property along Galois connections between suitable lattices. We instantiate this to the setting of coalgebras, in which we spell out the compatibility property ensuring that we can derive a behaviour function whose greatest fixpoint coincides with the logical conformance. We then concentrate on the linear-time case, for which we study coalgebras based on the machine functor living in Eilenberg-Moore categories, a scenario for which we obtain a particularly simple logic and fixpoint equation. The theory is instantiated to concrete examples, both in the branching-time case (bisimilarity and behavioural metrics) and in the linear-time case (trace equivalences and trace distances).

2012 ACM Subject Classification Theory of computation → Modal and temporal logics

Keywords and phrases modal logics, coalgebras, behavioural equivalences, behavioural metrics, linear-time semantics, Eilenberg-Moore categories

Digital Object Identifier 10.4230/LIPIcs.STACS.2024.10

Related Version *Full Version*: `https://arxiv.org/abs/2310.05711` [5]

Funding The first author was supported by the EPSRC NIA Grant EP/X019373/1, while the remaining authors were supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – project number 434050016 (SpeQt).

1 Introduction

Behavioural equivalences (such as bisimilarity and trace equivalence) are an important technique to identify states with the same behaviour in a transition system [39]. They have been complemented by notions of behavioural metrics [14, 38, 10] measuring the distance between states, in particular in a quantitative setting. We work in a coalgebraic setting [32] that allows us to answer generic questions about behavioural equivalences and metrics, parametrized over various branching types (non-deterministic, probabilistic, weighted, etc.).

There are various ways to characterize behavioural equivalences or metrics, which we illustrate using trace equivalence as an example: (i) *direct specification:* Two states x, y are trace equivalent if they admit the same traces; (ii) *logic:* x, y are trace equivalent if they cannot be distinguished in a modal logic based on diamond modalities and the constant

41st International Symposium on Theoretical Aspects of Computer Science (STACS 2024).
Editors: Olaf Beyersdorff, Mamadou Moustapha Kanté, Orna Kupferman, and Daniel Lokshtanov; Article No. 10; pp. 10:1–10:19
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

true; (iii) *fixpoint equation:* x, y are trace equivalent if the pair $(\{x\}, \{y\})$ is contained in the greatest fixpoint of the bisimulation function on the determinized transition system; (iv) *games:* there is an attacker-defender game characterizing the equivalence.

Our focus is on (ii) and (iii), and our take is quite different from the usual approach: Instead of first defining the behavioural equivalence/metric and then setting up an expressive logic for it, we start by defining the logic and derive a fixpoint equation from the logic. Fixpoint equations are of great interest since efficient algorithms for computing behavioural conformances are almost always based on fixpoint characterizations; in future work, we aim to exploit such characterizations for algorithmic purposes. However, for a given logic, corresponding fixpoint equations do not always exist, and we give conditions for ensuring that they do. We use the Galois connection approach from [4] as a starting point, and instead of instantiating it for each case study, we propose a generic coalgebraic framework. By employing fibrations (resp. indexed categories) [17, 18], we parameterize over the notion of *conformance* (e.g. equivalence, metric) that is our focus of attention. Moreover, we parametrize over a quantale in which both conformances and formulae take their values.

One interest, particularly, is in linear-time notions of conformance (such as trace/language equivalences and their quantitative cousins), for which we work in an Eilenberg-Moore category where the coalgebras live. We exploit the generalized powerset construction [20, 34] and characterize those (trace) logics that can be turned into suitable fixpoint equations on the determinized coalgebra, using a notion of compatibility [4] that has its roots in up-to techniques [30]. We also study the relation of compatibility to the notion of depth-1 separation used in (quantitative) graded logics [11, 13].

More concretely, we work with coalgebras of the form $c\colon X \to FTX$, living in some category $\mathbf{C}$, where a monad T intuitively specifies the implicit branching (or side effects) and a functor F describes the explicit branching type. For instance, for a non-deterministic automaton we choose $T = \mathcal{P}$ and $F = \_^{\Sigma} \times \mathbf{2}$. We fix an *EM-law* $\zeta\colon TF \Rightarrow FT$ [20] allowing us to obtain a determinized coalgebra, i.e., a coalgebra of the form $c^{\#}\colon TX \to FTX$ that can be viewed as a coalgebra in the Eilenberg-Moore category of T. We can then define a logic function log that is defined on sets of $\mathcal{V}$-valued predicates on X and whose least fixpoint induces a behavioural conformance on TX. Alternatively, given the determinization $c^{\#}$, we can – in a fibrational style – define a conformance on TX as the greatest fixpoint of a Kantorovich lifting followed by a reindexing via $c^{\#}$. The aim is to show that both conformances coincide.

We allow arbitrary constants in the logic, which – in particular in the linear-time case – are able to add extra distinguishing power to the logic. Along the way we give an answer to the question of why, unlike branching-time logics, linear-time logics often do not need any additional (boolean) operators, only modalities and constants.

As examples we consider bisimilarity and branching-time pseudometrics for probabilistic transition systems, as well as linear-time conformances such as trace equivalence and trace distance.

Roadmap. After reviewing preliminaries in Section 2, we summarize the approach based on Galois connections (adjunctions) in Section 3. The instantiation to generic coalgebras is presented in Section 4 and a concrete quantale-valued branching-time logic spelled out in Section 5. Section 6 specializes to coalgebras in Eilenberg-Moore categories, leading to results strengthening those for the general case. Finally, Section 7 details the linear-time case studies mentioned above, and we conclude in Section 8.

The proofs can be found in the full version of this paper [5].

2 Preliminaries

We recall some basic definitions and facts on lattices, quantales, generalized distances, coalgebra, monads and their algebras and on indexed categories. We do assume basic familiarity with category theory (e.g. [1]).

2.1 Lattices, Fixpoints and Galois Connections

A *complete lattice* $(\mathbb{L}, \sqsubseteq)$ consists of a set $\mathbb{L}$ with a partial order $\sqsubseteq$ such that each $Y \subseteq \mathbb{L}$ has a least upper bound $\bigsqcup Y$ (also called supremum, join) and a greatest lower bound $\bigsqcap Y$ (also called infimum, meet). The Knaster-Tarski theorem [36] guarantees that any monotone function $f \colon \mathbb{L} \to \mathbb{L}$ on a complete lattice $\mathbb{L}$ has a *least fixpoint* μf and a *greatest fixpoint* νf.

Let $\mathbb{L}$, $\mathbb{B}$ be two lattices. A *Galois connection* from $\mathbb{L}$ to $\mathbb{B}$ is a pair $\alpha \dashv \gamma$ of monotone functions $\alpha \colon \mathbb{L} \to \mathbb{B}$, $\gamma \colon \mathbb{B} \to \mathbb{L}$ such that for all $\ell \in L$, $b \in \mathbb{B}$: $\alpha(\ell) \sqsubseteq b \iff \ell \sqsubseteq \gamma(b)$.

A *closure* $\mathsf{cl} \colon \mathbb{L} \to \mathbb{L}$ is a monotone, idempotent and extensive (i.e. $\forall_{x\in\mathbb{L}}\ x \sqsubseteq \mathsf{cl}(x)$) function on a lattice. A *co-closure* is monotone, idempotent and extensive wrt. $\sqsupseteq$. Given a Galois connection $\alpha \dashv \gamma$, $\gamma \circ \alpha$ is always a closure and $\alpha \circ \gamma$ a co-closure.

2.2 Quantales and Generalized Distances

▶ **Definition 1.** A *(unital, commutative) quantale* $(\mathcal{V}, \otimes, 1)$, or just $\mathcal{V}$, is a complete lattice with an associative, commutative operation $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ with unit 1 that distributes over arbitrary (possibly infinite) joins $\bigvee$. If 1 is the top element of $\mathcal{V}$, then $\mathcal{V}$ is *integral*.

In a quantale $\mathcal{V}$, the functor $- \otimes y$ has a right adjoint $[y, -]$ for every $y \in \mathcal{V}$; that is, $x \otimes y \le z \iff x \le [y, z]$ for all $x, y, z \in \mathcal{V}$.

▶ **Example 2.**
1. The Boolean algebra $\mathbf{2}$ with $\otimes = \wedge$ and unit 1 is an integral quantale; for $y, z \in \mathbf{2}$, we have $[y, z] = y \to z$.
2. The complete lattice $[0, 1]$ ordered by the reversed order of the reals, i.e. $\le\, = \,\ge_{\mathbb{R}}$, and equipped with truncated addition $r \otimes s = \min(r + s, 1)$, is an integral quantale; for $r, s \in [0, 1]$, we have $[r, s] = s \dot{-} r = \max(s - r, 0)$ (truncated subtraction).

Given a quantale $\mathcal{V}$, a *directed ($\mathcal{V}$-valued) pseudometric* (*on* X) is a function $d \colon X \times X \to \mathcal{V}$ where (i) $\forall_{x\in X}\ d(x, x) \ge 1$ (reflexivity); (ii) $\forall_{x,y,z\in X}\ d(x, z) \ge d(x, y) \otimes d(y, z)$ (transitivity/triangle inequality). Moreover, d is a *pseudometric* if additionally, (iii) $\forall_{x,y\in X}\ d(x, y) = d(y, x)$ (symmetry). We write $\mathbf{DPMet}_{\mathcal{V}}(X)$ to denote the lattice of all directed pseudometrics on X, while for pseudometrics we use $\mathbf{PMet}_{\mathcal{V}}(X)$. Given $d_X \in \mathbf{DPMet}_{\mathcal{V}}(X)$, $d_Y \in \mathbf{DPMet}_{\mathcal{V}}(Y)$, a function $f \colon X \to Y$ is *non-expansive* (wrt. d_X, d_Y) if $d_X(x, x') \le d_Y(f(x), f(x'))$ for all $x, x' \in X$. Note that due to the choice of order in the quantale, the inequality in the definitions above is reversed wrt. the standard definitions that are typically given in the order on the reals. As originally observed by Lawvere [26], one may see directed $\mathcal{V}$-valued pseudometrics as $\mathcal{V}$-enriched categories, or just $\mathcal{V}$-categories, and non-expansive functions as $\mathcal{V}$-functors.

2.3 Coalgebras and Eilenberg-Moore Categories

Given a functor $F \colon \mathbf{C} \to \mathbf{C}$, an *$F$-coalgebra* (X, c) (or simply c) consists of an object $X \in \mathbf{C}$ and a $\mathbf{C}$-arrow $c \colon X \to FX$. In the paradigm of *universal coalgebra* [32], we understand X as the state space of a transition system, F as specifying the branching type of the system, and c as a transition map that assigns to each state a collection of successors structured according to F. For instance, when $\mathbf{C} = \mathbf{Set}$ is the category of sets and functions, then the powerset functor $\mathcal{P}$ assigns to each set its powerset, and $\mathcal{P}$-coalgebras are just sets equipped with a transition relation. On the other hand, the (finitely supported) distribution functor $\mathcal{D}$ assigns to each set X the set of finitely supported probability distributions on X, given in terms of maps $p\colon X \to [0,1]$ with finite support such that $\sum_{x\in X} p(x) = 1$. A $\mathcal{D}$-coalgebra thus is precisely a Markov chain.

Recall that a *monad* $(T, \eta\colon \mathrm{Id} \Rightarrow T, \mu\colon TT \Rightarrow T)$ on $\mathbf{C}$, usually denoted by just T, consists of a functor $T\colon \mathbf{C} \to \mathbf{C}$ and natural transformations $\eta\colon \mathrm{Id} \Rightarrow T$ (the *unit*) and $\mu\colon TT \Rightarrow T$ (the *multiplication*), subject to certain coherence laws. Monads abstractly capture notions of algebraic theory, with TX being thought of as terms modulo provable equality over variables in X. Correspondingly, a *T-algebra* (X, a) consists of an object X of $\mathbf{C}$ and a $\mathbf{C}$-arrow $a\colon TX \to X$ such that $a \circ \eta_X = id_X$ and $a \circ Ta = a \circ \mu_X$; we may think of T-algebras as algebras for the algebraic theory encapsulated by T. A homomorphism between T-algebras (X, a), (Y, b) is a $\mathbf{C}$-arrow $f\colon X \to Y$ such that $b \circ Tf = f \circ a$. The *Eilenberg-Moore category* of T, denoted $\mathbf{EM}(T)$, is the category of T-algebras and their homomorphisms. There is a free-forgetful adjunction $L \dashv R\colon \mathbf{C} \to \mathbf{EM}(T)$, where the forgetful functor R maps an algebra to its underlying $\mathbf{C}$-object and L maps an object $X \in \mathbf{C}$ to the free algebra (TX, μ_X).

A *(monad-over-functor) distributive law (or EM-law)* of a monad T over a functor F is a natural transformation $\zeta\colon TF \Rightarrow FT$ satisfying $\zeta_X \circ \eta_{FX} = F\eta_X$ and $\zeta_X \circ \mu_{FX} = F\mu_X \circ \zeta_{TX} \circ T\zeta_X$. This is equivalent to saying that the assignment $\tilde{F}(X, a) = (FX, Fa \circ \zeta_X)$ defines a *lifting* $\tilde{F}\colon \mathbf{EM}(T) \to \mathbf{EM}(T)$ of F (where *lifting* means that $R\tilde{F} = FR$). Then, the *determinization* [34] of a coalgebra $c\colon X \to FTX$ in $\mathbf{C}$ is the transpose $c^\#\colon LX \to \tilde{F}LX$ of c under $L \dashv R$. More concretely the determinization can be obtained as $c^\# = F\mu_X \circ \zeta_{TX} \circ Tc$. For instance, when $FX = X^\Sigma \times \mathbf{2}$ and $T = \mathcal{P}$, then this yields exactly the standard powerset construction for the determinization of non-deterministic automata.

2.4 Indexed Categories and Fibrations

Our aim is to equip objects of a category with additional information, e.g., consider sets with (equivalence) relations or metrics. Formally, this is done by working with fibrations, in particular we will consider fibrations arising from the Grothendieck construction for indexed categories [17, 18]. For us it is sufficient to consider as indexed categories functors $\Phi\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Pos}$, where $\mathbf{Pos}$ is the category of posets (ordered by $\preceq$) with monotone maps. Such functors induce a fibration $U\colon \int \Phi \to \mathbf{C}$ where U is the forgetful functor and $\int \Phi$ is the category whose objects and arrows are characterized as follows:

$$\frac{X \in \mathbf{C} \ \wedge\ d \in \Phi X}{(X, d) \in \int \Phi} \qquad \frac{X \xrightarrow{f} Y \in \mathbf{C} \ \wedge\ d \preceq (\Phi f)d'}{(X, d) \xrightarrow{f} (Y, d') \in \int \Phi}$$

Here, $f^* = \Phi f$ is also called *reindexing operation* and d is called a *conformance*.

Typical examples are functors $\Phi\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Pos}$ mapping a set X to the lattice of equivalence relations or pseudometrics on X.

3 Adjoint Logic: the General Framework

We summarize previous results on relating logical and behavioural conformances using Galois connections [4]. These results are based on the following well-known property that shows how fixpoints are preserved by Galois connections (e.g. [3, 8, 9]). The formulation of these properties involves a notion of compatibility studied for coinductive up-to techniques [30].

▶ **Definition 3.** Let $\mathsf{log}, \mathsf{cl}\colon \mathbb{L} \to \mathbb{L}$ be monotone endofunctions on a partial order $(\mathbb{L}, \sqsubseteq)$. Then log is cl-*compatible* if $\mathsf{log} \circ \mathsf{cl} \sqsubseteq \mathsf{cl} \circ \mathsf{log}$.

▶ **Theorem 4** ([4])**.** *Let $\alpha\colon \mathbb{L} \to \mathbb{B}$, $\gamma\colon \mathbb{B} \to \mathbb{L}$ be a Galois connection between complete lattices $\mathbb{L}, \mathbb{B}$, and let $\mathsf{log}\colon \mathbb{L} \to \mathbb{L}$, $\mathsf{beh}\colon \mathbb{B} \to \mathbb{B}$ be monotone. Then the following holds.*

1. *If $\mathsf{beh} = \alpha \circ \mathsf{log} \circ \gamma$ then $\alpha(\mu\, \mathsf{log}) \sqsubseteq \mu\, \mathsf{beh}$.*
2. *If $\alpha \circ \mathsf{log} = \mathsf{beh} \circ \alpha$, then $\alpha(\mu\, \mathsf{log}) = \mu\, \mathsf{beh}$. If log reaches its fixpoint in ω steps, i.e., $\mu\, \mathsf{log} = \mathsf{log}^\omega(\bot)$, then so does beh.*
3. *Let $\mathsf{cl} = \gamma \circ \alpha$ be the closure operator of the Galois connection, and suppose that $\mathsf{beh} = \alpha \circ \mathsf{log} \circ \gamma$. If log is cl-compatible, then $\alpha(\mu\, \mathsf{log}) = \mu\, \mathsf{beh}$.*

▶ **Remark 5.** In fact, there is a weaker notion than compatibility that ensures the same result, i.e., $\alpha(\mu\, \mathsf{log}) = \mu\, \mathsf{beh}$. In particular, it is sufficient to show the following condition:

- $\mathsf{log}(\mathsf{cl}(\ell)) \sqsubseteq \mathsf{cl}(\mathsf{log}(\ell))$ for all $\ell \in \mathbb{S} \subseteq \mathbb{L}$ where $\mathbb{S}$ is an invariant of log, i.e. $\mathsf{log}[\mathbb{S}] \subseteq \mathbb{S}$, $\bot \in \mathbb{S}$, and $\mathbb{S}$ is closed under directed joins. (If the fixpoint is reached in ω steps, closure under directed joins is not required.)

We apply this in a scenario where $\mathbb{L}$ consists of logical formulas (or more precisely their semantics, in the shape of sets of definable predicates) and $\mathbb{B}$ consists of conformances. These Galois connections will be contra-variant when we consider the quantalic ordering in $\mathbb{B}$. Then, log is the "logic function" that adds a layer of modalities and propositional operators to a given set of predicates, so that $\mu\, \mathsf{log}$ is the semantics of the set of formulas of the logic. On the other hand, beh is the "behaviour function", whose greatest fixpoint $\nu\, \mathsf{beh}$ (remember the contra-variance) is behavioural conformance.

▶ **Example 6.** The simplest Galois connection used in [4] for characterizing behavioural equivalence is between $\mathbb{L} = \mathcal{P}(\mathbf{2}^X)$ (sets of predicates on X) and $\mathbb{B} = \mathbf{PMet}_{\mathbf{2}}(X)$ (equivalences on X), where α maps every set of predicates to the equivalence relation induced by it and γ maps an equivalence to the set of predicates closed under it.

Moving to pseudometrics we obtain a Galois connection between $\mathbb{L} = \mathcal{P}([0,1]^X)$ (sets of real-valued predicates on X) and $\mathbb{B} = \mathbf{PMet}_{[0,1]}(X)$ (pseudometrics on X) where α maps every set of functions $X \to [0,1]$ to the least pseudometric making all these functions non-expansive and γ takes a pseudometric and produces all its non-expansive functions.

So first, define a logical universe $\mathbb{L}$ and a logic function $\mathsf{log}\colon \mathbb{L} \to \mathbb{L}$. Second, choose a suitable Galois connection $\alpha \dashv \gamma$ to a behaviour universe $\mathbb{B}$ and show that log is cl-compatible. Third, derive the behaviour function $\mathsf{beh} = \alpha \circ \mathsf{log} \circ \gamma\colon \mathbb{B} \to \mathbb{B}$. From the results above, we automatically obtain the equality $\alpha(\mu\, \mathsf{log}) = \mu\, \mathsf{beh}$, which tells us that logical and behavioural equivalence respectively distance coincide (Hennessy-Milner theorem).

4 Adjoint Logic for Coalgebras

In this section we will describe a general framework where the adjoint logic is instantiated to the setting of coalgebraic modal logic.

4.1 Setting up the Adjunction

One can generalize from Example 6 and instead of a set X take an object in a (locally small) category $\mathbf{C}$. Furthermore we fix an object $\Omega \in \mathbf{C}$ (the *truth value* object), which in all our applications will be a quantale. Predicates are represented by the indexed category $\mathbf{C}(\_, \Omega)$; thus, sets of predicates (lattice $\mathbb{L}$) are given by the indexed category $\mathcal{P} \circ \mathbf{C}(\_, \Omega)$ (where the order is inclusion). In addition, we use an indexed category Φ specifying the notion of conformance on X (lattice $\mathbb{B}$) and work with the following assumptions:

A1 Each fibre ΦX is a poset having arbitrary meets (thus, a complete lattice) and the reindexing map preserves these meets (i.e. $\int \Phi$ has fibred limits).

A2 Let $d_\Omega \in \Phi\Omega$ be a fixed conformance on the truth value object Ω.

For an arrow $f \in \mathbf{C}(X, Y)$, we write $f^\bullet$ for the reindexing in $\mathbf{C}(\_, \Omega)$ ($f^\bullet g = g \circ f$, where $g \in \mathbf{C}(Y, \Omega)$) and f^* for the reindexing in Φ ($f^* = \Phi f$).

▶ **Theorem 7.** *Let X be an object of $\mathbf{C}$. Under Assumptions* **A1** *and* **A2***, there is a dual adjoint situation (contravariant Galois connection) $\alpha_X \dashv \gamma_X$ between the underlying fibres:*

$$\alpha_X : \mathcal{P}(\mathbf{C}(X, \Omega)) \to \Phi(X)^{op} \quad S \subseteq \mathbf{C}(X, \Omega) \mapsto \bigwedge_{k \in S} k^*(d_\Omega)$$

$$\gamma_X : \Phi(X)^{op} \to \mathcal{P}(\mathbf{C}(X, \Omega)) \quad d \in \Phi X \mapsto \{k \in \mathbf{C}(X, \Omega) \mid d \preceq k^*(d_\Omega)\}.$$

More concretely: α_X, γ_X are both antitone ($S \subseteq S' \implies \alpha_X(S) \succeq \alpha_X(S')$, $d \preceq d' \implies \gamma_X(d) \supseteq \gamma_X(d')$) and we have $d \preceq \alpha_X(S) \iff S \subseteq \gamma_X(d)$ for $d \in \Phi X$ and $S \in \mathcal{P}\mathbf{C}(X, \Omega)$.

Thus, for $X \in \mathbf{C}$, the fibres $\mathcal{P}\mathbf{C}(X, \Omega)$ and $(\Phi X)^{\mathrm{op}}$ will take the role of $\mathbb{B}$ and $\mathbb{L}$ (respectively) as in [4, Theorem 3.2]. Moreover, Theorem 7 will be instantiated to obtain the desired Galois connections between predicates and conformances for our case studies.

▶ **Example 8.** Let $\mathbf{C} = \mathbf{Set}$ and $\Omega = \mathcal{V}$ be a quantale. We consider $\Phi X = \mathbf{DPMet}_\mathcal{V}(X)$ (resp., $\Phi X = \mathbf{PMet}_\mathcal{V}(X)$) with the order $\preceq$ on ΦX induced by the pointwise lifting of the order $\leq$ on $\mathcal{V}$. The reindexing functor f^* for a function $f : X \to Y$ is given by $f^* d = d \circ (f \times f)$; thus, satisfying **A1**. As conformance d_Ω on $\mathcal{V}$ we take the internal hom $[\_, \_]$ (resp., its symmetrization: $d_\Omega(x, y) = [x, y] \wedge [y, x]$); thus, satisfying **A2**. Then we have $\alpha_X \dashv \gamma_X : \mathcal{P}(\mathbf{Set}(X, \mathcal{V})) \rightleftarrows \mathbf{DPMet}_\mathcal{V}(X)$, where:

$$\alpha_X(S)(x, x') = \bigwedge_{h \in S} d_\Omega(h(x), h(x'))$$

$$\gamma_X(d) = \{h : X \to \mathcal{V} \mid \forall_{x, x' \in X}\ d(x, x') \leq d_\Omega(h(x), h(x'))\}$$

In both cases, α_X assigns to a set of maps the greatest (directed) pseudometric making all these maps non-expansive, while γ_X maps a pseudometric all its non-expansive maps.

4.2 Characterizing Closure

Given that the key condition imposed on the logic function in Theorem 4 is compatibility with the closure of the Galois connection, it is important to understand how this closure operates. In the setting of Theorem 7 we can characterize the closure in terms of non-expansive propositional operators, provided that γ is natural. We note first that α is always natural:

▶ **Proposition 9.** *In Theorem 7, the transformation α is natural in $X \in \mathbf{C}$, that is, for $f \in \mathbf{C}(X, Y)$, we have $\alpha_X \circ \mathcal{P}(f^\bullet) = f^* \circ \alpha_Y$.*

For the right adjoint γ, naturality need not hold in general. It does hold for **Set** and generalized (directed) metrics over the quantales in Example 2. A counterexample can however be constructed for $\mathbf{C} = \mathbf{EM}(\mathcal{P})$ and $\mathcal{V} = [0, 1]$ (see [4]).

If γ is natural, then we can characterize the closure $\gamma \circ \alpha$ using the internal language of indexed categories. To this end, suppose that $\mathbf{C}$ has (small) products. Then for every $S \subseteq \mathbf{C}(X, \Omega)$. we have a unique tupling $\langle S \rangle : X \to \Omega^S$ such that $\pi_k \circ \langle S \rangle = k$ for all $k \in S$, where $\pi_k : \Omega^S \to \Omega$ is the product projection for k.

▶ **Lemma 10.** *The right adjoint* γ *in Theorem 7 is laxly natural, i.e.* $\mathcal{P}(f^\bullet) \circ \gamma_Y \subseteq \gamma_X \circ f^*$ *for* $f\colon X \to Y \in \mathbf{C}$*. If* γ *is natural (i.e. the inclusion is an equality) and* $\mathbf{C}$ *has products, then* $\gamma_X(\alpha_X(S)) = \mathcal{P}(\langle S\rangle^\bullet)(\gamma_{\Omega^S}(d_{\Omega^S}))$*, where* $d_{\Omega^S} = \bigwedge_{k\in S} \pi_k^* d_\Omega$*.*

This result can be interpreted as follows: $\gamma_{\Omega^S}(d_{\Omega^S})$ is the set of all non-expansive functions $\Omega^S \to \Omega$, hence all non-expansive operators of arbitrary arity on Ω. Reindexing via $\langle S\rangle$ means to combine all predicates in S via those operators, hence we describe the closure under all non-expansive operations on Ω.

4.3 Towards a Generic Logic Function

Since our slogan is to generate the behaviour function from the logic function, we start by setting up our logical framework first. Following [4, 12], we adopt a semantic approach to defining a (modal) logic, i.e., we specify the operators (including modalities) as a transformation of predicates; formally, as a (natural) transformation log of type $\mathcal{P}\mathbf{C}(\_,\Omega) \Rightarrow \mathcal{P}\mathbf{C}(\_,\Omega)$. The idea is that the logic function log_X adds one "layer" of modal depth; in particular, the least fixpoint $\mu\,\mathsf{log}_X$ of log_X can be seen as the set of (interpretations of) all modal formulas.

In particular, we require

A3 Fix a family $(ev_\lambda \in \mathbf{C})_{\lambda\in\Lambda}$ of evaluation maps $ev_\lambda\colon F\Omega \to \Omega$.

As noted in [33], such evaluation maps – commonly used in coalgebraic modal logic – correspond to natural transformations of type $\mathbf{C}(\_,\Omega) \to \mathbf{C}(F\_,\Omega)$ by the Yoneda lemma.

▶ **Proposition 11.** *A family of evaluation maps* $(ev_\lambda)_{\lambda\in\Lambda}$ *induces a natural transformation* $\Lambda\colon \mathcal{P}\mathbf{C}(\_,\Omega) \Rightarrow \mathcal{P}\mathbf{C}(F\_,\Omega)$ *given by* $S \mapsto \{\lambda_X(h) \mid \lambda \in \Lambda, h \in S\}$*, where* $\lambda_X(h) = ev_\lambda \circ Fh$*.*

Apart from modalities, a logic typically needs operators and constants. We do not consider constants as 0-ary operators, which allows us to distinguish between operators that arise as in Lemma 10 from the closure of the Galois connection (that is, non-expansive operators) and the remaining (constant) operators that bring additional distinguishing power. This is, for instance, needed in the case of trace equivalence on a determinized transition system to distinguish the empty set of states from sets of states having no transitions. We need an additional (constant) predicate for this task that can neither be provided by the closure nor by a constant modality (see Appendix A.1).

A4 We assume a set $\Theta_X \subseteq \mathbf{C}(X,\Omega)$ of *constants* (which is later restricted to consist of free extensions of constant maps).

To model the propositional operators, we introduce a closure cl'_X:

A5 For each $X \in \mathbf{C}$ we assume that there is a closure $\mathsf{cl}'_X\colon \mathcal{P}\mathbf{C}(X,\Omega) \to \mathcal{P}\mathbf{C}(X,\Omega)$ (not necessarily natural), specifying the propositional operators.

We say that cl'_X is a *subclosure* of cl_X whenever $\mathsf{cl}'_X \subseteq \mathsf{cl}_X$, which means that the propositional operators implemented by cl'_X are already contained in the closure induced by the Galois connection (cf. Lemma 10).

Now we can define the logic function for a coalgebra c as

$$\mathsf{log}_X = \mathcal{P}(c^\bullet) \circ \Lambda_X \circ \mathsf{cl}'_X \cup \Theta_X : \mathcal{P}\mathbf{C}(X,\Omega) \to \mathcal{P}\mathbf{C}(X,\Omega).$$

Its least fixpoint contains all predicates that can be described by modal formulas.[1]

[1] In our setup a formula is either a constant or starts with a modality, which still results in an expressive logic. One could slightly modify log_X and obtain all formulas by adding another closure cl'.

▶ **Example 12.** Let $\mathbf{C} = \mathbf{Set}$, $c\colon X \to FX$ and $\Phi X = \mathbf{PMet}_{\mathcal{V}}(X)$. Recall the Galois connection from Example 8 and consider the following two examples, where in both cases no constants are needed, i.e., Θ_X is empty; thus, **A4** vacuously holds.

1. *Bisimilarity on (unlabelled) transition systems:* we let $F = \mathcal{P}_{\mathit{fin}}$ (finite powerset functor), $\mathcal{V} = 2$, and consider the evaluation map $ev_{\Diamond}\colon \mathcal{P}_{\mathit{fin}}2 \to 2$ encoding the usual diamond modality: $ev_{\Diamond}(U) = 1 \iff 1 \in U$. This can be extended to a logic by choosing as cl' (Assumption **A5**) the closure under all (finitary) Boolean operators.
2. *Behavioural metrics for probabilistic transition systems with termination:* we let $FX = \mathcal{D}X + 1$ (where $1 = \{\checkmark\}$) and $\mathcal{V} = ([0,1], \geq_{\mathbb{R}})$. Define two evaluation maps: $ev_E\colon D\,[0,1] + 1 \to [0,1]$ corresponds to expectation, i.e. $ev_E(p) = \sum_{r\in[0,1]} r\cdot p(r)$ if $p \in \mathcal{D}X$ (0 otherwise). Furthermore, $ev_*\colon \mathcal{D}[0,1] + 1 \to [0,1]$ with $ev_*(p) = 1$ if $p = \checkmark$ (0 otherwise). We extend this to a logic by defining cl' (Assumption **A5**): we add as operators the constant 1, $\min(\varphi, \varphi')$, $1 - \varphi$ and $\varphi \dot{-} q$ for a rational q (where φ is a formula), as in similar logics for probabilistic transition systems [38].

To ensure that the requirements of Section 3 are met, we have to show compatibility of the logic function. To this end, we introduce the notion of compability of cl'.

▶ **Definition 13.** Given a closure $\mathsf{cl}'_X\colon \mathcal{P}\mathbf{C}(X,\Omega) \to \mathcal{P}\mathbf{C}(X,\Omega)$, we say that cl'_X is *compatible* if the map $\Lambda_X \circ\ \mathsf{cl}'_X$ (for each $X \in \mathbf{C}$) is compatible with the closure $\mathsf{cl}_X = \gamma_X \circ \alpha_X$ induced by the adjoint situation in Theorem 7, i.e., $\Lambda_X \circ \mathsf{cl}'_X \circ \mathsf{cl}_X \subseteq \mathsf{cl}_{FX} \circ \Lambda_X \circ \mathsf{cl}'_X$.

The following results hold under Assumptions **A1-A5** and thus, we avoid stating them in various lemma/theorem statements.

▶ **Proposition 14.** *For a given compatible closure cl'_X, the above logic function log_X is cl_X-compatible, i.e., $\mathsf{log}_X \circ \mathsf{cl}_X \subseteq \mathsf{cl}_X \circ \mathsf{log}_X$.*

We will now study equivalent conditions and special cases in which compatibility holds. First, it is easy to see that $\mathsf{cl}' = \mathsf{cl}$ is always compatible, but typically introduces infinitary operators. Moreover, if cl' is the identity (that is, there are no propositional operators), then compatibility of cl' reduces to cl-compatibility of Λ.

▶ **Lemma 15.** *Let cl'_X be a subclosure of cl_X. It holds that cl'_X is compatible if and only if $\alpha_{FX} \circ \Lambda_X \circ \mathsf{cl}'_X \preceq \alpha_{FX} \circ \Lambda_X \circ \mathsf{cl}_X$.*

We next adapt the separation property establishing expressiveness of *graded logics* w.r.t. *graded semantics* [11, 13] to the present setting, an additional twist being that the conformance w.r.t. which modalities must be separating is the one induced by the modalities themselves.

▶ **Definition 16** (Depth-1 self-separation)**.** A set $S \subseteq \mathbf{C}(X,\Omega)$ of predicates is *initial* for $d \in \Phi X$ if $\alpha_X(S) = d$. Let cl'_X be a subclosure of cl_X. The *depth-1 self-separation* property holds if for every S that is closed under cl'_X (i.e., $S = \mathsf{cl}'_X(S)$) and initial for d, it follows that $\Lambda_X(S)$ is initial for $\kappa_X(d)$ where $\kappa_X = \alpha_{FX} \circ \Lambda_X \circ \gamma_X$.

▶ **Lemma 17.** *Let cl'_X be a subclosure of cl_X. Then cl'_X is compatible if and only if the depth-1 self-separation property holds.*

Finally, we study a sufficient condition on evaluation maps ensuring cl-compatibility of Λ.

▶ **Lemma 18.** *If each evaluation map ev_λ arises as a natural transformation $\eta\colon F \Rightarrow Id$ or $\eta\colon F \Rightarrow \mathbf{\Omega}$ ($\mathbf{\Omega}$ is the constant functor mapping every object to Ω), that is $ev_\lambda = \eta_\Omega$, then Λ is compatible with cl.*

▶ **Example 19.** We establish compatibility for the logics considered in Example 12. In branching-time logics in general, depth-1 self-separation usually boils down to establishing a Stone-Weierstraß type property saying that if $S \subseteq \mathbf{C}(X, \Omega)$ is initial and closed under cl'_X, then S is dense in $\mathbf{C}(X, \Omega)$, for suitably restricted X [12]. For finitary set functors such as $\mathcal{P}_{\mathit{fin}}$ or $\mathcal{D}$, it suffices to prove self-separation on finite X. Additional details are as follows.

1. In the case of unlabelled transition systems, we are given an equivalence relation R on a finite set X, a set $S \subseteq \mathbf{Set}(X, 2)$ that is initial for R and closed under Boolean combinations, and $A, B \in \mathcal{P}_{\mathit{fin}}(X)$ that are distinguished by some predicate $\Diamond_X f \colon \mathcal{P}_{\mathit{fin}}(X) \to 2$ where f is invariant under R. We then have to show that A, B are distinguished by $\Diamond g$ for some $g \in S$. But by functional completeness of Boolean logic and because X is finite, S is in fact the set of *all* R-invariant functions $X \to 2$, so we can just take $g = f$.
2. The argument is similar for probabilistic transition systems, with some additional considerations necessitated by the quantitative setting. We are now given a set $S \subseteq \mathbf{Set}(X, [0,1])$ that is initial for d and closed under propositional operators as per Example 12.2. By a variant of the Stone-Weierstraß theorem, this implies that S is dense in the space of non-expansive maps $(X, d) \to [0,1]$ (see [43]), which means that in an argument as in the previous item, we can take g to range over functions in S that approximate the given non-expansive function $f \colon (X, d) \to [0,1]$ arbitrarily closely, using additionally that the predicate lifting induced by ev_E as in Example 12(2) is non-expansive [42].

4.4 Towards a Generic Behaviour Function

Building on the previous section, we define the behaviour function $\mathsf{beh}_X \colon \Phi X \to \Phi X$ as $\mathsf{beh}_X = \alpha_X \circ \mathsf{log}_X \circ \gamma_X$ and – under the assumption of compatibility – we have[2] $\alpha_X(\mu \, \mathsf{log}_X) = \nu \, \mathsf{beh}_X$. In other words, the notions of logical and behavioural conformances coincide.

This motivates a closer investigation of $\nu \, \mathsf{beh}_X$: in what sense does it coincide with known behavioural equivalences or metrics? Defining a behavioural conformance (that is, an element of ΦX) in a fibrational setting is typically done by taking the greatest fixpoint of a function defined in two steps: the lifting of a conformance ΦX to $\Phi(FX)$, followed by a reindexing via c. Here, we consider Kantorovich-style [2] or codensity [22] liftings based on the evaluation maps. Kantorovich liftings have originally been used to lift metrics on a set X to metrics of probability distributions over X. In the probabilistic case, an alternative characterization is given via optimal transport plans in transportation theory (earth mover's distance) [40].

We use the natural transformation Λ introduced earlier and consider the composite $\kappa_X = \alpha_{FX} \circ \Lambda_X \circ \gamma_X$, the mentioned Kantorovich lifting. If $F = \mathcal{D}$ and the evaluation map is expectation, then we obtain exactly the classical Kantorovich lifting.

Given a coalgebra $c \colon X \to FX$, we use the *behaviour function* $\mathsf{beh}_X = c^* \circ \kappa_X \wedge \alpha_X(\Theta_X)$; here, Θ_X is a set of constants as in Section 4.3 (Assumption **A4**).

▶ **Example 20.** We derive the behaviour functions for Examples 12 and 19.

1. In the case $F = \mathcal{P}$ and $\mathcal{V} = 2$, the lifting $\kappa_X(R) \subseteq \mathcal{P}X \times \mathcal{P}X$ of an equivalence relation $R \subseteq X \times X$ is the Egli-Milner lifting, i.e. $U \; \kappa_X(R) \; V \iff \forall_{x \in U} \exists_{y \in V} x \; R \; y \wedge \forall_{y \in V} \exists_{x \in U} \; x \; R \; y$. It is well-known that the greatest fixpoint of $\mathsf{beh}_X = c^* \circ \kappa_X$ is precisely Park-Milner bisimilarity.

[2] Note that the adjunction defined in Theorem 7 is contravariant. Hence the least fixpoint of beh from Theorem 4 becomes the greatest fixpoint $\nu \, \mathsf{beh}_X$ wrt. the lattice order $\preceq$.

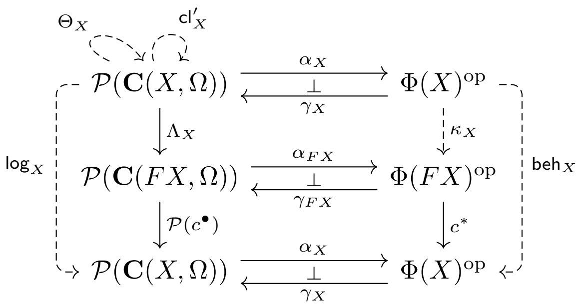

Figure 1 The adjoint setup with $\mathsf{log}_X = \mathcal{P}(c^\bullet) \circ \Lambda_X \circ \mathsf{cl}'_X \cup \Theta_X$ and $\mathsf{beh}_X = c^* \circ \kappa_X \cup \alpha_X(\Theta_X)$.

2. In the case $FX = \mathcal{D}X + 1$ and $\mathcal{V} = [0,1]$, we obtain the lifting $\kappa_X(d) \in \mathbf{PMet}(\mathcal{D}X + 1)$ of a pseudometric $d \in \mathbf{PMet}(X)$. It is easy to see that $\kappa_X(d)(p_1, p_2)$ is the distance given by the classical Kantorovich lifting of d if $p_1, p_2 \in \mathcal{D}X$. If $p_1 = \checkmark = p_2$, then the distance is 0, otherwise 1. The least fixpoint (under the usual order on $[0,1]$) of the behaviour function $\mathsf{beh}_X = c^* \circ \kappa_X$ agrees with standard notions of bisimulation distance (e.g. [38]).

We conclude the section by showing that behaviour functions defined in this way are actually the ones obtained from the logic function. For the diagram underlying the proof see Figure 1.

▶ **Theorem 21.** *Assume that cl'_X is a subclosure of cl_X and compatible. For a set Θ_X of constants and a coalgebra $c\colon X \to FX$, the logic function $\mathsf{log}_X = \mathcal{P}(c^\bullet) \circ \Lambda_X \circ \mathsf{cl}'_X \cup \Theta_X$ induces $\mathsf{beh}_X = c^* \circ \kappa_X \wedge \alpha_X(\Theta_X)$, i.e., $\alpha_X \circ \mathsf{log}_X \circ \gamma_X = \mathsf{beh}_X$.*

Putting everything together via Theorem 4, if cl'_X is a compatible closure, then we have $\alpha_X(\mu\, \mathsf{log}_X) = \nu\, \mathsf{beh}_X$, that is, logical conformance coincides with behavioural conformance.

5 Logics for Quantale-valued Simulation Distances

We next consider a quantitative modal logic $\mathcal{L}_\Lambda$ that we show to be expressive for similarity distance (a behavioural directed metric) under certain conditions. Throughout this section, our working category $\mathbf{C}$ is $\mathbf{Set}$, we have a fixed functor F on $\mathbf{Set}$, and a fixed quantale $\Omega = \mathcal{V}$ with distance $d_\mathcal{V} = [\_,\_]$ being the internal hom. Furthermore $\Phi X = \mathbf{DPMet}_\mathcal{V}(X)$. In this section we assume naturality of γ, which holds for the quantales given in Example 2.

$$\varphi \in \mathcal{L}_\Lambda ::= \bigwedge\nolimits_{i\in I} \varphi_i \;\mid\; \varphi \otimes v \;\mid\; d_\mathcal{V}(v,\varphi) \;\mid\; [\lambda]\varphi \quad (\text{for } v \in \mathcal{V}, \lambda \in \Lambda, I \in \mathbf{Set})$$

This logic is the positive fragment of quantale-valued coalgebraic modal logic [41, 12], and generalizes logics for real-valued simulation distance [42] to the quantalic setting. The first three operators are regarded as propositional operators of cl', while the $[\lambda]$ are the modalities. We do not use explicit constants ($\Theta_X = \emptyset$), but note that constant truth $\top$ is included as the empty meet. On a coalgebra $c\colon X \to FX \in \mathbf{Set}$, we interpret a formula φ as a function $[\![\varphi]\!]\colon X \to \mathcal{V}$ as usual (by structural induction on terms). Note that we do not allow negation, as we aim to characterize similarity distance. Disjunction could be included but, as in the two-valued case [39], is not needed to characterize simulation. Meet is infinitary, so the logic function log does not reach its least fixpoint in ω steps.

The next results show that the three operators (as well as join) are all non-expansive and hence cl' is a subclosure of cl (cf. Lemma 10), which moreover is compatible.

▶ **Proposition 22.** *Infinitary meets, infinitary joins, negative scaling $d_\mathcal{V}(v,\_)$, and positive scaling $\_ \otimes v$ (for $v \in \mathcal{V}$) are non-expansive.*

▶ **Proposition 23.** *If each $\lambda \in \Lambda$ is sup-preserving (i.e. $\lambda_X(\bigvee P) = \bigvee \mathcal{P}(\lambda_X)(P)$, for every subset $P \subseteq \mathbf{Set}(X, \mathcal{V})$,) then the sub-closure cl' of cl as above is compatible.*

Diamond-like modalities (for powerset or fuzzy powerset) are typically sup-preserving. In such cases, Theorem 21 yields expressiveness of the above logic for similarity distance, defined as the greatest fixpoint of $\mathsf{beh}_X(d) = c^* \circ \kappa_X(d)$ where κ_X is the directed Kantorovich lifting.

6 The Adjoint Setup in an Eilenberg-Moore Category

While we have seen in the examples of the previous sections that the framework can be instantiated to coalgebras living in **Set**, thus providing Hennessy-Milner theorems for bisimilarity, we are now interested in tackling trace equivalences and trace metrics. To this end, we work in Eilenberg-Moore categories [34], which also allows us to determinize a coalgebra using the generalized powerset construction (cf. 2.3).

In particular, we instantiate the adjoint setup to the category $\mathbf{EM}(T)$ of T-algebras (for some monad T on $\mathbf{C}$), provide conditions guaranteeing compatibility, and characterize the behaviour function. Furthermore, taking inspiration from [33], we also introduce a general syntax for modal formulas that can be interpreted over coalgebras living in $\mathbf{EM}(T)$. As introduced in Section 2.3, we fix a coalgebra $c\colon X \to FTX$ living in $\mathbf{C}$ and its determinization $c^{\#}\colon LX \to \tilde{F}LX$ in $\mathbf{EM}(T)$ via a distributive law $\zeta\colon TF \Rightarrow FT$.

We assume an indexed category $\Psi\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Pos}$ (that has fibred limits) and lift it to the category $\mathbf{EM}(T)$ of T-algebras by postcomposition, that is, $\Phi = \Psi \circ R$ (thus ensuring **A1**):

$$\mathbf{EM}(T)^{\mathrm{op}} \xrightarrow{R} \mathbf{C}^{\mathrm{op}} \xrightarrow{\Psi} \mathbf{Pos}.$$

Here, R is the forgetful functor in the free-forgetful adjunction $L \dashv R\colon \mathbf{C} \to \mathbf{EM}(T)$ from Section 2.3. To handle **A2**, we fix a truth value object $\Omega \in \mathbf{C}$ equipped with a T-algebra structure $o\colon T\Omega \to \Omega$ and $d_\Omega \in \Phi\Omega$. These assumptions ensure that Theorem 7 becomes applicable. We will denote the reindexing for Φ by $\_^*$, while we overload the notation and specify the reindexing in both $\mathbf{C}$ and $\mathbf{EM}(T)$ by $\_^\bullet$.

We focus on free algebras $LX = (TX, \mu_X)$ (over $X \in \mathbf{C}$) and apply Theorem 7 to the above-mentioned indexed category $\Psi \circ R$, which gives the adjoint situations depicted by shaded rectangles in Figure 2. We note that the middle hom-set $\mathbf{EM}(T)(LX, (\Omega, o))$ is isomorphic to $\mathbf{C}(X, \Omega)$ at the top left – due to the free-forgetful adjunction – with the bijection between the respective powersets witnessed by α', γ'. This allows us to define the logic function on the lattice $\mathcal{P}(\mathbf{C}(X, \Omega))$, which is a simpler structure than $\mathcal{P}(\mathbf{EM}(T)(LX, (\Omega, o)))$. In particular, formulas can then be evaluated directly on the state space X.

6.1 Logic and Behaviour Function for Coalgebras in Eilenberg-Moore

Recall from Section 4.3 that we need evaluation maps in the working category to define a logic function. So, to ensure **A3**, we assume a set Λ of evaluation maps for $\tilde{F}$, i.e., a family $(\tilde{F}(\Omega, o) \xrightarrow{ev_\lambda} (\Omega, o) \in \mathbf{EM}(T))_{\lambda \in \Lambda}$ of algebra homomorphisms. More concretely, a $\mathbf{C}$-arrow $ev_\lambda\colon F\Omega \to \Omega$ is such an algebra homomorphism if it satisfies $o \circ Tev_\lambda = ev_\lambda \circ Fo \circ \zeta_\Omega$.

As in Section 4, this induces a natural transformation Λ. Since every homomorphism is also a map in $\mathbf{C}$, we can define Λ'_X, the predicate lifting on $\mathcal{P}\mathbf{C}(X, \Omega)$:

$$\mathcal{P}\mathbf{C}(X, \Omega) \xrightarrow{\alpha'_X} \mathcal{P}\mathbf{EM}(T)(LX, (\Omega, o)) \xrightarrow{\Lambda_{LX}} \mathcal{P}\mathbf{EM}(T)(\tilde{F}LX, (\Omega, o)) \xrightarrow{\mathcal{P}(R)} \mathcal{P}\mathbf{C}(FTX, \Omega).$$

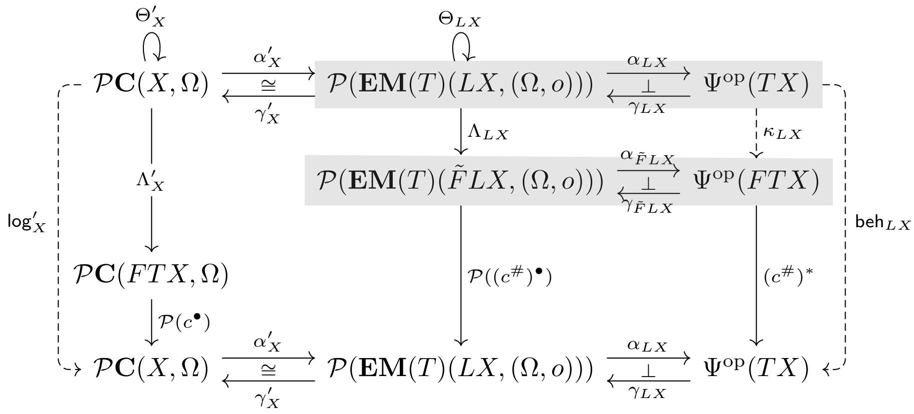

Figure 2 The adjoint setup for algebras, where $\mathsf{beh}_{LX}=(c^\#)^*\circ\kappa_{LX}\wedge\alpha_{LX}(\Theta_{LX})$ and $\mathsf{log}'_X=\mathcal{P}(c^\bullet)\circ\Lambda'_X\cup\Theta_X$.

Note that Λ' is a natural transformation (since α,Λ are natural transformations and R is a functor); on components it can be easily characterized as follows:

▶ **Lemma 24.** *We have that* $\Lambda'_X(S)=\{ev_\lambda\circ Fo\circ FTh\mid h\in S\}$ *where* $S\subseteq\mathbf{C}(X,\Omega)$.

That is, $\Lambda'_X(S)$ is obtained by first lifting the predicates from $\mathbf{C}(X,\Omega)$ to $\mathbf{C}(TX,\Omega)$ via the evaluation map $o\colon T\Omega\to\Omega$ and then to $\mathbf{C}(FTX,\Omega)$ via $ev_\lambda\colon F\Omega\to\Omega$. This process can be seen as applying a "double modality" for T and F.

We can now invoke the results of the previous chapter and assume that Λ is compatible with the closure induced by the adjunction, that is, we work without propositional operators (hence cl', as mentioned in Assumption **A5**, is the identity), only constants, at first sight a strong property. We will however see in the next section that this always holds when F is a machine functor and we choose suitable evaluation maps.

The next theorem focusses on free algebras and is partly a corollary of Proposition 14 and Theorem 21. However there is a new component: instead of defining the logic function on (free) Eilenberg-Moore categories, reindexing via the determinized coalgebra $c^\#$, it is possible – as indicated above – to define it directly on arrows of type $X\to\Omega$ living in $\mathbf{C}$, reindexing with c. This coincides with the view that formulas should be evaluated on states in X rather than elements of TX. The diagram in Figure 2 outlines how to show this result.

▶ **Theorem 25.** *We fix a coalgebra* $(c\colon X\to FTX)\in\mathbf{C}$. *Assume that* Λ_{LX} *is compatible with the closure* cl_{LX} *and fix* $\Theta_{LX}\subseteq\mathbf{EM}(T)(LX,\Omega)$ *to ensure that* **A4** *holds.*

1. *Then the logic function* $\mathsf{log}_{LX}=\mathcal{P}((c^\#)^\bullet)\circ\Lambda_{LX}\cup\Theta_{LX}$ *is* cl_{LX}*-compatible.*
2. *For the behaviour function* $\mathsf{beh}_{LX}=(c^\#)^*\circ\kappa_{LX}\wedge\alpha_{LX}(\Theta_{LX})$ *(where* $\kappa_{LX}=\alpha_{\tilde{F}LX}\circ\Lambda_{LX}\circ\gamma_{LX}$*), we have* $\alpha_{LX}(\mu\,\mathsf{log}_{LX})=\nu\,\mathsf{beh}_{LX}$.
3. *Now define another logic function* $\mathsf{log}'_X=\mathcal{P}(c^\bullet)\circ\Lambda'_X\cup\Theta'_X$ *with* $\Theta_{LX}=\alpha'_X(\Theta'_X)$. *It holds that* $\alpha'_X\circ\mathsf{log}'_X\circ\gamma'_X=\mathsf{log}_{LX}$ *and we obtain* $\alpha_{LX}(\alpha'_X(\mu\,\mathsf{log}'_X))=\nu\,\mathsf{beh}_X$.

We hence consider a simple logic $\mathcal{L}_{\mathrm{EM}}$ for $\mathbf{EM}(T)$, where T is a monad on $\mathbf{Set}$:

$$\varphi\in\mathcal{L}_{\mathrm{EM}}::=\theta\ \mid\ [\lambda]\varphi\qquad(\text{where }\theta\in\Theta,\lambda\in\Lambda)$$

Given a coalgebra $c\colon X\to FTX\in\mathbf{Set}$, each formula $\varphi\in\mathcal{L}_{\mathrm{EM}}$ is interpreted as a function $[\![\varphi]\!]\colon X\to\Omega$, which is defined by structural induction as follows:

- Let $\varphi=\theta$. Then $[\![\varphi]\!]$ is given by a predefined constant $X\to\Omega$.
- Let $\varphi=[\lambda]\varphi'$. Then $[\![\varphi]\!]=ev_\lambda\circ Fo\circ FT[\![\varphi]\!]\circ c$ (see definition of Λ'_X in Lemma 24).

▶ **Corollary 26.** *Under the requirements of Theorem 25, the logic* $\mathcal{L}_{EM}$ *is expressive for the behavioural conformance* beh_{LX}*, i.e.,* $\alpha_{LX}(\alpha'_X(\{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_{EM}\})) = \nu\,\mathsf{beh}_{LX}$.

6.2 The Machine Functor

Our next aim is to show that the machine functor has certain natural evaluation maps ensuring that the predicate lifting is cl-compatible (one of the conditions of Theorem 25). Throughout this section, we restrict ourselves to a monad T on **Set** and fix the *machine functor* $M = \_^{\Sigma} \times B$ with $\Sigma \in \mathbf{Set}$ and $(B, b) \in \mathbf{EM}(T)$. Since all monads in **Set** are strong and B is endowed with a T-algebra structure $b\colon TB \to B$, there is a canonical distributive law ζ [19, Exercise 5.4.4]:

$$\zeta_X : T(X^{\Sigma} \times B) \xrightarrow{\langle a \mapsto T(\pi_a \circ \pi_1),\, b \circ T\pi_2 \rangle} (TX)^{\Sigma} \times B, \tag{1}$$

where $(\pi_i)_{i \in \{1,2\}}$ are the usual projections and $\pi_a\colon X^{\Sigma} \to X$ is the evaluation map ($\pi_a(g) = g(a)$ where $g\colon \Sigma \to X$). Now let $\tilde{M}$ be the lifting of M to $\mathbf{EM}(T)$, induced by the ζ. We observe that the evaluation maps suggested by it arise from natural transformations in the sense of Lemma 18.

▶ **Proposition 27.**
1. *Let* $a \in \Sigma$*. Then* $\eta_a\colon \tilde{M} \Rightarrow Id$ *given by the composite* $MX \xrightarrow{\pi_1} X^{\Sigma} \xrightarrow{\pi_a} X$ *is a natural transformation.*
2. *Let* $f\colon (B, b) \to (\Omega, o)$ *be a homomorphism. Then* $\eta'_f\colon \tilde{M} \Rightarrow \mathbf{\Omega}$ *given by the composite* $MX \xrightarrow{\pi_2} B \xrightarrow{f} \Omega$ *is a natural transformation.*

Thus, $ev_a = \eta_\Omega$*,* $ev_f = \eta'_\Omega$ *satisfy the properties of Lemma 18, and if each evaluation map is of this form, then* Λ *is* cl*-compatible.*

6.3 Alternative Formulation of Kantorovich Lifting

The behaviour function in Section 6.1 is based on the generalized Kantorovich lifting κ [2], which works as follows: given a pseudometric d on Y (here $Y = TX$), generate *all* non-expansive functions $Y \to \Omega$ wrt. d, lift these functions to $FY \to \Omega$ and from there generate a pseudometric on FY. However, κ_{LX} – since it is defined in an Eilenberg-Moore category – works subtly differently: it takes *all non-expansive functions that are algebra homomorphisms.* This looks natural in the categorical setting, but may pose problems if we implement the procedures. The standard (probabilistic) Kantorovich lifting can for instance be computed based on the Kantorovich-Rubinstein duality, by determining optimal transport plans [40].

Here, both types of liftings coincide at least on relevant metrics. To show this result, we first define an alternative way of lifting, as opposed to defining the lifting on T-algebra maps. Applying Theorem 7 on Ψ (rather than on $\Psi \circ R$) gives the adjunction $\alpha^{\mathbf{C}}_X \dashv \gamma^{\mathbf{C}}_X\colon \mathcal{P}\mathbf{C}(X, \Omega) \rightleftarrows \Psi X^{\mathrm{op}}$. Now consider the lifting $\kappa^{\mathbf{C}}_{TX} = \alpha^{\mathbf{C}}_{FTX} \circ \Lambda_{TX} \circ \gamma^{\mathbf{C}}_{TX}$, where $\Lambda_{TX}\colon \mathcal{P}\mathbf{C}(TX, \Omega) \to \mathcal{P}\mathbf{C}(FTX, \Omega)$ is defined identically to Λ_{LX}.

▶ **Theorem 28.** *Assume that* d *is preserved by the co-closure, i.e.* $d = \alpha_{LX}(\gamma_{LX}(d))$*, the co-closure* $\alpha^{\mathbf{C}}_X \circ \gamma^{\mathbf{C}}_X$ *is the identity, and each evaluation map* ev_λ *arises from some natural transformation either of type* $F \Rightarrow Id$ *or* $F \Rightarrow \mathbf{\Omega}$*. Then the two liftings* $\kappa_{LX}, \kappa^{\mathbf{C}}_{TX}$ *coincide on* d*, i.e.,* $\kappa_{LX}(d) = \kappa^{\mathbf{C}}_{TX}(d)$*.*

The condition $d = \alpha_{LX}(\gamma_{LX}(d))$ is a necessary, but not a serious restriction: this property is typically satisfied by the $\top$ metric and is preserved by the behaviour function. Hence during fixpoint iteration this invariant is preserved and the greatest fixpoints of beh_X based on either version of the Kantorovich lifting coincide (if the fixpoint is reached in ω steps).

In addition, if $\mathbf{C} = \mathbf{Set}$ and $\mathcal{V}$ is an integral quantale, it can easily be shown that the co-closure $\alpha_X^{\mathbf{C}} \circ \gamma_X^{\mathbf{C}}$ is the identity. This enables us to concretely spell out the behaviour function for the case of the machine functor $M = \_^\Sigma \times B$, provided that the conformances ΨX are (directed) pseudometrics.

▶ **Theorem 29.** *Assume that $\mathbf{C} = \mathbf{Set}$, $\Psi X = \mathbf{DPMet}_{\mathcal{V}}(X)$ (resp. $\Psi X = \mathbf{PMet}_{\mathcal{V}}(X)$) for an integral quantale $\mathcal{V}$ and let $d_{\mathcal{V}} = [\_,\_]$ (resp. the symmetrized variant of $[\_,\_]$). Let $d\colon LX \times LX \to \mathcal{V}$ be a pseudometric that is preserved by the co-closure $\alpha_{LX} \circ \gamma_{LX}$. Assume that M is the machine functor and the family of evaluation maps is*

$$\{ev_a \mid a \in \Sigma\} \cup \{ev_f \mid f \in \mathcal{F} \subseteq \mathbf{EM}(T)((B, b), (\mathcal{V}, o))\}.$$

Then the corresponding behaviour function $\mathsf{beh}_{LX}\colon \Psi(LX) \to \Psi(LX)$ is defined as follows: let $t_1, t_2 \in LX$ with $c^\#(t_i) = (b_i, g_i) \in B \times LX^\Sigma$:

$$\mathsf{beh}_{LX}(d)(t_1, t_2) = \bigwedge_{a \in \Sigma} d(g_1(a), g_2(a)) \wedge \bigwedge_{f \in \mathcal{F}} d_{\mathcal{V}}(f(b_1), f(b_2)) \wedge \bigwedge_{\theta \in \Theta_{LX}} d_{\mathcal{V}}(\theta(t_1), \theta(t_2)),$$

The above function beh is co-continuous and fixpoint iteration terminates after ω steps.

7 Case Studies for the Linear-time Case

7.1 Workflow

We recall the parameters of our framework and set out a workflow that we follow in our case studies. Let F be a machine functor and T a monad on a category $\mathbf{C}$.

- Model systems as coalgebras of type $c\colon X \to FTX$ with a distributive law $\zeta\colon TF \Rightarrow FT$.
- Fix a truth value object $(\Omega, o) \in \mathbf{EM}(T)$ and $d_\Omega \in \Psi\Omega$.
- Define a fibration (indexed category) $\Phi = \Psi \circ R\colon \mathbf{EM}(T)^{\mathrm{op}} \to \mathbf{Pos}$ by fixing an indexed category $\Psi\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Pos}$ to define the conformances.
- Fix a set Λ of evaluation maps (predicate liftings) as homomorphisms $\tilde{F}(\Omega, o) \to (\Omega, o)$.
- Fix a set of constants $\Theta_X \subseteq \mathbf{C}(X, \Omega)$.

Note that the last four conditions correspond to Assumptions **A1-A5**, which are necessary to set up a logic and a behaviour function beh as defined in Section 6.1 and guarantee expressiveness of the resulting logic. Whenever we choose $\Psi X = \mathbf{DPMet}_{\mathcal{V}}(X)$ or $\Psi X = \mathbf{PMet}_{\mathcal{V}}(X)$ for an integral quantale $\mathcal{V}$, we can rely on the characterization of the fixpoint equation in Theorem 29.

We present one worked out case studies based on this workflow, two others are given in Appendix A.

7.2 Trace Distance for Probabilistic Automata

A *probabilistic automaton* [31] is a quadruple (X, Σ, μ, p) where for each state $x \in X$ and each possible action $a \in \Sigma$ there is a probability distribution $\mu_{x,a}$ on the possible successors in X, and where each state $x \in X$ has a payoff value $p(x) \in [0, 1]$. Following [34, 35], we model them as coalgebras in the Eilenberg-Moore setting as detailed in the table below.

$\mathbf{C} = \mathbf{Set}$, $F = \_^{\Sigma} \times [0,1]$, $T = \mathcal{D}$ ($B = [0,1]$, b expectation) $c\colon X \to (\mathcal{D}X)^{\Sigma} \times [0,1]$ $c^{\#}\colon \mathcal{D}X \to (\mathcal{D}X)^{\Sigma} \times [0,1]$	**Logic:** evaluation maps: ev_a (Prop. 27(1)) $ev_*(f,r) = r$ (Prop. 27(2)) constants $\Theta_X = \emptyset$ formulas: $\varphi = [a_1]\cdots[a_n]*$
$\Omega = [0,1]$ (Ex. 2(2)) $o\colon \mathcal{D}[0,1] \to [0,1]$ expectation $\Psi(X) = \mathbf{PMet}_{[0,1]}(X)$ $d_\Omega(r,s) = \lvert r-s \rvert$	**Behaviour function:** $\mathsf{beh}_{\mathcal{D}X}\colon \Psi(\mathcal{D}X) \to \Psi(\mathcal{D}X)$ $\mathsf{beh}_{\mathcal{D}X}(d)(p_1,p_2) = \max\{\sup_{a\in\Sigma} d(g_1(a), g_2(a)), d_\Omega(r_1,r_2)\}$

Thus, given a formula $\varphi = [a_1]\dots[a_n]*$ and a state $x \in X$, $[\![\varphi]\!](x)$ gives us the expected payoff after choosing actions according to the word $a_1 \dots a_n$. The distance of two states x_1, x_2 is hence the supremum of the difference of payoffs, over all words.

Expressiveness again follows from Corollary 26.

8 Conclusion

Related work. By now there is a large number of papers considering coalgebraic semantics beyond branching-time, for instance [15, 34, 27, 7]. Furthermore in the same period quite a wealth of results on the treatment of behavioural metrics in coalgebraic generality has been published [38, 2, 23, 12]. However, there is little work combining both linear-time semantics and behavioural metrics in the setting of coalgebra. In that respect we want to mention [13] that is based on the graded monad framework [27] and which investigates exactly this combination. However, different from the present paper, the focus is on the expressiveness of the logics with respect a graded semantics (that intuitively specifies the traces of a state). Hence, using the classification of the introduction, it studies the relationship of (i) and (ii).

Unlike other approaches, our main focus is on exploiting an adjunction (Galois connection) and fixpoint preservation results to obtain Hennessy-Milner theorems "for free". We start by setting up a logic, characterizing the behavioural equivalence, and investigate under which circumstances we can derive a corresponding fixpoint characterization. The fixpoint equation might be defined on an infinite state space, but often there are finitary techniques that can be employed, such as reducing the state space to a finite subset, linear programming, up-to techniques, etc. In particular, for systems as in Appendix A.2 we are working on promising results (based on [3]) for deriving bounds for behavioural distances via finite witnesses using up-to techniques, even for infinite state spaces. The algorithmic angle of our approach is not yet fully worked out in the present paper but establishing fixpoint equations as we do here is a necessary first step in this direction.

Note that our concept deviates from the dual adjunction approach [21, 24, 25, 28] to coalgebraic modal logic. There the functor on the "logic universe" characterizes the *syntax* of the logics, while the semantics is instead given by a natural transformation. Nevertheless, it complements (at least when restricted to the classical case of Boolean predicates) the recent approach [37] that combines fibrations in the dual adjunction setup since having contravariant Galois connections between fibres (at a "local" level) is equivalent to having dual adjunctions between certain fibred categories (at a "global" level). It is unclear how to establish this correspondence in the setting of quantitative $\mathcal{V}$-valued predicates.

Future work. Currently the operators of the logic, given by cl', are rather generic, although we instantiated them in special cases to ensure expressiveness (see in particular Sections 5 and 6.2). We envision a general theory to ensure expressiveness of the logics, similar to

Post's functional completeness theorem [29], which characterizes complete sets of operators for the boolean case. This question is strongly related to the notion of an approximating family in [23] that has again close connections to compatibility as discussed in [4].

We will also study the condition requiring that a conformance (pseudometric) is preserved by the co-closure ($d = \alpha_{LX}(\gamma_{LX}(d))$). Previous results [4] suggest that this is related to the notion of (metric) congruence, as e.g. defined in [6], but the connection seems to be non-trivial.

Another avenue of research is to further investigate the quantale-valued logic for the branching case introduced in Section 5, to extend it to the undirected case and restrict to finitary operators.

References

1 Jiří Adámek, Horst Herrlich, and George E. Strecker. *Abstract and concrete categories: The joy of cats*. Wiley, 1990. Republished in: Reprints in Theory and Applications of Categories, No. 17 (2006) pp. 1–507. URL: `http://tac.mta.ca/tac/reprints/articles/17/tr17abs.html`.

2 Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. Coalgebraic behavioral metrics. *Logical Methods in Computer Science*, 14(3), 2018. Selected Papers of the 6th Conference on Algebra and Coalgebra in Computer Science (CALCO 2015).

3 Paolo Baldan, Barbara König, and Tommaso Padoan. Abstraction, up-to techniques and games for systems of fixpoint equations. In *Proc. of CONCUR '20*, volume 171 of *LIPIcs*, pages 25:1–25:20. Schloss Dagstuhl – Leibniz Center for Informatics, 2020.

4 Harsh Beohar, Sebastian Gurke, Barbara König, and Karla Messing. Hennessy-Milner theorems via Galois connections. In *Proc. of CSL '23*, volume 252 of *LIPIcs*, pages 12:1–12:18. Schloss Dagstuhl – Leibniz Center for Informatics, 2023.

5 Harsh Beohar, Sebastian Gurke, Barbara König, Karla Messing, Jonas Forster, Lutz Schröder, and Paul Wild. Expressive quantale-valued logics for coalgebras: an adjunction-based approach, 2023. arXiv:2310.05711. URL: `https://arxiv.org/abs/2310.05711`.

6 Filippo Bonchi, Alexandra Silva, and Ana Sokolova. The Power of Convex Algebras. In *Proc. of CONCUR '17*, volume 85 of *LIPIcs*, pages 23:1–23:18. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017.

7 Corina Cîrstea. From branching to linear time, coalgebraically. *Fundamenta Informaticae*, 150(3-4):379–406, 2017.

8 Patrick Cousot and Radhia Cousot. Systematic design of program analysis frameworks. In *Proc. of POPL '79*, pages 269–282. ACM Press, 1979.

9 Patrick Cousot and Radhia Cousot. Temporal abstract interpretation. In Mark N. Wegman and Thomas W. Reps, editors, *Proc. of POPL '00*, pages 12–25. ACM, 2000.

10 Josée Desharnais, Vineet Gupta, Radha Jagadeesan, and Prakash Panangaden. Metrics for labelled Markov processes. *Theoretical Computer Science*, 318:323–354, 2004.

11 Ulrich Dorsch, Stefan Milius, and Lutz Schröder. Graded monads and graded logics for the linear time - branching time spectrum. In Wan J. Fokkink and Rob van Glabbeek, editors, *Concurrency Theory, CONCUR 2019*, volume 140 of *LIPIcs*, pages 36:1–36:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019. `doi:10.4230/LIPIcs.CONCUR.2019.36`.

12 Jonas Forster, Sergey Goncharov, Dirk Hofmann, Pedro Nora, Lutz Schröder, and Paul Wild. Quantitative Hennessy-Milner Theorems via Notions of Density. In *Proc. of CSL '23*, volume 252 of *LIPIcs*, pages 22:1–22:20, 2023.

13 Jonas Forster, Lutz Schröder, Paul Wild, Harsh Beohar, Sebastian Gurke, Barbara König, and Karla Messing. Graded semantics and graded logics for Eilenberg-Moore coalgebras, 2023. arXiv:2307.14826. URL: `https://arxiv.org/abs/2307.14826`.

14 Alessandro Giacalone, Chi-Chang Jou, and Scott A. Smolka. Algebraic reasoning for probabilistic concurrent systems. In Manfred Broy and Cliff B. Jones, editors, *Programming concepts and methods*, pages 443–458. North-Holland, 1990.

15 Ichiro Hasuo, Bart Jacobs, and Ana Sokolova. Generic trace semantics via coinduction. *Logical Methods in Computer Science*, 3(4:11):1–36, 2007.

16 Dirk Hofmann, Gavin J. Seal, and Walter Tholen, editors. *Lax algebras*, pages 145–283. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2014. doi:10.1017/CBO9781107517288.005.

17 Bart Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 1st edition, January 1999.

18 Bart Jacobs. Predicate logic for functors and monads. Available from author's website, 2010. URL: http://www.cs.ru.nl/~bart/PAPERS/predlift-indcat.pdf.

19 Bart Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2016. doi:10.1017/CBO9781316823187.

20 Bart Jacobs, Alexandra Silva, and Ana Sokolova. Trace semantics via determinization. In *Proc. of CMCS '12*, pages 109–129. Springer, 2012. LNCS 7399.

21 Bartek Klin. Coalgebraic modal logic beyond sets. In *Proc. of MFPS '07*, volume 173 of *ENTCS*, pages 177–201, 2007.

22 Yuichi Komorida, Shin-ya Katsumata, Nick Hu, Bartek Klin, and Ichiro Hasuo. Codensity games for bisimilarity. In *Proc. of LICS '19*, pages 1–13, 2019. doi:10.1109/LICS.2019.8785691.

23 Yuichi Komorida, Shin-ya Katsumata, Clemens Kupke, Jurriaan Rot, and Ichiro Hasuo. Expressivity of quantitative modal logics: Categorical foundations via codensity and approximation. In *Proc. LICS '21*, pages 1–14. IEEE, 2021.

24 Clemens Kupke and Dirk Pattinson. Coalgebraic semantics of modal logics: An overview. *Theoretical Computer Science*, 412:5070–5094, 2011.

25 Clemens Kupke and Jurriaan Rot. Expressive logics for coinductive predicates. In *Proc. of CSL '20*, volume 152 of *LIPIcs*, pages 26:1–26:18. Schloss Dagstuhl – Leibniz Center for Informatics, 2020.

26 F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matemàtico e Fisico di Milano*, 43(1):135–166, 1973. Republished in Reprints in Theory and Applications of Categories 1 (2002), 1–37. doi:10.1007/bf02924844.

27 Stefan Milius, Dirk Pattinson, and Lutz Schröder. Generic trace semantics and graded monads. In *Proc. of CALCO '15*, volume 35 of *LIPIcs*, pages 253–269. Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, 2015.

28 Dusko Pavlovic, Michael Mislove, and James Worrell. Testing semantics: Connecting processes and process logics. In *Proc. of AMAST '06*, pages 308–322. Springer, 2006. LNCS 4019.

29 Francis Jeffry Pelletier and Norman M. Martin. Post's functional completeness theorem. *Notre Dame Journal of Formal Logic*, 31(2), 1990.

30 Damien Pous. Complete lattices and up-to techniques. In *Proc. of APLAS '07*, pages 351–366. Springer, 2007. LNCS 4807.

31 Michael O. Rabin. Probabilistic automata. *Information and Control*, 6(3):230–245, 1963.

32 Jan Rutten. Universal coalgebra: A theory of systems. *Theoretical Computer Science*, 249:3–80, 2000. doi:10.1016/S0304-3975(00)00056-6.

33 Lutz Schröder. Expressivity of coalgebraic modal logic: The limits and beyond. *Theoretical Computer Science*, 390(2):230–247, 2008. Foundations of Software Science and Computational Structures. doi:10.1016/j.tcs.2007.09.023.

34 Alexandra Silva, Filippo Bonchi, Marcello M. Bonsangue, and Jan J. M. M. Rutten. Generalizing the powerset construction, coalgebraically. In *Proc. of FSTTCS '10*, volume 8, pages 272–283. Schloss Dagstuhl – Leibniz Center for Informatics, 2010. LIPIcs.

35 Alexandra Silva, Filippo Bonchi, Marcello M. Bonsangue, and Jan J. M. M. Rutten. Generalizing determinization from automata to coalgebras. *Logical Methods in Computer Science*, 9(1:09), 2013.

36 Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.

37 Ruben Turkenburg, Harsh Beohar, Clemens Kupke, and Jurriaan Rot. Forward and Backward Steps in a Fibration. In Paolo Baldan and Valeria de Paiva, editors, *10th Conference on Algebra and Coalgebra in Computer Science (CALCO 2023)*, volume 270 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 6:1–6:18, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.CALCO.2023.6.

38 Franck van Breugel and James Worrell. A behavioural pseudometric for probabilistic transition systems. *Theoretical Computer Science*, 331:115–142, 2005.

39 Rob van Glabbeek. The linear time – branching time spectrum I. In J.A. Bergstra, A. Ponse, and S.A. Smolka, editors, *Handbook of Process Algebra*, chapter 1, pages 3–99. Elsevier, 2001.

40 Cédric Villani. *Optimal Transport – Old and New*, volume 338 of *A Series of Comprehensive Studies in Mathematics*. Springer, 2009.

41 Paul Wild and Lutz Schröder. A quantified coalgebraic van Benthem theorem. In *Proc. of FOSSACS '21*, volume 12650 of *LNCS*, pages 551–571. Springer, 2021.

42 Paul Wild and Lutz Schröder. Characteristic logics for behavioural hemimetrics via fuzzy lax extensions. *Logical Methods in Computer Science*, 18(2), 2022. doi:10.46298/lmcs-18(2:19)2022.

43 Paul Wild, Lutz Schröder, Dirk Pattinson, and Barbara König. A van Benthem theorem for fuzzy modal logic. In Anuj Dawar and Erich Grädel, editors, *Logic in Computer Science, LICS 2018*, pages 909–918. ACM, 2018. doi:10.1145/3209108.3209180.

A Case Studies for the Linear-time Case

A.1 Trace Equivalence for Labelled Transition Systems

We spell out a simple case study: trace equivalence [39] for labelled transition systems. The main ingredients are summarized in the table below.

$\mathbf{C} = \mathbf{Set}$, $F = \_^{\Sigma}$, $T = \mathcal{P}$ $(B = 1)$ $c\colon X \to (\mathcal{P}X)^{\Sigma}$ $c^{\#}\colon \mathcal{P}X \to (\mathcal{P}X)^{\Sigma}$	**Logic:** evaluation maps: ev_a (Prop. 27(1)) constants $\Theta_X = \{1\}$, constant 1-function formulas: $\varphi = [a_1]\cdots[a_n]1$
$\Omega = \mathbf{2}$ (Ex. 2(1)) $o\colon \mathcal{P}\mathbf{2} \to \mathbf{2}$ supremum $\Psi(X)$: equivalences on X d_Ω: equality on Ω	**Behaviour function:** $\mathsf{beh}_{\mathcal{P}X}\colon \Psi(\mathcal{P}X) \to \Psi(\mathcal{P}X)$ $\mathsf{beh}_{\mathcal{P}X}(R)(U,V) = (U = \emptyset \Leftrightarrow V = \emptyset) \land$ $\forall_{a\in\Sigma}(c^{\#}(U)(a), c^{\#}(V)(a)) \in R$

The modality $[a]$ boils down to the standard diamond modality (due to the definition of o); a state $x \in X$ satisfies $\varphi = [a_1]\cdots[a_n]1$ iff there exists a trace $a_1\cdots a_n$ from x. The constant 1 is needed to start building formulas and to distinguish the empty set from a non-empty set on $LX = \mathcal{P}X$. (Note that $\Theta_{LX} = \alpha'_X(\Theta_X) = \{\tilde{1}\}$ with $\tilde{1}(Y) = 0$ iff $Y = \emptyset$.) Its role cannot be taken by a constant modality or an operator, since those have to be homomorphisms in $\mathbf{EM}(\mathcal{P})$, hence sup-preserving.

Expressiveness of trace logic $\mathcal{L}_{\mathrm{EM}}$ now directly follows from Corollary 26.

A.2 Directed Fuzzy Trace Distance

We now consider directed trace distances for weighted transition systems over a generic quantale.

We work with the "fuzzy" monad $T = \mathcal{P}_{\mathcal{V}}$ (aka $\mathcal{V}$-valued powerset monad [16, Remark 1.2.3]) on **Set** that is defined as $\mathcal{P}_{\mathcal{V}} = \mathcal{V}^X$ on objects and as $Tf(g)(y) = \bigvee_{f(x)=y} g(x)$ (for $f\colon X \to Y$) on arrows. Its unit $\eta_X\colon X \to \mathcal{P}_{\mathcal{V}}X$ is given by $\eta_X(x)(x') = 1$ if $x = x'$

and 0 (the empty join) otherwise. Multiplication $\mu_X\colon \mathcal{P}_\mathcal{V}\mathcal{P}_\mathcal{V}X \to \mathcal{P}_\mathcal{V}X$ is defined as $\mu_X(G)(x) = \bigvee_{g\in\mathcal{P}_\mathcal{V}X} G(g)\otimes g(x)$. Note that for $\mathcal{V} = 2$ (cf. Example 2(1)) T corresponds to the powerset monad $\mathcal{P}$.

$\mathbf{C} = \mathbf{Set}$, $F = \_^\Sigma$, $T = \mathcal{P}_\mathcal{V}$ $(B = 1)$ $c\colon X \to (\mathcal{P}_\mathcal{V}X)^\Sigma$ $c^\#\colon \mathcal{P}_\mathcal{V}X \to (\mathcal{P}_\mathcal{V}X)^\Sigma$	**Logic:** evaluation maps: ev_a (Prop. 27(1)) constants $\Theta_X = \{1\}$, constant 1-function formulas: $\varphi = [a_1]\cdots[a_n]1$
$\Omega = \mathcal{V}$, $o\colon \mathcal{P}_\mathcal{V}\mathcal{V} \to \mathcal{V}$ $g \mapsto \bigvee_{v\in\mathcal{V}} g(v)\otimes v$ $\Psi(X) = \mathbf{DPMet}_\mathcal{V}(X)$ $d_\Omega(v, v') = [v, v']$	**Behaviour function:** $\mathsf{beh}_{\mathcal{P}_\mathcal{V}X}\colon \Psi(\mathcal{P}_\mathcal{V}X) \to \Psi(\mathcal{P}_\mathcal{V}X)$ $\mathsf{beh}_{\mathcal{P}_\mathcal{V}X}(d)(g_1, g_2) = \bigwedge_{a\in\Sigma} d(c^\#(g_1)(a), c^\#(g_2)(a)) \wedge$ $\left[\bigvee_{x\in X} g_1(x), \bigvee_{x\in X} g_2(x)\right]$

Evaluating a formula $\varphi = [a_1]\dots[a_n]1$ on a state $x_0 \in X$ results in

$$[\![\varphi]\!](x_0) = \bigvee\{\bigotimes_{0\le i<n} c(x_i)(a_{i+1})(x_{i+1}) \mid x_1\dots x_n \in X^n\}.$$

This follows directly from structural induction on φ, from distributivity and from the evaluation of the modality $[a]\varphi'$:

$$[\![[a]\varphi']\!](x) = \bigvee_{y\in X} c(x)(a)(y)\otimes[\![\varphi']\!](y).$$

Intuitively we check how well x can match the trace $a_1\dots a_n$, where $c(x)(a)(y)$ measures the degree to which x can make an a-transition to y.

The second part of the minimum in the definition of beh stems from the constants $\Theta_X = \{1\}$, since $\Theta_{LX} = \alpha'_X(\Theta_X) = \{\tilde{1}\}$ with $\tilde{1}(h) = \bigvee_{x\in X} h(x)$ for $h\colon X \to \mathcal{V}$. Without it, the fixpoint iteration would stabilize at the constant 1-pseudometric.

Expressiveness again follows from Corollary 26. Expressiveness of a logic for *symmetric* fuzzy trace distance has already been shown in previous work [13].