Graph Isomorphism Completeness for Chordal bipartite graphs and Strongly Chordal Graphs

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Abstract

This paper deal with the graph isomorphism (GI) problem for two graph classes: chordal bipartite graphs and strongly chrdal graphs. It is known that GI problem is GI complete for some special graph classes including regular graphs, bipartite graphs, chordal graphs, comparability graphs, split graphs, and k-trees for unbounded k. On the other side, the relative complexity of the GI problem for the above classes was unknown. We prove that deciding isomorphism of the classes are GI complete.

Key words: Graph isomorphism problem, Graph isomorphism complete, Strongly chordal graphs, Chordal bipertite graphs

1 Introduction

The graph isomorphism (GI) problem is a well-known open problem and was listed as an important open problem in Karp (Kar72) three decades ago. Although the problem is trivially in NP, the problem is not known to be in P and not known to be NP-complete either (see (RC77; KST93)). In particular, Mathon (Mat79) showed that the counting version of GI problem is

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GI-complete. Thus, it is very unlikery that the GI problem is NP-complete as well as the unlikelihood that the problem is in P given the recent results coming from cryptography.

The GI problem is called *GI complete* for a graph class if the GI problem for the graph class is polynomial time equivalent to the problem for general graphs. The GI problem is GI complete for several graph classes including regular graphs, bipartite graphs, chordal graphs, comparability graphs, and split graphs, and k-trees for unbounded k (see (BC79) for a review). On the other hand, the GI problem is solvable in polynomial time when it is restricted to special graph classes, e.g., graphs of bounded degrees, planar graphs, interval graphs, permutation graphs, k-trees for fixed k (see (BPT96) for reference), and convex graphs (Che99).

Recently, many graph classes have been proposed and widely investigated (see (BLS99) for a comprehensive survey). However, relative complexity of the GI problem is not known for some graph classes. Among them, the graph classes of strongly chordal graphs and chordal bipartite graphs are on the border. We show that the GI problem for the graph classes is GI complete. This results solve the open problems listed by J.P. Spinrad (Spi95; BPT96).

2 Preliminaries

For given graph $G = (V, E)$, $G[U]$ denotes the subgraph of G induced by $U \subseteq V$. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if and only if there is a one-to-one mapping $\phi : V \to V'$ such that $\{u, v\} \in E$ if and only if $\{\phi(u), \phi(v)\} \in E'$ for every pair of vertices $u, v \in V$. We denote by $G \sim G'$ if G and G' are isomorphic. The *graph isomorphism (GI) problem* is to determine if $G \sim G'$ for given graphs G and G'. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. A graph is *chordal* if every cycle of length at least 4 has a chord. A graph is *chordal bipartite* if the graph is bipartite and every cycle of length at least 6 has a chord. A chord $\{x_i, x_j\}$ in a cycle $(x_1, x_2, \cdots, x_{2k}, x_1)$ of even length $2k$ is an *odd chord* if $|j - i| \equiv 1 \pmod 2$. A graph is *strongly chordal* if G is chordal and each cycle in G of even length at least 6 has an odd chord. I_n and K_n denote an independent set and a clique of size n, respectively. A graph $G = (V, E)$ is a *split graph* if V can be partitioned into two subsets X and Y such that $G[X]$ is $K_{|X|}$ and $G[Y]$ is $I_{|Y|}$.

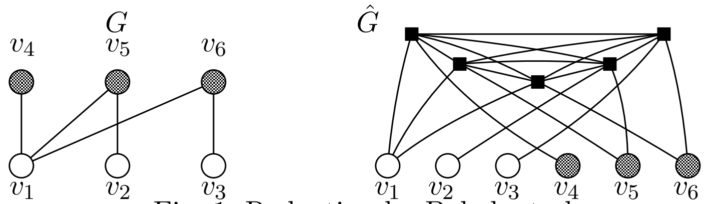

Fig. 1. Reduction by Babel, et al.

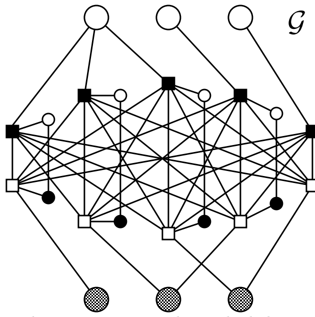

Fig. 2. Reduction to chordal bipartite graph

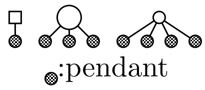

Fig. 3. Pendant vertices

3 Main Results

In (BPT96), Babel, et al. give the following reduction from a bipartite graph to a directed path (DP) graph such that two given bipartite graphs are isomorphic if and only if the reduced DP graphs are isomorphic: given bipartite graph $G = (X, Y, E)$ with $|X \cup Y| = n$ and $|E| = m$, the edge set $\hat{E}$ of the reduced graph $\hat{G} = (X \cup Y \cup E, \hat{E})$ contains $\{e, e'\}$ for all $e, e' \in E$, and $\{x, e\}$ and $\{y, e\}$ for each $e = \{x, y\} \in E$ (Figure 1). Our starting point is the DP graph $\hat{G}$ that is a split graph having the following properties: (a) $\hat{G}[E] \sim K_m$, (b) $\hat{G}[X \cup Y] \sim I_n$, and (c) each $e \in E$ has exactly one neighbor in X, and another one in Y (thus $d(e) = m+1$). Without loss of generality, we also assume that (d) $m > 1$ and (e) $|X| > |Y| > 1$ (if $|X| = |Y|$, construct new graph $(X_1 \cup Y_2 \cup \{v\}, X_2 \cup Y_1, E')$ from (X, Y, E) as follows: for each $e = \{x, y\} \in E$, $x_i \in X_i$, $y_i \in Y_i$, and $\{x_i, y_i\} \in E'$ for $i = 1, 2$, and for every $u \in X_2 \cup Y_1$, $\{v, u\} \in E'$).

We reduce the split graph $\hat{G} = (X \cup Y \cup E, \hat{E})$ to a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We set $\mathcal{V} = X \cup Y \cup E \cup E' \cup B \cup W$ such that each vertex $e \in E$ corresponds to three vertices $e' \in E'$, $e_b \in B$, and $e_w \in W$, respectively (hence $|E| = |E'| = |B| = |W| = m$). Vertices are connected as follows: (1) for each $e \in E$, $\{e, e'\}$, $\{e', e_b\}$, $\{e_b, e_w\}$, $\{e, e_w\} \in \mathcal{E}$, (2) for each $e_1, e_2 \in E$, $\{e_1, e_2'\}$, $\{e_1', e_2\} \in \mathcal{E}$ (thus $\mathcal{G}[E \cup E'] \sim K_{m,m}$), and (3) for each vertex $x \in X$, $\{x, e\} \in \mathcal{E}$ if $\{x, e\} \in \hat{E}$, and for each vertex $y \in Y$, $\{y, e'\} \in \mathcal{E}$ if $\{y, e\} \in \hat{E}$. The reduced graph $\mathcal{G}$ for $\hat{G}$ in Figure 1 is shown in Figure 2. The reduction can be done in polynomial time.

Lemma 1 *$\mathcal{G}$ is chordal bipartite.*

Proof. Dividing $\mathcal{V}$ into $\mathcal{V}_1 = X \cup E' \cup W$ and $\mathcal{V}_2 = Y \cup E \cup B$, $\mathcal{G}$ is bipartite. We show every cycle of length at least 6 has a chord. To derive a contradiction, we assume that we have a chordless cycle C of length at least 6. Then we show that any four consecutive vertices on C has a chord.

If C contains at least one vertex in $B \cup W$, that induces C_4. Thus C only contains vertices in $X \cup Y \cup E \cup E'$. Let v_0, v_1, v_2, v_3 be consecutive vertices on C. If they are all in $E \cup E'$, since $\mathcal{G}[E \cup E']$ is $K_{m,m}$, $\{v_0, v_3\} \in \mathcal{E}$. Thus at least one vertex is in $X \cup Y$. Without loss of generality, assume that $v_1 \in X$, and hence $v_0, v_2 \in E$. Then, by (c), no other vertex in X is incident to v_0 and v_2. Since no vertices in Y are incident to E, $v_3 \in E'$. Hence $\{v_0, v_3\} \in \mathcal{E}$, which is a contradiction. Thus $\mathcal{G}$ is chordal bipartite. □

Lemma 2 *Given bipartite graphs G_1 and G_2, $G_1 \sim G_2$ if and only if $\mathcal{G}_1 \sim \mathcal{G}_2$.*

Proof. It is sufficient to show that $\hat{G}_1 \sim \hat{G}_2$ if $\mathcal{G}_1 \sim \mathcal{G}_2$. Given $\hat{G}_1$, we assume that $\mathcal{G}_1 \sim \mathcal{G}_2$ for some $\hat{G}_2$. We show that we can reconstruct $\hat{G}_2$ isomorphic to $\hat{G}_1$ from $\mathcal{G}_2$ uniquely.

Let $e = \{u, v\}$ be any edge with $d(u) = d(v) = 2$, and u' and v' be the (unique) vertices with $u' \in N(u) - \{v\}$ and $v' \in N(v) - \{u\}$. Then, by (c) and (d) and the reduction, $u, v \in B \cup W$ and $u'v' \in E \cup E'$. Therefore, finding all four-tuples using the *handles* e, and contracting each four-tuple into a vertex, we can reconstruct the graph $\hat{G}_2 \sim \hat{G}_1$. □

Theorem 3 *The GI problem is GI complete for chordal bipartite graphs and strongly chordal graphs.*

Proof. Lemmas 1 and 2 imply the claim for chordal bipartite graphs. We next reduce the chordal bipartite graph $\mathcal{G} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$ constructed above to a graph $\mathcal{G}'$ as follows: first add edges to change the independent set $\mathcal{V}_1$ into a clique. More precisely, we add $e = \{u, v\}$ for each $u, v \in \mathcal{V}_1$. We note that we can uniquely determine the set $\mathcal{V}_1$ since $|\mathcal{V}_1| > |\mathcal{V}_2|$ by (e). We then add *pendant* vertices to each vertex in $\mathcal{V}_1$ as follows (see Figure 3): (1) for each vertex $e' \in E'$, we add vertex e'' into $\mathcal{V}$ and an edge $\{e', e''\}$ into $\mathcal{E}$, (2) for each vertex x in X, we add vertices x_i with $1 \leq i \leq 3$ into $\mathcal{V}$ and edges $\{x, x_i\}$ with $1 \leq i \leq 3$ into $\mathcal{E}$, and (3) for each vertex w in W, we add vertices w_i with $1 \leq i \leq 4$ into $\mathcal{V}$ and edges $\{w, w_i\}$ with $1 \leq i \leq 4$ into $\mathcal{E}$. Then $\mathcal{G}'$ is strongly chordal if $\mathcal{G}$ is chordal bipartite (BLS99, Theorem 3.4.3). On $\mathcal{G}'$, the following claims are easy to see: each vertex having four neighbors of degree 1 is in W, each vertex having three neighbors of degree 1 is in X, and each vertex having one or two neighbors of degree 1 is in E'. Moreover, for each vertex having two neighbors of degree 1, one of two neighbors is in Y. Thus, deleting the pendant vertices and the additional edges to make the clique $\mathcal{G}[\mathcal{V}_1]$, we can reconstruct $\mathcal{G}$ from $\mathcal{G}'$. Thus $\mathcal{G}_1 \sim \mathcal{G}_2$ iff $\mathcal{G}'_1 \sim \mathcal{G}'_2$, which completes the proof. □

4 Concluding Remarks

The class of strongly chordal graphs is between chordal graphs and interval graphs. Babel, et al. show that the GI problem for directed path (DP) graphs is GI complete, while the GI problem for rooted directed path (RDP) graphs is polynomial time solvable in (BPT96). The class of the RDP graphs is between the strongly chordal graphs and interval graphs, although the class of the DP graphs is incomparable to strongly chordal graphs. In the paper, we draw a line between the RDP graphs and strongly chordal graphs for GI completeness, which answers the open problem stated in (BPT96).

The class of chordal bipartite graphs is between bipartite graphs and interval bigraphs. Recently, Hell and Huang show that any interval bigraph is the complement of a circular arc graph (HH03). Thus, combining the result by Hsu (Hsu95), we can see that the GI problem for interval bigraphs can be solved in polynomial time. Therefore we draw a line between the interval bigraphs and chordal bipartite graphs for GI completeness, which improves the GI completeness results.

As mentioned in Introduction, we have many graph classes, which are proposed recently, and we do not know whether the GI problem is GI complete or polynomial time solvable on some classes. In order to clarify the complexity of the GI problem, considering the GI problem on such graph classes is future work. For example, trapezoid graphs are the natural and classic graph class such that the complexity of the GI problems is still unknown, which is mentioned by Spinrad (Spi03).

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