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Affine Formation Maneuver Control of Multi-Agent Systems

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Abstract—**A multi-agent formation control task usually consists of two subtasks. The first is to steer the agents to form a desired geometric pattern and the second is to achieve desired collective maneuvers so that the centroid, orientation, scale, and other geometric parameters of the formation can be changed continuously. This paper proposes a novel affine formation maneuver control approach to achieve the two subtasks simultaneously. The proposed approach relies on stress matrices, which can be viewed as generalized graph Laplacian matrices with both positive and negative edge weights. The proposed control laws can track any target formation that is a time-varying affine transformation of a nominal configuration. The centroid, orientation, scales in different directions, and even geometric pattern of the formation can all be changed continuously. The desired formation maneuvers are only known by a small number of agents called leaders, and the rest agents called followers only need to follow the leaders. The proposed control laws are globally stable and do not require global reference frames if the required measurements can be measured in each agent's local reference frame.**

Index Terms—**Formation control, multi-agent systems, affine transformation, stress matrices**

I. INTRODUCTION

A multi-agent formation control task is usually constituted by two subtasks. The first is formation shape control, which is to steer a group of mobile agents to form a desired geometric pattern given any initial configuration. The second is formation maneuver control, which is to steer the mobile agents to maneuver as a whole such that the centroid, orientation, scale, and other geometric parameters of the formation can be changed continuously. Formation maneuver control is important for a formation of agents to achieve desired navigation tasks or dynamically respond to the environment to, for example, avoid obstacles.

Multi-agent formation control has been studied by various approaches in the last two decades. The approaches proposed in the early stage such as behavior-based ones can handle complicated formation tasks subject to various agent dynamics and constraints (see, for example, [1]–[4]). However, the system convergence of these approaches is difficult to prove mathematically [4]. From the practical point of view, system convergence is vital for a multi-agent control system because it guarantees the system to behave as expected.

Since the successful application of the consensus theory in formation control [5], [6], tremendous research efforts have been devoted to developing convergence-guaranteed formation control approaches (see [7], [8] for recent surveys). These existing formation control approaches can be classified by how the target formation is defined. For example, displacement-based, distance-based, and bearing-based approaches are three conventional approaches that define target formations by using *constant* constraints on inter-agent displacements, distances, and bearings, respectively [8]–[10]. The invariance of the constant constraints of the target formation has critical impact on the formation maneuverability. In particular, inter-agent displacement constraints are invariant to formation translation. As a result, displacement-based formation control laws can be applied to track target formations with time-varying translations [11], [12]. However, the scale or orientation of the formation is difficult to control using this approach because changing the scale or orientation requires changing the displacement constraints. As a comparison, distance-based control laws can be applied to track target formations with time-varying translations and orientations [13], [14], but it is difficult to track time-varying formation scales. Bearing-based control laws can track formations with time-varying translations and scales [9], [10], but it is difficult to track time-varying orientations.

Motivated by the limitations of the three approaches, researchers have proposed some methods to modify them in order to achieve desired formation maneuvers. For example, the work in [15] modified the displacement-based formation control approach by adding a formation scale estimation mechanism, and the work in [16] modified the distance-based formation control approach to allow the final formation has an unspecified scale. These modifications, however, usually result in complicated control and estimation problems, and may require additional sensing or communication abilities for each agent. An approach that can track general time-varying formations has been proposed recently in [17]. However, the desired maneuver of each agent must be pre-specified in this approach.

Very recently, researchers have proposed some approaches defining target formations using new types of constant constraints such as local bearings [18], barycentric coordinates [19], complex Laplacians [20], [21], and stress matrices [22]. These approaches are appealing due to the enhanced invariance of the new constraints. For example, a complex Laplacian is invariant to the translation, rotation, and scaling variations of a formation. As a result, the approach based on complex Laplacians can be applied to simultaneously achieve translational, rotational, and scaling formation maneuvers. This approach is, however, merely applicable to formation control in two dimensions.

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Among these new approaches, the one based on stress matrices is promising to achieve general formation maneuvers. The stress matrix of a formation can be viewed as a generalized graph Laplacian. Its structure is determined by the underlying graph, but the values of the entries are jointly determined by the formation configuration. Unlike conventional graph Laplacian matrices, in a stress matrix the weight of an edge may be positive, negative, or zero. Stress matrices have been applied in stabilization of stationary target formations in [22], but their great potential to solve formation maneuver control has not been explored yet. In fact, the stress matrix is invariant to any affine transformation of the formation configuration. An affine transformation is a general linear transformation that may correspond to a translation, rotation, scaling, shear, or compositions of them. As a result, stress matrices provide a powerful tool to achieve various formation maneuver behaviors.

In this paper, we adopt the leader-follower strategy to solve the problem of formation maneuver control based on stress matrices. The main contributions of this paper are threefold. First, we address the leader selection problem and introduce the notion of affine formation localizability that indicates whether or not the selected leaders can fully control the entire formation to achieve desired affine transformations. Necessary and sufficient conditions for affine localizability are proved. Second, we propose a variety of distributed control laws for single- and double-integrator agent dynamics based on different types of measurements. With the proposed control laws, not only the desired formation pattern can be achieved, any time-varying affine transformation such as a translation, rotation, scaling, or even shape deformation of the formation can be tracked. The proposed control laws are globally stable and applicable to formation control in arbitrary dimensions. Third, we propose control laws for unicycle models subject to linear and angular velocity saturation constraints. The proposed nonlinear control laws are proved to be globally stable in the case of stationary leaders. It is worth mentioning that the proposed control laws do not require global reference frames if the desired measurements can be measured in each agent's local reference frame.

The paper is organized as follows. Notations and preliminaries are given in Section II. In Section III, the problem of affine formation control is described and necessary results are presented. The problem of leader selection and affine localizability are studied in Section IV. Control laws for single- and double-integrator agent dynamics are proposed in Section V. Nonlinear control laws for unicycle agents are proposed in Section VI. The implementation of the control laws and simulation examples are given in Section VII. Conclusions are drawn in Section VIII.

II. Notations and Preliminaries

This section presents some notations and preliminary results that will be used throughout this paper.

A. Notations for Formations

Consider a group of n mobile agents in $\mathbb{R}^d$ where $d \geq 2$ and $n \geq d+1$. Let $p_i \in \mathbb{R}^d$ be the position of agent i and $p = [p_1^T, \dots, p_n^T]^T \in \mathbb{R}^{dn}$ be the *configuration* of all the agents. The interaction among the agents is described by a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which consists of a vertex set $\mathcal{V} = \{1, \dots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The edge $(i,j) \in \mathcal{E}$ indicates that agent i can receive information from agent j, and agent j is a neighbor of i. The set of neighbors of vertex i is $\mathcal{N}_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$. This paper only consider undirected graphs where $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$. Let m be the number of undirected edges. An orientation of an undirected graph is the assignment of a direction to each undirected edge. An oriented graph is an undirected graph together with an orientation. The incidence matrix $H \in \mathbb{R}^{m \times n}$ of an oriented graph is the $\{0, \pm 1\}$-matrix with rows indexed by edges and columns by vertices [9].

A formation, denoted as $(\mathcal{G}, p)$, is the graph $\mathcal{G}$ with its vertex i mapped to point p_i. Without loss of generality, suppose the first n_ℓ agents are leaders and the rest $n_f = n - n_\ell$ agents are followers. Let $\mathcal{V}_\ell = \{1, \dots, n_\ell\}$ and $\mathcal{V}_f = \mathcal{V} \setminus \mathcal{V}_\ell$ be the sets of leaders and followers, respectively. The positions of the leaders and followers are denoted as $p_\ell = [p_1^T, \dots, p_{n_\ell}^T]^T$ and $p_f = [p_{n_\ell+1}^T, \dots, p_n^T]^T$, respectively.

Denote $\otimes$ as the Kronecker product and $\mathrm{vec}(\cdot)$ the vector obtained by stacking all the columns of a matrix. A useful property of $\mathrm{vec}(\cdot)$ is that $\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$, where A, B, C are real matrices of appropriate dimensions. As a special yet useful consequence, $x \otimes y = \mathrm{vec}(yx^T)$ for any real vectors x, y, because $\mathrm{vec}(yx^T) = \mathrm{vec}(y1x^T) = (x \otimes y)\mathrm{vec}(1) = x \otimes y$. The two properties will be frequently used in this paper.

Let $\mathrm{Null}(\cdot)$ and $\mathrm{Col}(\cdot)$ be the null and column spaces of a matrix, respectively. Let $\|\cdot\|$ be the Euclidian norm of a vector or the spectral norm of a matrix, $I_d \in \mathbb{R}^{d \times d}$ the identity matrix, $\mathbf{1}_n \in \mathbb{R}^n$ the vector with all entries equal to one, and $\dim(\cdot)$ the dimension of a linear space. For any vector x, $\mathrm{diag}(x)$ denotes the diagonal matrix whose iith diagonal entry is the ith entry of x.

B. Affine Span and Affine Dependence

Given a set of points $\{p_i\}_{i=1}^n$ in $\mathbb{R}^d$, the *affine span* of these points, denoted as $\mathcal{S}$, is

$$\mathcal{S} = \left\{ \sum_{i=1}^n a_i p_i : a_i \in \mathbb{R} \text{ for all } i \text{ and } \sum_{i=1}^n a_i = 1 \right\}.$$

For example, the affine span of two distinct points is the 1-dimensional line passing through the two points. The affine span of three points that are not collinear is the 2-dimensional plane passing through the three points. The affine span of four points that are not coplanar is $\mathbb{R}^3$. If a_i is restricted to be nonnegative, affine span degenerates to convex hull.

Given any affine span, we can always translate it to contain the origin to obtain a linear space. The dimension of the obtained linear space is defined as the *dimension* of the affine span. If the dimension of the affine span is d, then we say that these points *affinely span* $\mathbb{R}^d$.

The set of points $\{p_i\}_{i=1}^n$ are called *affinely dependent* if there exists scalars $\{a_i\}_{i=1}^n$ that are not all zero such that $\sum_{i=1}^n a_i p_i = 0$ and $\sum_{i=1}^n a_i = 0$, and *affinely independent* otherwise. Define the *configuration matrix* $P \in \mathbb{R}^{n\times d}$ and an augmented matrix $\bar{P} \in \mathbb{R}^{n\times(d+1)}$ as

$$P(p) = \begin{bmatrix} p_1^T \\ \vdots \\ p_n^T \end{bmatrix}, \quad \bar{P}(p) = \begin{bmatrix} p_1^T & 1 \\ \vdots & \vdots \\ p_n^T & 1 \end{bmatrix} = [P(p), \mathbf{1}_n],$$

where $\mathbf{1}_n \triangleq [1,\ldots,1]^T \in \mathbb{R}^n$. By definition, $\{p_i\}_{i=1}^n$ are affinely dependent if and only if the rows of $\bar{P}(p)$ are linearly dependent, i.e., there exists $a = [a_1,\ldots,a_n]^T$ such that $\bar{P}^T(p)a = 0$; and $\{p_i\}_{i=1}^n$ are affinely independent if and only if the rows of $\bar{P}(p)$ are linearly independent. Since $\bar{P}(p)$ has $d+1$ columns, there exist at most $d+1$ points that are affinely independent in $\mathbb{R}^d$.

If $\{p_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$, there must exist $d+1$ points that are affinely independent. As a result, $\bar{P}(p)$ has $d+1$ rows that are linearly independent and consequently $\operatorname{rank}(\bar{P}(p)) = d+1$. This useful result is given as a lemma.

Lemma 1 (Rank Condition for Affine Span). *The set of points $\{p_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$ if and only if $n \geq d+1$ and* $\operatorname{rank}(\bar{P}(p)) = d+1$.

C. Stress Matrices

For formation $(\mathcal{G}, p)$, a *stress* is a set of scalars, $\{\omega_{ij}\}_{(i,j)\in\mathcal{E}}$ where $\omega_{ij} = \omega_{ji} \in \mathbb{R}$, assigned to all the edges. A stress is called an *equilibrium stress* [23]–[25] if it satisfies

$$\sum_{j\in\mathcal{N}_i} \omega_{ij}(p_j - p_i) = 0, \qquad i \in \mathcal{V}. \tag{1}$$

The mechanical interpretation of equilibrium stresses is as follows. The value ω_{ij} represents an attracting force in edge (i,j) when $\omega_{ij} > 0$ and a repelling force when $\omega_{ij} < 0$. The vector $\omega_{ij}(x_j - x_i)$ represents the force applied on agent i by agent j through edge (i,j). Thus, equation (1) means that the forces applied on joint i by joints $j \in \mathcal{N}_i$ are balanced. See Fig. 1 for an illustration. Denote $\omega = [\omega_1,\ldots,\omega_m] \in \mathbb{R}^m$ as the stress vector where ω_k corresponds to the kth undirected edge ($k = 1,\ldots,m$). Note that equilibrium stresses can be only determined up to a scalar factor. That means if ω is an equilibrium stress, then $k\omega$ is also an equilibrium stress for any $k \in \mathbb{R}_{\neq 0}$.

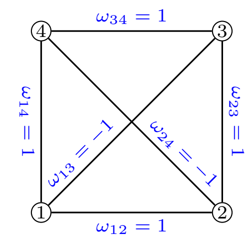
Equation (1) can be expressed in a matrix form as

$$(\Omega \otimes I_d)p = 0,$$

where $\Omega \in \mathbb{R}^{n\times n}$ is the *stress matrix* satisfying

$$[\Omega]_{ij} = \begin{cases} 0, & i \neq j,\ (i,j) \notin \mathcal{E}, \\ -\omega_{ij}, & i \neq j,\ (i,j) \in \mathcal{E}, \\ \sum_{k\in\mathcal{N}_i} \omega_{ik}, & i = j. \end{cases}$$

The stress matrix has a similar structure as graph Laplacian matrices. The difference is that the weight for an edge in a stress matrix may be positive, negative, or zero whereas the weight for an edge in a graph Laplacian is usually positive. See Fig. 1 for an illustrative example of stress matrices.



$$\Omega = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Fig. 1: An example to illustrate equilibrium stresses and stress matrices. In this example, the four points form a square where the length of each side is equal to 1 and the length of each diagonal chord is equal to $\sqrt{2}$. The corresponding stress matrix is positive semi-definite and its eigenvalues are $\{4,0,0,0\}$.

The properties of stress matrices have intimate connections to the structural rigidity of the formation. We next review some necessary notions in the distance rigidity theory [23]–[25]. In $\mathbb{R}^d$, two formations $(\mathcal{G}, p)$ and $(\mathcal{G}, p')$ are *equivalent* if $\|p_i - p_j\| = \|p'_i - p'_j\|$ for all $(i,j) \in \mathcal{E}$, and *congruent* if $\|p_i - p_j\| = \|p'_i - p'_j\|$ for all $i,j \in \mathcal{V}$. Formation $(\mathcal{G}, p)$ is *globally rigid* if an arbitrary formation that is equivalent to $(\mathcal{G}, p)$ is also congruent to it. Formation $(\mathcal{G}, p)$ in $\mathbb{R}^d$ is *universally rigid* if it is globally rigid in any $\mathbb{R}^{d_1}$ where $d_1 \geq d$. A configuration is generic if the coordinates of all the nodes do not satisfy any nontrivial equations with rational coefficients [25, Section 7.2]. The following result establishes the connection between stress matrices and universal rigidity.

Lemma 2 (Generic Universal Rigidity [23], [26], [27]). *Given an undirected graph $\mathcal{G}$ and a generic configuration p, formation $(\mathcal{G}, p)$ is universally rigid if and only if there exists a stress matrix Ω such that Ω is positive semi-definite and* $\operatorname{rank}(\Omega) = n - d - 1$.

III. Problem Statement of Affine Formation Maneuver Control

This section first defines the time-varying target formation and then explores the properties of an important notion termed affine image.

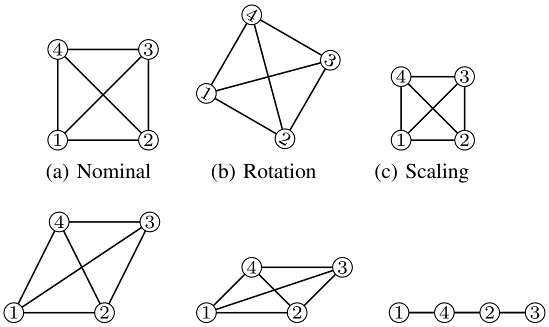
A. Time-Varying Target Formation

The objective of affine formation maneuver control is to steer a group of agents to track the time-varying target formation defined below.

Definition 1 (Target Formation). *The time-varying configuration of the target formation has the form of*

$$p^*(t) = [I_n \otimes A(t)]r + \mathbf{1}_n \otimes b(t),$$

where $r = [r_1^T,\ldots,r_n^T]^T = [r_\ell^T, r_f^T]^T \in \mathbb{R}^{dn}$ is a constant configuration, and $A(t) \in \mathbb{R}^{d\times d}$ and $b(t) \in \mathbb{R}^d$ are continuous of t. The desired position of agent $i \in \mathcal{V}$ in the target formation is $p_i^*(t) = A(t)r_i + b(t)$.



(a) Nominal (b) Rotation (c) Scaling (d) Shear (e) (f)

Fig. 2: An illustration of affine transformations of a nominal configuration. The formations in (b), (c), and (d) are obtained by rotating, scaling, and shearing the original formation in (a), respectively. The formation (e) is obtained from (d) by reducing the scale in the vertical direction. The formation (f), where the four points are collinear, is obtained from (e) by reducing the scale in the vertical direction to zero.

The constant configuration r represents a typical geometric pattern that the formation would like to maintain. Here, r is called the *nominal configuration* and $(\mathcal{G}, r)$ the *nominal formation*. The target configuration is actually a time-varying affine transformation of the nominal configuration. Affine transformation is a general linear transformation that may correspond to a translation, rotation, scaling, shear, or compositions of them. Note that shearing or scaling of the formation in different directions would deform the formation shape (see Fig. 2 for an illustration). Affine transformation preserves straight lines and planes. As a result, collinear (or coplanar) points remain collinear (or coplanar) after any affine transformations. Parallel lines are also preserved by affine transformations.

With the notion of the target formation, the problem to be solved in this paper is to control the group of agents to track the time-varying target configuration so that $p(t) \to p^*(t)$ as $t \to \infty$. A trivial control strategy to solve this problem is to let each agent know $A(t)$, $b(t)$, and r_i so that each agent can track its individual reference trajectory. The disadvantage of the strategy is that it requires $A(t)$ and $b(t)$ for all t to be specified in advance and stored on each agent, which is impractical because the formation is not able to dynamically respond to unexpected situations such as pop-up obstacles.

In order to achieve the target formation in a distributed manner, we adopt the leader-follower strategy, where the desired formation maneuvers are merely known by a small number of agents, called *leaders*, and the other agents, called *followers*, only need to follow the motion of the leaders. As will be shown later, the leaders' positions will have a one-to-one correspondence to the affine transformation (A, b). Therefore, the affine transformation of the entire formation is achieved by controlling the positions of the leaders. Since the number of the leaders is usually small, in this work we do not specifically design coordination control laws for the leaders, and simply assume that they can be controlled properly. In practice, the leaders may be controlled by human operators or intelligent decision making programs. Suppose the position of each leader is equal to the desired value in the target formation, i.e., $p_\ell(t) = p_\ell^*(t)$ for all t. Then, the control objective becomes steering the followers such that $p_f(t) \to p_f^*(t)$ as $t \to \infty$. In order to achieve the control objective, we need to study an important notion termed *affine image* in the rest of the section.

B. Affine Image of Nominal Configuration

The *affine image* of the nominal configuration is defined as [22]

$$\begin{aligned}\mathcal{A}(r) &= \big\{p \in \mathbb{R}^{dn} : p = (I_n \otimes A)r + \mathbf{1}_n \otimes b, \\ &\qquad\qquad A \in \mathbb{R}^{d\times d}, b \in \mathbb{R}^d\big\} \\ &= \big\{p = [p_1^T, \dots, p_n^T]^T \in \mathbb{R}^{dn} : p_i = Ar_i + b, \\ &\qquad\qquad A \in \mathbb{R}^{d\times d}, b \in \mathbb{R}^d, i = 1, \dots, n.\big\}\,.\end{aligned}$$

The affine image is a set consisting of all the affine transformations of the nominal configuration r. The time-varying target configuration $p^*(t)$ is in $\mathcal{A}(r)$ for all t.

The affine image $\mathcal{A}(r)$ is a linear subspace because it is closed under addition and scalar multiplication. The dimension of $\mathcal{A}(r)$ is analyzed in the following lemma, which is a fundamental result for the subsequent analysis in the paper.

Lemma 3 (Dimension of Affine Image). *The dimension of $\mathcal{A}(r)$ equals $d^2 + d$ if and only if $\{r_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$.*

Proof. Denote $E_{ij} \in \mathbb{R}^{d\times d}$ as a matrix with its ijth entry equal to one and the others zero, and $e_i \in \mathbb{R}^d$ a vector with its ith entry equal to one and the others zero. Consider the following $d^2 + d$ vectors

$$(I_n \otimes E_{ij})r, \;\; i, j = 1, \dots, d; \quad \mathbf{1}_n \otimes e_i, \;\; i = 1, \dots, d. \quad (2)$$

It is easy to verify that these vectors are all in $\mathcal{A}(r)$ and any other vectors in $\mathcal{A}(r)$ can be expressed as a linear combination of them. As a result, $\dim(\mathcal{A}(r))$ is equal to the number of linearly independent vectors in (2).

Consider the set of coefficients α_{ij} $(i, j = 1, \dots, d)$ and β_i $(i = 1, \dots, d)$ that satisfy

$$\sum_{i=1}^{d}\sum_{j=1}^{d} \alpha_{ij}(I_n \otimes E_{ij})r + \sum_{i=1}^{d} \beta_i(\mathbf{1}_n \otimes e_i) = 0. \quad (3)$$

By using the properties that $\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$ for any real matrices A, B, C of appropriate dimensions and $x \otimes y = \mathrm{vec}(yx^T)$ for any real vectors x, y, we have

$$\begin{aligned}(I_n \otimes E_{ij})r &= \mathrm{vec}[(I_n \otimes E_{ij})r] \\ &= \mathrm{vec}(E_{ij}P^T(r)I_n) = \mathrm{vec}(E_{ij}P^T(r)) \\ \mathbf{1}_n \otimes e_i &= \mathrm{vec}(\mathbf{1}_n \otimes e_i) = \mathrm{vec}(e_i\mathbf{1}_n^T).\end{aligned}$$

As a result, equation (3) is equivalent to

$$\sum_{i=1}^{d}\sum_{j=1}^{d} \alpha_{ij}E_{ij}P^T(r) + \sum_{i=1}^{d} \beta_i e_i \mathbf{1}_n^T = 0,$$

which can be rewritten as

$$\underbrace{\left[\begin{array}{cc}\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}E_{ij} & \sum_{i=1}^{d}\beta_i e_i\end{array}\right]}_{M\in\mathbb{R}^{d\times(d+1)}}\underbrace{\left[\begin{array}{c}P^T(r)\\ \mathbf{1}_n^T\end{array}\right]}_{\bar{P}^T(r)}=0.$$

Note that $M\bar{P}^T(r)=0 \Leftrightarrow \bar{P}(r)M^T=0$.

(Sufficiency) If $\{r_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$, it follows from Lemma 1 that $\mathrm{rank}(\bar{P}(r))=d+1$ and hence $\mathrm{Null}(\bar{P}(r))=0$. As a result, M^T must be zero and hence all the coefficients α_{ij},β_i are zero. It then follows that all the vectors in (2) are linearly independent and hence $\dim(\mathcal{A}(r))=d^2+d$. (Necessity) If $\{r_i\}_{i=1}^n$ do not affinely span $\mathbb{R}^d$, there exist nonzero vectors in $\mathrm{Null}(\bar{P}(r))$. As a result, there exist nonzero values of α_{ij},β_i such that $\bar{P}(r)M^T=0$, and consequently the vectors in (2) are linearly dependent. Since there are less than d^2+d linearly independent vectors in (2), $\dim(\mathcal{A}(r))<d^2+d$. □

Remark 1. *The dimension of $\mathcal{A}(r)$ has also been analyzed in [22, Lemma 3.1]. However, the conclusion in [22] that $\dim(\mathcal{A}(r))=d^2+d$ if $\{r_i\}_{i=1}^n$ "linearly" span $\mathbb{R}^d$ is inaccurate, because the proof of [22, Lemma 3.1] merely considers the linear dependency of $(I_n\otimes E_{ij})r$ $(i,j=1,\dots,d)$ without incorporating $\mathbf{1}_n\otimes e_i$ $(i=1,\dots,d)$. Specifically, if $\{r_i\}_{i=1}^n$ linearly span $\mathbb{R}^d$, it can be proved that $(I_n\otimes E_{ij})r$ $(i,j=1,\dots,d)$ are linearly independent, but it is not sufficient to show all the vectors in (2) are linearly independent. Lemma 3 corrects this inaccuracy and generalizes the condition to be both necessary and sufficient.*

When $\dim(\mathcal{A}(r))=d^2+d$, any point in $\mathcal{A}(r)$ will correspond to a *unique* pair of (A,b). When $\dim(\mathcal{A}(r))<d^2+d$, for any $p\in\mathcal{A}(r)$, there exist an infinite number of (A,b) satisfying $p=(I_n\otimes A)r+\mathbf{1}_n\otimes b$. More information on how to compute A and b given any $p\in\mathcal{A}(r)$ can be found later in Theorem 1 and Corollary 1.

Motivated by Lemma 3, we make the following assumption on the nominal formation.

Assumption 1 (**Affine Span of Nominal Formation**). *For the nominal formation $(\mathcal{G},r)$, assume $\{r_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$.*

C. Affine Image as Null Space

This subsection explores under what conditions $\mathcal{A}(r)$ is the null space of a matrix. In the sequel of the paper, we write $\Omega(r)$ as Ω in short, and Ω always represents the stress matrix of the nominal formation.

Lemma 4. *For any nominal configuration r, it always holds that*

$$\mathcal{A}(r)\subseteq\mathrm{Null}(\Omega\otimes I_d), \tag{4}$$
$$\mathrm{Col}(\bar{P}(r))\subseteq\mathrm{Null}(\Omega). \tag{5}$$

Proof. First, since $\{r_i\}_{i=1}^n$ satisfies (1), it can be verified that $\{Ar_i+b\}_{i=1}^n$ also satisfies (1) for any $A\in\mathbb{R}^{d\times d}$ and $b\in\mathbb{R}^d$. As a result, any point in $\mathcal{A}(r)$ is also in $\mathrm{Null}(\Omega\otimes I_d)$ and consequently $\mathcal{A}(r)\subseteq\mathrm{Null}(\Omega\otimes I_d)$. Second, since $r=\mathrm{vec}(P^T(r))$, it follows from $(\Omega\otimes I_d)r=0$ that $(\Omega\otimes I_d)\mathrm{vec}(P^T(r))=\mathrm{vec}(I_dP^T(r)\Omega^T)=0$. As a result, $P^T(r)\Omega^T=0\Leftrightarrow\Omega P(r)=0$. Since $\Omega\mathbf{1}_n=0$, we have $\Omega\bar{P}(r)=0$ and consequently $\mathrm{Col}(\bar{P}(r))\subseteq\mathrm{Null}(\Omega)$. □

Next we show when the equalities in (4)-(5) hold. In order to do that, we make the following assumption on the nominal formation.

Assumption 2 (**Stress Matrix of Nominal Formation**). *Assume that the nominal formation $(\mathcal{G},r)$ has a positive semi-definite stress matrix Ω satisfying $\mathrm{rank}(\Omega)=n-d-1$.*

Assumption 2 is satisfied if $(\mathcal{G},r)$ is generically universally rigid according to Lemma 2. This assumption may still be valid even if r is not generic [28]. Figure 1 shows a nominal formation that satisfies Assumption 2. The configuration of this formation is not generic because the four agents are located on a circle [25, Section 7.2].

The next result shows when the equalities in (4)-(5) hold.

Lemma 5 (**Null Space of Stress Matrix**). *Under Assumption 2, the following conditions are equivalent to each other:*
1) $\{r_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$.
2) $\mathrm{Null}(\Omega\otimes I_d)=\mathcal{A}(r)$.
3) $\mathrm{Null}(\Omega)=\mathrm{Col}(\bar{P}(r))$.

Proof. First, since $\mathcal{A}(r)\subseteq\mathrm{Null}(\Omega\otimes I_d)$ as shown in Lemma 4, we have that $\mathrm{Null}(\Omega\otimes I_d)=\mathcal{A}(r)$ if and only if $\dim(\mathrm{Null}(\Omega\otimes I_d))=\dim(\mathcal{A}(r))$. Note that $\dim(\mathrm{Null}(\Omega\otimes I_d))=d(d+1)$ by Assumption 2. Since $\dim(\mathcal{A}(r))=d(d+1)$ if and only if $\{r_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$ according to Lemma 3, the equivalence between 1) and 2) follows. Second, since $\mathrm{Col}(\bar{P}(r))\subseteq\mathrm{Null}(\Omega)$ as shown in Lemma 4, we have that $\mathrm{Col}(\bar{P}(r))=\mathrm{Null}(\Omega)$ if and only if $\dim(\mathrm{Col}(\bar{P}(r))=\dim(\mathrm{Null}(\Omega))$. Note that $\dim(\mathrm{Null}(\Omega))=d+1$ by Assumption 2. Since $\dim(\mathrm{Col}(\bar{P}(r)))=\mathrm{rank}(\bar{P}(r))$ and $\mathrm{rank}(\bar{P}(r))=d+1$ if and only if $\{r_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$ by Lemma 1, the equivalence between 1) and 3) follows. □

IV. Affine Localizability and Leader Selection

This section studies the problem of leader selection. In order to manipulate the entire formation through the leaders, we must select sufficient and appropriate leaders. First of all, we define a notion termed *affine localizability*.

Definition 2 (**Affine Localizability**). *The nominal formation $(\mathcal{G},r)$ is* affinely localizable *by the leaders if for any $p=[p_\ell^T,p_f^T]^T\in\mathcal{A}(r)$, p_f can be uniquely determined by p_ℓ.*

Affine localizability indicates that if a configuration is in $\mathcal{A}(r)$, then the positions of the leaders can uniquely determine those of the followers. As will be shown later, it is the key property to ensure the followers track any desired affine transformation maneuvers. We next give a necessary and sufficient condition of affine localizability.

Theorem 1 (**Leader Selection for Affine Localizability**). *Under Assumption 1, the nominal formation $(\mathcal{G},r)$ is affinely localizable if and only if $\{r_i\}_{i\in\mathcal{V}_\ell}$ affinely span $\mathbb{R}^d$.*

Proof. For any $p \in \mathcal{A}(r)$, there exist (A, b) such that

$$p_1 = Ar_1 + b,$$
$$\vdots$$
$$p_n = Ar_n + b.$$

Since $Ar_i = \text{vec}(Ar_i) = [r_i^T \otimes I_d]\text{vec}(A)$, the above equations can be rewritten as

$$\underbrace{\begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}}_{p} = \underbrace{\begin{bmatrix} r_1^T & 1 \\ \vdots & \vdots \\ r_n^T & 1 \end{bmatrix}}_{\bar{P}(r)} \otimes I_d \underbrace{\begin{bmatrix} \text{vec}(A) \\ b \end{bmatrix}}_{z \in \mathbb{R}^{d^2+d}},$$

which can be partitioned to be

$$p_\ell = (\bar{P}(r_\ell) \otimes I_d)z, \tag{6}$$
$$p_f = (\bar{P}(r_f) \otimes I_d)z. \tag{7}$$

(Sufficiency) If $\{r_i\}_{i\in\mathcal{V}_\ell}$ affinely span $\mathbb{R}^d$, it follows from Lemma 1 that $\text{rank}(\bar{P}(r_\ell)) = d+1$. Then, z can be uniquely determined as

$$z = \left[\left(\bar{P}^T(r_\ell)\bar{P}(r_\ell)\right)^{-1} \bar{P}^T(r_\ell)\right] \otimes I_d p_\ell. \tag{8}$$

Then, p_f can be uniquely determined using (7) and hence the nominal formation is affinely localizable. (Necessity) If $\{r_i\}_{i\in\mathcal{V}_\ell}$ do not affinely span $\mathbb{R}^d$, $\text{rank}(\bar{P}(r_\ell)) < d+1$ and there will be an infinite number of z satisfying (6). In particular, if z^* is a solution of (6), then $z = z^* + z_0$ with $z_0 \neq 0$ and $z_0 \in \text{Null}(\bar{P}(r_\ell) \otimes I_d)$ is another solution of (6). Assumption 1 implies that $\bar{P}(r) \otimes I_d$ is of full column rank. As a result, $z_0 \notin \text{Null}(\bar{P}(r_f) \otimes I_d)$ (otherwise, $\bar{P}(r) \otimes I_d$ is not of full column rank). Therefore, $z = z^* + z_0$ and $z = z^*$ would yield different values of p_f. Hence, p_f cannot be uniquely determined and hence the nominal formation is not affinely localizable. □

Theorem 1 suggests that any agents in the nominal formation that affinely span $\mathbb{R}^d$ can be selected as leaders to ensure affine localizability. Since the affine span of $\mathbb{R}^d$ requires at least $d+1$ points, the minimum number of leaders is $d+1$. For example, we need at least 3 leaders in $\mathbb{R}^2$, and at least 4 leaders in $\mathbb{R}^3$. When there are exactly $d+1$ leaders, given any leader positions p_ℓ, there always exists (A, b) solving (6). When there are more than $d+1$ leaders, the positions of the leaders must be dependent on each other; otherwise, there may not exist (A, b) solving (6), because (6) is an overdetermined linear system in this case.

The leader selection problem has been studied in [22, Theorems 7.1 and 7.2]. These results address under what conditions A can be uniquely determined by some agents as a rotational or identity matrix. Theorem 1 is a generalization of these results in the sense that it addresses under what conditions a general matrix A can be uniquely determined.

When the leaders affinely span $\mathbb{R}^d$, there is an one-to-one correspondence between the positions of the leaders and the affine transformation (A, b). The next result shows how to calculate (A, b) using the positions of the leaders.

Corollary 1 (**Calculation of Affine Transformation**). *If $\{r_i\}_{i\in\mathcal{V}_\ell}$ affinely span $\mathbb{R}^d$, for any $p \in \mathcal{A}(r)$, the corresponding A and b can be uniquely determined by*

$$A = \left(\sum_{i\in\mathcal{V}_\ell} p_i \tilde{r}_i^T\right)\left(\sum_{i\in\mathcal{V}_\ell} \tilde{r}_i \tilde{r}_i^T\right)^{-1}, \tag{9}$$

$$b = \frac{1}{n_\ell}\sum_{i\in\mathcal{V}_\ell} p_i - \left(\sum_{i\in\mathcal{V}_\ell} p_i \tilde{r}_i^T\right)\left(\sum_{i\in\mathcal{V}_\ell} \tilde{r}_i \tilde{r}_i^T\right)^{-1} \bar{r} \tag{10}$$

where $\bar{r} = \sum_{i\in\mathcal{V}_\ell} r_i/n_\ell$ and $\tilde{r}_i = r_i - \bar{r}$.

Proof. This result can be proved in two ways. The first is to solve (8) to obtain A and b. In this direction, note that

$$\bar{P}^T(p_\ell)\bar{P}(p_\ell) = \begin{bmatrix} \sum_{i\in\mathcal{V}_\ell} r_i r_i^T & \sum_{i\in\mathcal{V}_\ell} r_i \\ \sum_{i\in\mathcal{V}_\ell} r_i^T & n_\ell \end{bmatrix}.$$

The Schur complement of n_ℓ in the above matrix is $\Delta = \sum_{i\in\mathcal{V}_\ell} r_i r_i^T - (\sum_{i\in\mathcal{V}_\ell} r_i)(\sum_{i\in\mathcal{V}_\ell} r_i)^T/n_\ell$. It can be verified that $\Delta = \sum_{i\in\mathcal{V}_\ell} \tilde{r}_i \tilde{r}_i^T$. By using the inverse of block matrices [29, Equation 2.3], we obtain

$$(\bar{P}^T(p_\ell)\bar{P}(p_\ell))^{-1} = \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}\bar{r} \\ -\bar{r}^T\Delta^{-1} & 1/n_\ell + \bar{r}^T\Delta^{-1}\bar{r} \end{bmatrix}.$$

It follows that

$$\begin{aligned} z &= [(\bar{P}^T(p_\ell)\bar{P}(p_\ell))^{-1}\bar{P}^T(p_\ell)] \otimes I_d p_\ell \\ &= \begin{bmatrix} \sum_{i\in\mathcal{V}_\ell}(\Delta^{-1}\tilde{r}_i) \otimes I_d p_i \\ \sum_{i\in\mathcal{V}_\ell} p_i/n_\ell - \sum_{i\in\mathcal{V}_\ell}(\bar{r}^T\Delta^{-1}\tilde{r}_i) \otimes I_d p_i \end{bmatrix}. \end{aligned}$$

As a result, $\text{vec}(A) = \sum_{i\in\mathcal{V}_\ell}(\Delta^{-1}\tilde{r}_i) \otimes I_d p_i = \text{vec}(\sum_{i\in\mathcal{V}_\ell} p_i \tilde{r}_i^T \Delta^{-1})$, which implies (9), and $b = \sum_{i\in\mathcal{V}_\ell} p_i/n_\ell - \sum_{i\in\mathcal{V}_\ell}(\bar{r}^T\Delta^{-1}\tilde{r}_i) \otimes I_d p_i = \sum_{i\in\mathcal{V}_\ell} p_i/n_\ell - \sum_{i\in\mathcal{V}_\ell} p_i \tilde{r}_i^T \Delta^{-1}\bar{r}$, which is (10).

The second way to prove is to directly substitute $p_i = Ar_i + b$ into (9)-(10) to verify. In particular, rewrite p_i as $p_i = A(r_i - \bar{r}) + A\bar{r} + b := A\tilde{r}_i + c$. Note that $\sum_{i\in\mathcal{V}_\ell} \tilde{r}_i = 0$. Substituting $p_i = A\tilde{r}_i + c$ into the right hand side of (9) leads to A, which verifies (9). Substituting it into the right hand side of (10) leads to $c - A\bar{r} = b$, which verifies (10). □

Remark 2. *Corollary 1 also implies that $\{r_i\}_{i\in\mathcal{V}_\ell}$ affinely span $\mathbb{R}^d$ if and only if $\sum_{i\in\mathcal{V}_\ell} \tilde{r}_i \tilde{r}_i^T$ is nonsingular.*

While Theorem 1 gives an intuitive condition for affine localizability, we next give another mathematical condition expressed in terms of stress matrices. This mathematical condition will be widely used in the stability analysis of the control laws proposed in the following sections. In the sequel of the paper, denote $\bar{\Omega} = \Omega \otimes I_d$ for notational simplicity. Partition $\bar{\Omega}$ according to the partition of leaders and followers as

$$\bar{\Omega} = \begin{bmatrix} \bar{\Omega}_{\ell\ell} & \bar{\Omega}_{\ell f} \\ \bar{\Omega}_{f\ell} & \bar{\Omega}_{ff} \end{bmatrix},$$

where $\bar{\Omega}_{ff} \in \mathbb{R}^{(dn_f)\times(dn_f)}$ and $\bar{\Omega}_{f\ell} \in \mathbb{R}^{(dn_f)\times(dn_\ell)}$.

Theorem 2 (**Stress Condition for Affine Localizability**). *Under Assumptions 1 and 2, the nominal formation $(\mathcal{G}, r)$ is affinely localizable if and only if $\bar{\Omega}_{ff}$ is nonsingular. When*

$\bar{\Omega}_{ff}$ is nonsingular, for any $p = [p_\ell^T, p_f^T]^T \in \mathcal{A}(r)$, p_f can be uniquely calculated as $p_f = -\bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}p_\ell$.

Proof. (Sufficiency) Since any $p \in \mathcal{A}(r)$ is also $\mathrm{Null}(\bar{\Omega})$ by Lemma 4, any $p \in \mathcal{A}(r)$ satisfies $\bar{\Omega}p = 0$ which implies $\bar{\Omega}_{ff}p_f + \bar{\Omega}_{f\ell}p_\ell = 0$. If $\bar{\Omega}_{ff}$ is nonsingular, p_f can be uniquely determined as $p_f = -\bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}p_\ell$ and hence the nominal formation is affinely localizable.

(Necessity) Assume that $\bar{\Omega}_{ff}$ is singular and hence there exists a nonzero vector $x_0 \in \mathbb{R}^{dn_f}$ such that $\bar{\Omega}_{ff}x_0 = 0$. Let $x = [0, x_0^T]^T \in \mathbb{R}^{dn}$. Then, $x^T\bar{\Omega}x = x_0^T\bar{\Omega}_{ff}x_0 = 0$. Under Assumptions 1 and 2, it follows from Lemma 5 that $\mathcal{A}(r) = \mathrm{Null}(\bar{\Omega})$. As a result, for any $p \in \mathcal{A}(r) = \mathrm{Null}(\bar{\Omega})$, we have $(p+x)^T\bar{\Omega}(p+x) = 0$, and consequently $p+x \in \mathrm{Null}(\bar{\Omega}) = \mathcal{A}(r)$. Therefore, for any $p \in \mathcal{A}(r)$, $p+x$ is also in $\mathcal{A}(r)$. Note that p and $p+x$ have the same leaders' positions but different followers' positions because the first dn_ℓ elements of x are zero. As a result, it is impossible to distinguish p from $p+x$ merely using the leaders' positions, and consequently the nominal formation is not affinely localizable. □

Now we are ready to make the third assumption of the nominal formation.

Assumption 3 (Affine Localizability of Nominal Formation). *Assume that the nominal formation $(\mathcal{G}, r)$ is affinely localizable by the leaders.*

Up to now, we have made three assumptions on the nominal formation. Assumption 1 requires that the nominal configuration affinely span $\mathbb{R}^d$ so that $\dim(\mathcal{A}(r)) = d^2 + d$. Assumption 2 requires that the nominal formation satisfies some rigidity constraints so that $\Omega(r)$ is positive semi-definite and $\mathrm{rank}(\Omega(r)) = n - d - 1$. Assumption 3 requires that the selected leaders in the nominal formation affinely span $\mathbb{R}^d$. According to Theorem 2, the three assumptions imply an important mathematical conation: $\bar{\Omega}_{ff}$ is positive definite.

Recall that the control objective is to achieve $p_f(t) \to p_f^*(t)$ as $t \to \infty$ where $p_f^*(t)$ is the desired position of the followers in the target formation. If $\bar{\Omega}_{ff}$ is positive definite, we have $p_f^*(t) = -\bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}p_\ell^*(t)$. Define the tracking error as

$$\delta_{p_f}(t) = p_f(t) - p_f^*(t) = p_f(t) + \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}p_\ell^*(t).$$

As a result, the control objective becomes steering the followers so that $\delta_{p_f}(t) \to 0$ as $t \to \infty$. The subsequent sections will present distributed control laws to achieve this objective.

V. Affine Formation Maneuver Control Laws

In this section, we propose distributed affine formation maneuver control laws for single- or double-integrator agent dynamics based on different types of measurements.

A. Single-Integrator Agent Dynamics

We first consider the case where each mobile agent can be modeled by a single integrator: $\dot{p}_i = u_i$ where u_i is the control input to be designed.

1) Stationary Leaders: We start by considering the simplest case where the leaders are stationary, i.e., $\dot{p}_i = 0$ for $i \in \mathcal{V}_\ell$. In this case, the target formation is also stationary and the affine formation control problem can be solved by the following control law,

$$\dot{p}_i = -\sum_{j \in \mathcal{N}_i} \omega_{ij}(p_i - p_j), \quad i \in \mathcal{V}_f. \tag{11}$$

The matrix-vector form of (11) is

$$\dot{p}_f = -\bar{\Omega}_{ff}p_f - \bar{\Omega}_{f\ell}p_\ell^*. \tag{12}$$

Since (12) can be rewritten as $\dot{p}_f = -\bar{\Omega}_{ff}\delta_{p_f}$, it can be viewed as a gradient-decent control law for the Lyapunov function $V = 1/2\delta_{p_f}^T\bar{\Omega}_{ff}\delta_{p_f}$. When there are no leaders, (12) becomes $\dot{p} = -\bar{\Omega}p$, which is the control law studied in [22].

Control law (11) can be implemented in each agent's local reference frame since ω_{ij} is a scalar. More specifically, denote $p_{ij} = p_i - p_j$ and suppose R_i is the rotational transformation from a global frame to the local frame of agent i. Then, $p_{ij}^{(i)} = R_i p_{ij}$ is the relative position of agent j expressed in agent i's local reference frame. Consider the following control law: $v_i^{(i)} = -\sum_{i\in\mathcal{N}_i}\omega_{ij}p_{ij}^{(i)}$, where $v_i^{(i)}$ is the velocity of agent i expressed in its own reference frame. This control law merely requires the relative position measured in agent i's local reference frame. On the other hand, since $v_i^{(i)} = R_i\dot{p}_i$, this control law can be written as $R_i\dot{p}_i = -\sum_{i\in\mathcal{N}_i}\omega_{ij}R_i p_{ij}$, which is the same as (11). It can be similarly shown that the control laws presented in the rest of the paper can also be implemented in each agent's local reference frame if the relative measurements can be measured in each agent's local reference frame.

The stability of control law (11) is analyzed below.

Theorem 3 (Zero Leader Velocities). *Under Assumptions 1–3, if the leader velocity $\dot{p}_\ell^*(t)$ is constantly zero, then the tracking error $\delta_{p_f}(t)$ under the action of control law (11) converges to zero globally and exponentially fast.*

Proof. Substituting (12) into $\dot{\delta}_{p_f}$ gives

$$\dot{\delta}_{p_f} = \dot{p}_f(t) + \bar{\Omega}_{f\ell}\dot{p}_\ell^* = -\bar{\Omega}_{ff}\delta_{p_f} + \bar{\Omega}_{f\ell}\dot{p}_\ell^*. \tag{13}$$

Since $\dot{p}_\ell^* = 0$, the tracking error δ_{p_f} is globally and exponentially stable if $\bar{\Omega}_{ff}$ is nonsingular. □

As shown in the error dynamics in (13), if $\dot{p}_\ell^*(t)$ is not identically zero, it may be viewed as a disturbance of the system and can cause nonzero tracking errors. However, since the control law is linear, if the leader velocities are sufficiently small, the tracking error would also be sufficiently small. We next present another two control laws that can eliminate the tracking error even when $\dot{p}_\ell^*(t)$ is nonzero.

2) Moving Leaders with Constant Velocities: If the leaders move with constant nonzero velocities, then control law (11) is not able to guarantee zero tracking errors. To handle this case, we introduce an additional integral term and propose the following proportional-integral (PI) control law,

$$\dot{p}_i = \underbrace{-\alpha \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j)}_{\text{proportional term}} \underbrace{- \beta \int_0^t \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i(\tau) - p_j(\tau))\mathrm{d}\tau}_{\text{integral term}}, \quad i \in \mathcal{V}_f, \quad (14)$$

where α, β are positive constant control gains. Note that control law (14) does not require additional measurements compared to (11). By defining a new state for the integral term, control law (14) can be rewritten as

$$\begin{aligned} \dot{p}_i &= -\alpha \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j) - \beta\xi_i, \\ \dot{\xi}_i &= \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j), \quad i \in \mathcal{V}_f. \end{aligned} \quad (15)$$

Let $\xi = [\cdots \xi_i^T \cdots]^T \in \mathbb{R}^{dn_f}$. The matrix-vector form of (15) is

$$\begin{aligned} \dot{p}_f &= -\alpha\bar{\Omega}_{ff}p_f - \alpha\bar{\Omega}_{f\ell}p_\ell^* - \beta\xi, \\ \dot{\xi} &= \bar{\Omega}_{ff}p_f + \bar{\Omega}_{f\ell}p_\ell^*. \end{aligned} \quad (16)$$

The stability of the control law is analyzed below.

Theorem 4 (**Constant Leader Velocities**). *Under Assumptions 1–3, if the leader velocity $\dot{p}_\ell^*(t)$ is constant, then the tracking error $\delta_{p_f}(t)$ under the action of control law* (14) *converges to zero globally and exponentially fast.*

Proof. Substituting control law (16) into the error dynamics gives

$$\begin{aligned} \dot{\delta}_{p_f} &= \dot{p}_f + \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\dot{p}_\ell^* \\ &= -\alpha\bar{\Omega}_{ff}p_f - \alpha\bar{\Omega}_{f\ell}p_\ell^* - \beta\xi + \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\dot{p}_\ell^* \\ &= -\alpha\bar{\Omega}_{ff}\delta_{p_f} - \beta\xi + \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\dot{p}_\ell^*. \end{aligned}$$

Together with the dynamics of ξ, we obtain the error dynamics as

$$\begin{bmatrix} \dot{\delta}_{p_f} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -\alpha\bar{\Omega}_{ff} & -\beta I_{dn_f} \\ \bar{\Omega}_{ff} & 0 \end{bmatrix} \begin{bmatrix} \delta_{p_f} \\ \xi \end{bmatrix} + \begin{bmatrix} \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell} \\ 0 \end{bmatrix} \dot{p}_\ell^*. \quad (17)$$

Suppose λ is an eigenvalue of the state matrix. By using the results in [29], we obtain

$$\begin{aligned} &\det\left(\begin{bmatrix} \lambda I + \alpha\bar{\Omega}_{ff} & \beta I \\ -\bar{\Omega}_{ff} & \lambda I \end{bmatrix}\right) = \det\left(\lambda^2 I + \alpha\lambda\bar{\Omega}_{ff} + \beta\bar{\Omega}_{ff}\right) \\ &= \det\left((\alpha\lambda + \beta)\left(\frac{\lambda^2 I}{\alpha\lambda + \beta} + \bar{\Omega}_{ff}\right)\right) = 0. \end{aligned} \quad (18)$$

It follows that either $\lambda = -\beta/\alpha < 0$ or

$$\frac{\lambda^2}{\alpha\lambda + \beta} = -\sigma,$$

where σ is the eigenvalue of $\bar{\Omega}_{ff}$. Since $\bar{\Omega}_{ff}$ is symmetric positive definite and hence $\sigma > 0$, the solution to the above equation satisfies $\lambda < -\beta/\alpha < 0$. As a result, the error dynamics is stable and the steady state satisfies

$$\begin{bmatrix} -\alpha\bar{\Omega}_{ff} & -\beta I_{dn_f} \\ \bar{\Omega}_{ff} & 0 \end{bmatrix} \begin{bmatrix} \delta_{p_f}(\infty) \\ \xi(\infty) \end{bmatrix} + \begin{bmatrix} \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell} \\ 0 \end{bmatrix} \dot{p}_\ell^* = 0. \quad (19)$$

It follows that $\delta_{p_f}(\infty) = 0$. □

As can be seen from the error dynamics (17), the constant leader velocity may be viewed as a constant disturbance. The role of the integral term is to eliminate this disturbance. This can be seen from (19) where $\xi(\infty)$ cancels the term containing $\dot{p}_\ell^*$.

3) Moving Leaders with Time-Varying Velocities: When the velocities of the leaders are time-varying, the PI control law in (14) is not able ensure zero tracking errors. In order to handle the time-varying case, we propose the following control law that requires absolute velocity feedback,

$$\dot{p}_i = -\frac{1}{\gamma_i} \sum_{j\in\mathcal{N}_i} \omega_{ij}\left[(p_i - p_j) - \dot{p}_j\right], \quad i \in \mathcal{V}_f \quad (20)$$

where $\gamma_i = \sum_{j\in\mathcal{N}_i} \omega_{ij}$. Although ω_{ij} may be negative, the nonsingularity of γ_i is guaranteed by the affine localizability as shown below.

Proposition 1 (**Nonsingularity of γ_i**). *Under Assumptions 1–3, $\gamma_i > 0$ for all $i \in \mathcal{V}_f$.*

Proof. Note that $\gamma_i = \sum_{j\in\mathcal{N}_i} \omega_{ij} = [\Omega]_{ii}$. Since Ω_{ff} is positive definite by Assumptions 1–3, all the diagonal entries of Ω_{ff} is positive and consequently $\gamma_i > 0$ for all $i \in \mathcal{V}_f$. □

The stability of control law (20) is analyzed below.

Theorem 5 (**Time-Varying Leader Velocities**). *Under Assumptions 1–3, if the leader velocity $\dot{p}_\ell^*(t)$ is time-varying and continuous, then the tracking error $\delta_{p_f}(t)$ under the action of control law* (20) *converges to zero globally and exponentially fast.*

Proof. Multiplying γ_i on both sides of (20) gives

$$\sum_{j\in\mathcal{N}_i} \omega_{ij}(\dot{p}_i - \dot{p}_j) = -\sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j), \quad i \in \mathcal{V}_f.$$

Denote $\epsilon_i = \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j)$ for $i \in \mathcal{V}_f$. Then we have $\dot{\epsilon}_i = -\epsilon_i$, which implies that ϵ_i converges to zero globally and exponentially fast. If $\epsilon_i = 0$ for all $i \in \mathcal{V}_f$, then we have $-\bar{\Omega}_{ff}p_f - \bar{\Omega}_{f\ell}p_\ell^* = 0$, which can be rewritten as $\bar{\Omega}_{ff}\delta_{p_f} = 0 \Rightarrow \delta_{p_f} = 0$. □

In practice, the absolute velocity measurement $\dot{p}_j$ may be transmitted from agent j to agent i via wireless communication or obtained by differentiating the position measurement p_j. Both of the methods will result in measurement errors due to, for example, communication delays. However, since the system is linear, if the velocity measurement errors are bounded (or sufficiently small), the tracking error would also be bounded (or sufficiently small). Note that control law (20) cannot be implemented in each agent's local reference frame due to the requirement of the absolute velocity measurement.

B. Double-Integrator Agent Dynamics

We now consider the case where each mobile agent can be modeled by a double integrator: $\dot{p}_i = v_i$ and $\dot{v}_i = u_i$ where v_i is the agent velocity and u_i is the control input to be designed.

1) Moving Leaders with Zero Accelerations: We start by considering the simplest case where the accelerations of the leaders are zero. The following control law can be used to handle this case,

$$\begin{aligned}\dot{p}_i &= v_i,\\ \dot{v}_i &= -\sum_{j\in\mathcal{N}_i}\omega_{ij}\left[k_p(p_i-p_j)+k_v(v_i-v_j)\right],\ i\in\mathcal{V}_f,\end{aligned} \quad (21)$$

where k_p and k_v are positive constant control gains. The matrix-vector form of (21) is

$$\begin{aligned}\dot{p}_f &= v_f,\\ \dot{v}_f &= -k_p(\bar{\Omega}_{ff}p_f+\bar{\Omega}_{f\ell}p_\ell^*)-k_v(\bar{\Omega}_{ff}v_f+\bar{\Omega}_{f\ell}v_\ell^*),\end{aligned} \quad (22)$$

where $v_f \in \mathbb{R}^{dn_f}$ and $v_\ell^* = \dot{p}_\ell^*$ are the velocities of the followers and leaders, respectively.

The stability of the control law is analyzed below.

Theorem 6 (Zero Leader Accelerations). *Under Assumptions 1–3, if the leader acceleration $\dot{v}_\ell^*(t)$ is constantly zero, then the tracking error $\delta_{p_f}(t)$ under the action of control law* (21) *converges to zero globally and exponentially fast.*

Proof. Define the velocity error as $\delta_{v_f} = \dot{\delta}_{p_f} = v_f + \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}v_\ell^*$. Substituting (22) into $\dot{\delta}_{v_f}$ gives

$$\dot{\delta}_{v_f} = -k_p\bar{\Omega}_{ff}\delta_{p_f} - k_v\bar{\Omega}_{ff}\delta_{v_f} + \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\dot{v}_\ell^*.$$

The position and velocity error dynamics can be expressed as

$$\begin{bmatrix}\dot{\delta}_{p_f}\\ \dot{\delta}_{v_f}\end{bmatrix} = \begin{bmatrix}0 & I_{dn_f}\\ -k_p\bar{\Omega}_{ff} & -k_v\bar{\Omega}_{ff}\end{bmatrix}\begin{bmatrix}\delta_{p_f}\\ \delta_{v_f}\end{bmatrix} + \begin{bmatrix}0\\ \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\end{bmatrix}\dot{v}_\ell^*. \quad (23)$$

Note that $\dot{v}_\ell^* = 0$. Let λ be an eigenvalue of the state matrix of (23). The characteristic equation of the state matrix is given by $\det(\lambda^2 I + \lambda k_v\bar{\Omega}_{ff} + k_p\bar{\Omega}_{ff}) = 0$. Similar to (18), it can be shown that $\lambda \le -k_p/k_v < 0$. As a result, the state matrix is Hurwitz and hence δ_p and δ_v globally and exponentially converge to zero. □

As can be seen from the error dynamics (23), when $\dot{v}_\ell^*$ is nonzero, it would cause nonzero tracking errors. Control laws that can eliminate the tracking errors in the presence of nonzero $\dot{v}_\ell^*$ will be proposed in the following subsections.

2) Moving Leaders with Constant Accelerations: In order to handle the case where the leaders move with nonzero constant accelerations, we propose the following PI control law,

$$\begin{aligned}\dot{p}_i &= v_i,\\ \dot{v}_i &= -\alpha\sum_{j\in\mathcal{N}_i}\omega_{ij}\left[k_p(p_i-p_j)+k_v(v_i-v_j)\right]\\ &\quad -\beta\int_0^t\sum_{j\in\mathcal{N}_i}\omega_{ij}\left[k_p(p_i-p_j)+k_v(v_i-v_j)\right]\mathrm{d}\tau\end{aligned} \quad (24)$$

for $i \in \mathcal{V}_f$. Note that control law (24) does not require additional measurements compared to control law (21). The stability of control law (24) is analyzed below.

Theorem 7 (Constant Leader Accelerations). *Under Assumptions 1–3, if the leader acceleration $\dot{v}_\ell^*(t)$ is constant for all t, then the tracking error $\delta_{p_f}(t)$ under the action of control law* (24) *converges to zero globally and exponentially fast.*

Proof. By denoting a new variable $\xi_i \in \mathbb{R}^d$ for the integral term, control law (24) can be rewritten as

$$\begin{aligned}\dot{p}_i &= v_i,\\ \dot{v}_i &= -\alpha\sum_{j\in\mathcal{N}_i}\omega_{ij}\left[k_p(p_i-p_j)+k_v(v_i-v_j)\right]-\beta\xi_i\\ \dot{\xi}_i &= \sum_{j\in\mathcal{N}_i}\omega_{ij}\left[k_p(p_i-p_j)+k_v(v_i-v_j)\right].\end{aligned}$$

Let $\xi = [\cdots\xi_i^T\cdots]^T \in \mathbb{R}^{dn_f}$. The matrix-vector form is

$$\begin{aligned}\dot{p}_f &= v_f,\\ \dot{v}_f &= -\alpha k_p\bar{\Omega}_{ff}\delta_{p_f} - \alpha k_v\bar{\Omega}_{ff}\delta_{v_f} - \beta\xi\\ \dot{\xi} &= k_p\bar{\Omega}_{ff}\delta_{p_f} + k_v\bar{\Omega}_{ff}\delta_{v_f}.\end{aligned}$$

The velocity error dynamics can be written as $\dot{\delta}_{v_f} = \dot{v}_f + \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\dot{v}_\ell^* = -\alpha k_p\bar{\Omega}_{ff}\delta_{p_f} - \alpha k_v\bar{\Omega}_{ff}\delta_{v_f} - \beta\xi + \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\dot{v}_\ell^*$. Then we obtain the following error dynamics,

$$\begin{bmatrix}\dot{\delta}_{p_f}\\ \dot{\delta}_{v_f}\\ \dot{\xi}\end{bmatrix} = \left[\begin{array}{c|cc}0 & I & 0\\ \hline -\alpha k_p\bar{\Omega}_{ff} & -\alpha k_v\bar{\Omega}_{ff} & -\beta I\\ k_p\bar{\Omega}_{ff} & k_v\bar{\Omega}_{ff} & 0\end{array}\right]\begin{bmatrix}\delta_{p_f}\\ \delta_{v_f}\\ \xi\end{bmatrix} + \begin{bmatrix}0\\ \bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\\ 0\end{bmatrix}\dot{v}_\ell^*.$$

Partition the state matrix into a two by two block matrix as depicted above. By using the results in [29], it can be verified that the state matrix is Hurwitz for any positive α, β, k_p, k_v. The details are omitted here due to space limitations. Then, by examining the steady state values, we obtain $\delta_{p_f}(\infty) = \delta_{v_f}(\infty) = 0$. □

3) Moving Leaders with Time-Varying Accelerations: In order to handle the case where the leaders move with time-varying velocities, we propose the following control law that requires absolute acceleration measurements,

$$\begin{aligned}\dot{p}_i &= v_i,\\ \dot{v}_i &= -\frac{1}{\gamma_i}\sum_{j\in\mathcal{N}_i}\omega_{ij}\left[k_p(p_i-p_j)+k_v(v_i-v_j)-\dot{v}_j\right],\end{aligned} \quad (25)$$

where $\gamma_i = \sum_{j\in\mathcal{N}_i}\omega_{ij}$. The nonsingularity of γ_i has been shown in Proposition 1. The design of control law (25) is inspired by the consensus protocols for tracking time-varying references in [11], [12].

The stability of control law (25) is analyzed below.

Theorem 8 (Time-Varying Leader Accelerations). *Under Assumptions 1–3, if the leader acceleration $\dot{v}_\ell^*(t)$ is time-varying and continuous, then the tracking error $\delta_{p_f}(t)$ under the action of control law* (25) *converges to zero globally and exponentially fast.*

Proof. Multiplying γ_i on both sides of (25) gives

$$\sum_{j\in\mathcal{N}_i}\omega_{ij}(\dot{v}_i-\dot{v}_j)=\sum_{j\in\mathcal{N}_i}\omega_{ij}\left[-k_p(p_i-p_j)-k_v(v_i-v_j)\right],$$

whose matrix-vector form is

$$\begin{aligned}&\bar{\Omega}_{ff}\dot{v}_f+\bar{\Omega}_{f\ell}\dot{v}_\ell^*\\&\quad=-k_p(\bar{\Omega}_{ff}p_f+\bar{\Omega}_{f\ell}p_\ell^*)-k_v(\bar{\Omega}_{ff}v_f+\bar{\Omega}_{f\ell}v_\ell^*)\\&\quad=-k_p\bar{\Omega}_{ff}\delta_{p_f}-k_v\bar{\Omega}_{ff}\delta_{v_f}.\end{aligned}$$

It follows that $\dot{v}_f=-k_p\delta_{p_f}-k_v\delta_{v_f}-\bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\dot{v}_\ell^*$. Consequently, $\dot{\delta}_{v_f}=\dot{v}_f+\bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}\dot{v}_\ell^*=-k_p\delta_{p_f}-k_v\delta_{v_f}$. Then, the error dynamics can be expressed as

$$\begin{bmatrix}\dot{\delta}_{p_f}\\\dot{\delta}_{v_f}\end{bmatrix}=\begin{bmatrix}0&I\\-k_pI&-k_vI\end{bmatrix}\begin{bmatrix}\delta_{p_f}\\\delta_{v_f}\end{bmatrix}.\tag{26}$$

The eigenvalue of the state matrix is $\lambda=(-k_v\pm\sqrt{k_v^2-4k_p})/2$, which always has negative real part for any $k_p,k_v>0$. The global and exponential convergence follows. □

As can be seen from the error dynamics (26), the role of the absolute acceleration measurement is to eliminate the term containing $\dot{v}_\ell^*$. In practice, the acceleration can be transmitted via wireless communication from agent j to agent i, or calculated using differentiation of the velocity. In either case, the acceleration measurement will be corrupted by errors. If the measurement error is bounded (or sufficiently small), the tracking error would be bounded (or sufficiently small). Note that control law (25) cannot be implemented in each agent's local reference frame due to the requirement of the absolute velocity measurement.

VI. Affine Formation Control subject to Constraints

This section studies affine formation control subject to nonholonomic motion and velocity saturation constraints. Here we only consider the case where the leaders are stationary. The case of moving leaders will be studied in the future.

A. Unicycle Agents in the Plane

Consider a group of unicycle agents moving in the plane. Let $p_i=[x_i,y_i]^T\in\mathbb{R}^2$ and $\theta_i\in\mathbb{R}$ be the position coordinate and heading angle of agent i, respectively. The motion of robot i is governed by the unicycle model

$$\begin{aligned}\dot{x}_i&=v_i\cos\theta_i,\\\dot{y}_i&=v_i\sin\theta_i,\\\dot{\theta}_i&=w_i,\end{aligned}\tag{27}$$

where $v_i\in\mathbb{R}$ and $w_i\in\mathbb{R}$ are the linear and angular velocities to be designed. Here $v_i>0$ means the agent moves forward, and $v_i<0$ backward; and $w_i>0$ means the agent turns its heading vector to the left (i.e., counterclockwise), and $w_i<0$ to the right (i.e., clockwise). Suppose v_i and w_i are constrained by

$$\begin{aligned}-\mathrm{v}_i^{\mathrm{b}}\le&v_i\le\mathrm{v}_i^{\mathrm{f}},\\-\mathrm{w}_i^{\mathrm{r}}\le&w_i\le\mathrm{w}_i^{\mathrm{l}},\end{aligned}$$

where $\mathrm{v}_i^{\mathrm{f}},\mathrm{v}_i^{\mathrm{b}}>0$ are the maximum forward and backward linear speeds, respectively. The constants $\mathrm{w}_i^{\mathrm{r}},\mathrm{w}_i^{\mathrm{l}}>0$ are the maximum left-turn and right-turn angular speeds, respectively. Define the saturation functions for the linear and angular speeds for agent i as

$$\begin{aligned}\mathrm{sat}_{v_i}(x)&=\begin{cases}-\mathrm{v}_i^{\mathrm{b}}, & x\in(-\infty,-\mathrm{v}_i^{\mathrm{b}}),\\x, & x\in[-\mathrm{v}_i^{\mathrm{b}},\mathrm{v}_i^{\mathrm{f}}],\\\mathrm{v}_i^{\mathrm{f}}, & x\in(\mathrm{v}_i^{\mathrm{f}},+\infty),\end{cases}\\\mathrm{sat}_{w_i}(x)&=\begin{cases}-\mathrm{w}_i^{\mathrm{r}}, & x\in(-\infty,-\mathrm{w}_i^{\mathrm{r}}),\\x, & x\in[-\mathrm{w}_i^{\mathrm{r}},\mathrm{w}_i^{\mathrm{l}}],\\\mathrm{w}_i^{\mathrm{l}}, & x\in(\mathrm{w}_i^{\mathrm{l}},+\infty).\end{cases}\end{aligned}\tag{28}$$

Note that the saturation bounds $\mathrm{v}_i^{\mathrm{f}},\mathrm{v}_i^{\mathrm{b}},\mathrm{w}_i^{\mathrm{r}},\mathrm{w}_i^{\mathrm{l}}$ may differ for different agents.

1) The Case without Saturation Constraints: First consider the case without velocity saturation constraints. Inspired by [30], the affine formation control law for the unicycle model is designed as

$$\begin{aligned}v_i&=[\cos\theta_i,\sin\theta_i]\sum_{j\in\mathcal{N}_i}\omega_{ij}(p_j-p_i),\\w_i&=[-\sin\theta_i,\cos\theta_i]\sum_{j\in\mathcal{N}_i}\omega_{ij}(p_j-p_i),\quad i\in\mathcal{V}_f.\end{aligned}\tag{29}$$

Let $h_i=[\cos\theta_i,\sin\theta_i]^T$, $h_i^\perp=[-\sin\theta_i,\cos\theta_i]^T$, and $f_i=\sum_{j\in\mathcal{N}_i}\omega_{ij}(p_j-p_i)$, where h_i represents the heading vector of the unicycle and $h_i^\perp$ is orthogonal to h_i. Note that f_i is the control law for the single-integrator model in (11). With these notations, control law (29) can be written as $v_i=h_i^Tf_i$ and $w_i=(h_i^\perp)^Tf_i$. Substituting the control law into the unicycle model in (27) yields

$$\begin{aligned}\dot{p}_i&=h_ih_i^Tf_i,\\\dot{h}_i&=h_i^\perp(h_i^\perp)^Tf_i.\end{aligned}\tag{30}$$

The geometric interpretation of (30) is that the linear and angular velocities are the orthogonal projections of f_i onto h_i and $h_i^\perp$, respectively. The angular velocity aims to turn the heading of the unicycle to align with f_i.

Control law (29) can be implemented in each agent's local reference frame. To see that, let $p_{ji}=p_j-p_i$ and $R_i=[h_i,h_i^\perp]^T$. Then, R_i is the rotational transformation from the global reference frame to agent i's local reference frame. As a result, $p_{ji}^{(i)}=R_ip_{ji}$ is the relative position of agent j measured in agent i's local frame. Consider the following control law

$$\begin{aligned}\dot{p}_i^{(i)}&=e_1e_1^T\sum_{j\in\mathcal{V}_i}\omega_{ij}p_{ji}^{(i)},\\\dot{h}_i^{(i)}&=e_2e_2^T\sum_{j\in\mathcal{V}_i}\omega_{ij}p_{ji}^{(i)},\end{aligned}$$

where $\dot{p}_i^{(i)}, \dot{h}_i^{(i)} \in \mathbb{R}^2$ are the linear and angular velocities expressed agent's local reference frame, and $e_1, e_2 \in \mathbb{R}^2$ are the first and second columns of the identity matrix, respectively. Note that the above control law merely requires locally measured relative positions, and it is equivalent to (30) due to $\dot{p}_i^{(i)} = R_i \dot{p}_i$, $\dot{h}_i^{(i)} = R_i \dot{h}_i$, $R_i^T e_1 = h_i$, and $R_i^T e_2 = h_i^\perp$.

The stability of control law (29) can be analyzed similar to [30]. However, since the leader-follower affine formation control law was not specifically analyzed in [30], we present a proof here by fully considering the specific properties of this control law.

Theorem 9 (**Unicycles without Saturation Constraints**). *Under Assumptions 1–3, if the leader velocity $\dot{p}_\ell^*$ is constantly zero, then the tracking error $\delta_{p_f}(t)$ under the action of control law* (29) *converges to zero globally and asymptotically.*

Proof. Consider the Lyapunov function

$$V = \frac{1}{2} \delta_{p_f}^T \bar{\Omega}_{ff} \delta_{p_f}.$$

Note that $f = -\bar{\Omega}_{ff}\delta_{p_f}$ where $f = [\cdots f_i^T \cdots]^T \in \mathbb{R}^{dn_f}$. The time derivative of V is

$$\dot{V} = \delta_{p_f}^T \bar{\Omega}_{ff} \dot{\delta}_{p_f} = -f^T \dot{\delta}_{p_f} = -\sum_{i \in \mathcal{V}_f} f_i^T h_i h_i^T f_i \leq 0.$$

Since $\dot{V} \leq 0$, V is nonincreasing and bounded from below. As a result, V converges as $t \to \infty$. Moreover, since $V(t) \leq V(0)$, $\|\delta_{p_f}\|$ is bounded from above for all t.

We next show that $\dot{V}$ is uniformly continuous[1] in t by showing that h_i and f_i are both uniformly continuous in t. First, since $f = -\bar{\Omega}_{ff}\delta_{p_f}$ and $\|\delta_{p_f}\|$ is always bounded, we know $\|f\|$ is always bounded. Second, since $\dot{h}_i = h_i^\perp (h_i^\perp)^T f_i$, we have $\|\dot{h}_i\| \leq \|f_i\|$ and hence $\dot{h}_i$ is always bounded. It then follows that h_i is uniformly continuous in t. Third, since $f = -\bar{\Omega}_{ff}\delta_{p_f}$, we have $\dot{f} = -\bar{\Omega}_{ff}\dot{\delta}_{p_f} = -\bar{\Omega}_{ff}\dot{p}_f = \bar{\Omega}_{ff} D \bar{\Omega}_{ff}\delta_{p_f}$, where $D = \mathrm{diag}(h_{n_\ell+1}h_{n_\ell+1}^T, \ldots, h_n h_n^T) \in \mathbb{R}^{(2n_f)\times(2n_f)}$. As a result, $\|\dot{f}\| \leq \|\bar{\Omega}_{ff}\|^2 \|D\| \|\delta_{p_f}\|$ and hence $\dot{f}$ is always bounded. It then follows that f is uniformly continuous.

The uniform continuity of h_i and f_i implies that $\dot{V}$ is uniformly continuous in t. It then follows from the Barbalat's Lemma [31, Lemma 8.2] that $\dot{V} \to 0$ as $t \to \infty$. Note that $\dot{V} \to 0$ implies $h_i^T f_i \to 0$ for all $i \in \mathcal{V}_f$. It is further implied that the system converges to either $f_i = 0$ or $h_i \perp f_i$ but $f_i \neq 0$. In the first case, it follows that $f = -\bar{\Omega}_{ff}\delta_{p_f} = 0 \Rightarrow \delta_{p_f} = 0$. The second case is impossible. To see that, assume $h_i \perp f_i$ but $f_i \neq 0$ for certain i. Since $\dot{p}_i = h_i h_i^T f_i = 0$ for all $i \in \mathcal{V}_f$, all the agents are stationary and hence f_i is time-invariant. However, when $h_i \perp f_i$, we have $\|\dot{h}_i\| = \|h_i^\perp (h_i^\perp)^T f_i\| = \|f_i\| \neq 0$, vector h_i keeps rotating. It is impossible to maintain $h_i \perp f_i$ if f_i is time-invariant whereas h_i is rotating. □

The initial heading angles $\{\theta_i(0)\}_{i\in\mathcal{V}_f}$ do not affect the global convergence. The final heading angles of $\{\theta_i(\infty)\}_{i\in\mathcal{V}_f}$ are not specified.

2) The Case with Saturation Constraints: We now consider the case with velocity saturation constraints. The proposed affine formation control law for unicycle $i \in \mathcal{V}_f$ is

$$\begin{aligned} v_i &= \mathrm{sat}_{v_i}\left\{ [\cos\theta_i, \sin\theta_i] \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_j - p_i) \right\}, \\ w_i &= \mathrm{sat}_{w_i}\left\{ [-\sin\theta_i, \cos\theta_i] \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_j - p_i) \right\}. \end{aligned} \tag{31}$$

Control law (31) can be rewritten as $v_i = \mathrm{sat}_{v_i}(h_i^T f_i)$ and $w_i = \mathrm{sat}_{w_i}((h_i^\perp)^T f_i)$. Substituting into the unicycle model in (27) yields

$$\begin{aligned} \dot{p}_i &= h_i \mathrm{sat}_{v_i}(h_i^T f_i), \\ \dot{h}_i &= h_i^\perp \mathrm{sat}_{w_i}((h_i^\perp)^T f_i). \end{aligned} \tag{32}$$

The global stability of the control law is proved below.

Theorem 10 (**Unicycles subject to Saturation Constraints**). *Under Assumptions 1–3, if the leader velocity $\dot{p}_\ell^*$ is constantly zero, then the tracking error $\delta_{p_f}(t)$ under the action of control law* (31) *converges to zero globally and asymptotically.*

Proof. First of all, rewrite the saturation function as

$$\mathrm{sat}_{v_i}(h_i^T f_i) = \kappa_i h_i^T f_i,$$

where

$$\kappa_i = \begin{cases} \dfrac{\mathrm{v}_i^{\mathrm{b}}}{-h_i^T f_i}, & h_i^T f_i \in (-\infty, -\mathrm{v}_i^{\mathrm{b}}), \\ 1, & h_i^T f_i \in [-\mathrm{v}_i^{\mathrm{b}}, \mathrm{v}_i^{\mathrm{f}}], \\ \dfrac{\mathrm{v}_i^{\mathrm{f}}}{h_i^T f_i}, & h_i^T f_i \in (\mathrm{v}_i^{\mathrm{f}}, +\infty). \end{cases} \tag{33}$$

It is easy to see that $0 < \kappa_i \leq 1$. Then, control law (32) can be rewritten as

$$\dot{p}_i = \kappa_i h_i h_i^T f_i.$$

The time derivative of the Lyapunov function $V = \delta_{p_f}^T \bar{\Omega}_{ff} \delta_{p_f}/2$ is

$$\dot{V} = -\sum_{i\in\mathcal{V}_f} \kappa_i f_i^T h_i h_i^T f_i \leq 0.$$

Since $\dot{V} \leq 0$, V is nonincreasing and bounded from below. As a result, V converges as $t \to \infty$. Moreover, since $V(t) \leq V(0)$, $\|\delta_{p_f}\|$ is bounded from above for all t. Since $f = -\bar{\Omega}_{ff}\delta_{p_f}$, $\|f\|$ is bounded from above and so is $\|h_i^T f_i\|$. As a result, there exists a lower bound $\kappa_{\min} \in (0,1)$ such that $\kappa_{\min} \leq \kappa_i \leq 1$ for all t.

We next show that $\dot{V}$ is uniformly continuous in t by showing that h_i, f_i, and κ_i are all uniformly continuous in t. First, since $\|\dot{h}_i\| = \|h_i^\perp \mathrm{sat}_{w_i}[(h_i^\perp)^T f_i]\| \leq \max\{\mathrm{w}_i^{\mathrm{l}}, \mathrm{w}_i^{\mathrm{r}}\}$, h_i is uniformly continuous in t for all

[1]A function $f(x)$ is uniformly continuous in x if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f(x_1) - f(x_2)\| < \epsilon$ for every pair of x_1 and x_2 satisfying $\|x_1 - x_2\| < \delta$. A useful sufficient (yet not necessary) condition for uniform continuity is that if a function is differentiable and its derivative is bounded, then the function is uniformly continuous. This sufficient condition is frequently used in the proof of Theorems 9 and 10.

$i \in \mathcal{V}_f$. Second, since $f = -\bar{\Omega}_{ff}\delta_{p_f}$, we have $\dot{f} = -\bar{\Omega}_{ff}\dot{\delta}_{p_f} = -\bar{\Omega}_{ff}\dot{p}_f = \bar{\Omega}_{ff}D\bar{\Omega}_{ff}\delta_{p_f}$, where $D = \mathrm{diag}(\kappa_{n_\ell+1}h_{n_\ell+1}h_{n_\ell+1}^T, \ldots, \kappa_n h_n h_n^T) \in \mathbb{R}^{(2n_f)\times(2n_f)}$. As a result, $\|\dot{f}\| \leq \|\bar{\Omega}_{ff}\|^2\|D\|\|\delta_{p_f}\|$. Since $\|D\| = \max_{i\in\mathcal{V}_f}\|\kappa_i h_i h_i^T\| = 1$, $\|\dot{f}\|$ is always bounded and hence f is uniformly continuous. Third, it can be easily verified that κ_i is uniformly continuous in $(h_i^T f_i)$ by the definition of uniform continuity (though κ_i is not differentiable). Since both h_i and f_i are uniformly continuous in t as proved above, κ_i is uniformly continuous in t.

The uniform continuity of h_i, f_i, κ_i implies that $\dot{V}$ is uniformly continuous in t. It then follows from the Barbalat's Lemma [31, Lemma 8.2] that $\dot{V} \to 0$ as $t \to \infty$. Since $\kappa_i \geq \kappa_{\min}$ for all t, $\dot{V} \to 0$ implies $h_i^T f_i \to 0$ for all $i \in \mathcal{V}_f$. The rest of the proof is similar to the proof of Theorem 9. □

Control law (32) can be further generalized to incorporate obstacle avoidance by replacing the variable f_i in $\dot{h}_i$ with another velocity vector. See Theorem 3 and Section V-A in [30] for more information.

B. Nonholonomic Agents in Three Dimensions

Consider a group of nonholonomic agents moving in $\mathbb{R}^3$. Let $p_i = [x_i, y_i, z_i]^T \in \mathbb{R}^3$ be the position of agent i. The velocity direction of agent i is characterized by the yaw and pitch angles α_i and β_i, respectively. The motion of agent i is governed by the three-dimensional (3D) nonholonomic model

$$\begin{aligned}
\dot{x}_i &= v_i \cos\beta_i \cos\alpha_i, \\
\dot{y}_i &= v_i \cos\beta_i \sin\alpha_i, \\
\dot{z}_i &= v_i \sin\beta_i, \\
\dot{\alpha}_i &= w_{\alpha_i}, \\
\dot{\beta}_i &= w_{\beta_i},
\end{aligned} \tag{34}$$

where $v_i, w_{\alpha_i}, w_{\beta_i} \in \mathbb{R}$ are the linear and angular velocities to be designed. Suppose v_i, w_{α_i}, and w_{β_i} are constrained by $-\mathrm{v}_i^{\min} \leq v_i \leq \mathrm{v}_i^{\max}$, $-\mathrm{w}_{\alpha_i}^{\min} \leq w_{\alpha_i} \leq \mathrm{w}_{\alpha_i}^{\max}$, and $-\mathrm{w}_{\beta_i}^{\min} \leq w_{\beta_i} \leq \mathrm{w}_{\beta_i}^{\max}$, where the bounds are constant. Let sat_{v_i}, $\mathrm{sat}_{w_{\alpha_i}}$, and $\mathrm{sat}_{w_{\beta_i}}$ be the saturation functions for v_i, w_{α_i}, and w_{β_i}, respectively. Their definitions are similar to (28).

1) The Case without Saturation Constraints: We first address the case without saturation constraints. The proposed affine formation control law for agent $i \in \mathcal{V}_f$ is

$$\begin{aligned}
v_i &= [\cos\beta_i\cos\alpha_i, \cos\beta_i\sin\alpha_i, \sin\beta_i] f_i, \\
w_{\alpha_i} &= \Big[-\frac{\sin\alpha_i}{\cos\beta_i}, \frac{\cos\alpha_i}{\cos\beta_i}, 0\Big] f_i, \\
w_{\beta_i} &= [-\sin\beta_i\cos\alpha_i, -\sin\beta_i\sin\alpha_i, \cos\beta_i] f_i,
\end{aligned} \tag{35}$$

where $f_i = -\sum_{j\in\mathcal{N}_i}\omega_{ij}(p_i - p_j)$. The global stability of the control law is proved below.

Theorem 11 (3D Nonholonomic Agents without Saturation Constraints). *Under Assumptions 1–3, if the leader velocity $\dot{p}_\ell^*$ is constantly zero, the tracking error $\delta_{p_f}(t)$ under the action of control law* (35) *converges to zero globally and asymptotically.*

Proof. The unit heading vector of agent i is

$$h_i = \begin{bmatrix} \cos\beta_i\cos\alpha_i \\ \cos\beta_i\sin\alpha_i \\ \sin\beta_i \end{bmatrix}.$$

Then the 3D nonholonomic model in (34) can be rewritten as

$$\begin{aligned}
\dot{p}_i &= v_i h_i, \\
\dot{h}_i &= \begin{bmatrix} -\cos\beta_i\sin\alpha_i & -\sin\beta_i\cos\alpha_i \\ \cos\beta_i\cos\alpha_i & -\sin\beta_i\sin\alpha_i \\ 0 & \cos\beta_i \end{bmatrix} \begin{bmatrix} \dot{\alpha}_i \\ \dot{\beta}_i \end{bmatrix}.
\end{aligned}$$

Substituting control law (35) into the above equations yields

$$\begin{aligned}
\dot{p}_i &= h_i h_i^T f_i, \\
\dot{h}_i &= \begin{bmatrix} -\cos\beta_i\sin\alpha_i & -\sin\beta_i\cos\alpha_i \\ \cos\beta_i\cos\alpha_i & -\sin\beta_i\sin\alpha_i \\ 0 & \cos\beta_i \end{bmatrix} \\
&\qquad \begin{bmatrix} -\frac{\sin\alpha_i}{\cos\beta_i} & \frac{\cos\alpha_i}{\cos\beta_i} & 0 \\ -\sin\beta_i\cos\alpha_i & -\sin\beta_i\sin\alpha_i & \cos\beta_i \end{bmatrix} f_i \\
&= (I_3 - h_i h_i^T) f_i.
\end{aligned}$$

Consider the Lyapunov function $V = 1/2\delta_{p_f}^T\bar{\Omega}_{ff}\delta_{p_f}$. The time derivative is $\dot{V} = -\sum_{i\in\mathcal{V}_f} f_i^T h_i h_i^T f_i \leq 0$. The rest of the proof is similar to the proof of Theorem 9. □

The initial values of the angles, $\{\alpha_i(0), \beta_i(0)\}_{i\in\mathcal{V}_f}$, do not affect the global convergence. The final values of the angles, $\{\alpha_i(\infty), \beta_i(\infty)\}_{i\in\mathcal{V}_f}$, are not specified.

2) The Case with Saturation Constraints: We now consider the saturation constraints and propose the following control law,

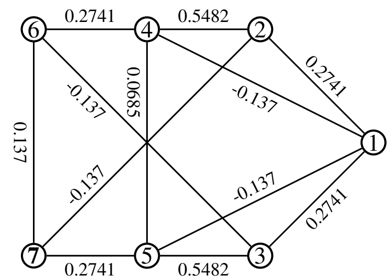
$$\begin{aligned}
v_i &= \mathrm{sat}_{v_i}\left\{[\cos\beta_i\cos\alpha_i, \cos\beta_i\sin\alpha_i, \sin\beta_i] f_i\right\}, \\
w_{\alpha_i} &= \mathrm{sat}_{w_{\alpha_i}}\left\{\Big[-\frac{\sin\alpha_i}{\cos\beta_i}, \frac{\cos\alpha_i}{\cos\beta_i}, 0\Big] f_i\right\}, \\
w_{\beta_i} &= \mathrm{sat}_{w_{\beta_i}}\left\{[-\sin\beta_i\cos\alpha_i, -\sin\beta_i\sin\alpha_i, \cos\beta_i] f_i\right\},
\end{aligned} \tag{36}$$

where $f_i = -\sum_{j\in\mathcal{N}_i}\omega_{ij}(p_i - p_j)$. The global stability of the control law is proved below.

Theorem 12 (3D Nonholonomic Agents with Saturation Constraints). *Under Assumptions 1–3, if the leader velocity $\dot{p}_\ell^*$ is constantly zero, then the tracking error $\delta_{p_f}(t)$ under the action of control law* (36) *converges to zero globally and asymptotically.*

Proof. The unit heading vector of agent i is $h_i = [\cos\beta_i\cos\alpha_i, \cos\beta_i\sin\alpha_i, \sin\beta_i]^T$. Under control law (36), we have $\dot{p}_i = h_i \mathrm{sat}_{v_i}(h_i^T f_i) = \kappa_i h_i h_i^T f_i$, where κ_i is given in (33). The time derivative of the Lyapunov function $V = 1/2\delta_{p_f}\bar{\Omega}_{ff}\delta_{p_f}$ is $\dot{V} = -\sum_{i\in\mathcal{V}_f}\kappa_i f_i^T h_i h_i^T f_i \leq 0$. The rest of the proof is similar to Theorem 10. □

Note that the 3D nonholonomic model in (34) is valid only if $\beta_i \neq \pm\pi/2$ because the yaw angle α_i is undefined when



$$P(r) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}$$

Fig. 3: The nominal formation in the simulation example. The equilibrium stress is plotted on each edge. Here the stress is normalized so that its norm is one. The stress matrix is positive semi-definite and the eigenvalues are $\{1.4432, 1.3218, 0.5967, 0.3383, 0, 0, 0\}$. Note that the configuration of the nominal formation is not generic because there exist collinear agents.

$\beta_i = \pm\pi/2$. This singularity corresponds to the special case where the agent's heading is parallel to the z-axis of the global reference frame. In this section, we simply assume $\beta_i \neq \pm\pi/2$ for all t. If this assumption is invalid, this model and the proposed control laws would become invalid. In order to eliminate the singularity, one may use a unit vector to represent the heading instead of parameterizing it by the yaw and pitch angles.

VII. Implementation and Simulation

To implement the proposed control laws, the first step is to design a nominal formation satisfying Assumptions 1–3. To satisfy Assumption 1, the nominal configuration must affinely span $\mathbb{R}^d$. To satisfy Assumption 2, the nominal formation may be designed to be generically universally rigid. To satisfy Assumption 3, at least $d+1$ agents that affinely span $\mathbb{R}^d$ in the nominal configuration must be selected as leaders. Once the nominal formation has been designed, the next step is to calculate the stress matrix. Calculating the stress matrix is nontrivial. It has been shown in [22] that this problem can be formulated as a dynamic programming problem. Here we present an alternative formulation.

A. Calculation of Equilibrium Stresses

Let ω be the stress vector of the nominal formation. Consider an arbitrary orientation of the undirected graph $\mathcal{G}$ and let $H \in \mathbb{R}^{m\times n}$ be the incidence matrix. Let $h_i \in \mathbb{R}^m$ be the ith column of H and hence $H = [h_1, \ldots, h_n]$. Define

$$E = \begin{bmatrix} \bar{P}^T(r) H^T \mathrm{diag}(h_1) \\ \vdots \\ \bar{P}^T(r) H^T \mathrm{diag}(h_n) \end{bmatrix} \in \mathbb{R}^{n(d+1)\times m}. \tag{37}$$

Let $z_1, \ldots, z_q \in \mathbb{R}^m$ be a basis of $\mathrm{Null}(E)$. In practice, an orthogonal basis of $\mathrm{Null}(E)$ can be obtained by calculating the singular value decomposition (SVD) of E. On the other hand, suppose the SVD of $\bar{P}(r)$ is $\bar{P}(r) = U\Sigma V^T$. Let $U = [U_1, U_2]$ where U_1 consists of the first $d+1$ columns of U. Define $M_i = U_2^T H^T \mathrm{diag}(z_i) H U_2$ for $i = 1, \ldots, q$. Then, the equilibrium stress can be calculated as below.

Proposition 2 (**Calculation of Stress Matrix**). *The equilibrium stress of the nominal formation is*

$$\omega = \sum_{i=1}^{q} c_i z_i,$$

where $c_1, \ldots, c_q$ *satisfy the linear matrix inequality*

$$\sum_{i=1}^{q} c_i M_i > 0. \tag{38}$$

Proof. Since $\Omega = H^T \mathrm{diag}(w) H$ and $\Omega \bar{P}(r) = 0$, we have $\bar{P}^T(r) H^T \mathrm{diag}(\omega) H = \bar{P}^T(r) H^T \mathrm{diag}(\omega)[h_1, \ldots, h_n] = 0$. Since $\mathrm{diag}(\omega) h_i = \mathrm{diag}(h_i)\omega$, we obtain $\bar{P}^T(r) H^T \mathrm{diag}(h_i)\omega = 0$ for all i and consequently $E\omega = 0$ where E is given in (37). As a result, $\omega \in \mathrm{Null}(E)$ and ω can be expressed as $\omega = \sum_{i=1}^{q} c_i z_i$ where $c_1, \ldots, c_q \in \mathbb{R}$ are the coefficients to be determined. According to [22, Theorem 3.3], $\mathrm{rank}(\Omega) = n - d - 1$ if and only if $U_2^T \Omega U_2 = U_2^T H^T \mathrm{diag}(\omega) H U_2 > 0$. Substituting $\omega = \sum_{i=1}^{q} c_i z_i$, into $U_2^T H^T \mathrm{diag}(\omega) H U_2$ gives $\sum_{i=1}^{q} c_i U_2^T H^T \mathrm{diag}(z_i) H U_2 = \sum_{i=1}^{q} c_i M_i > 0$. In order to calculate the coefficients, we only need to find $c_1, \ldots, c_q$ that satisfies the LMI in (38). □

The LMI problem in Proposition 2 is a feasibility problem that can be numerically solved using the Matlab LMI Toolbox.

B. Simulation Examples

We next present two simulation examples. The nominal formation for the two simulation examples is given in Fig. 3, where the first three agents are selected as leaders and the rest as followers. Since the three leaders in the nominal formation are not collinear, they affinely span the plane. By using the method proposed in Proposition 2, we calculate an equilibrium stress, which has been depicted in Fig. 3. The equilibrium stress is normalized so that its norm is unit. The corresponding stress matrix is positive semi-define and satisfies $\mathrm{rank}(\Omega) = n - d - 1 = 4$.

The first simulation example shown in Fig. 4 demonstrates the control law in (25) for double-integrator agent dynamics. As can be seen, the formation keeps maneuvering to change its centroid, orientation, scale, and geometric pattern to avoid obstacles such as passing through narrow passages. The tracking error remains zero when the formation maneuvers.

In the simulation, the trajectories of the three leaders are generated in advance. In practical applications, the leaders may generate proper trajectories in real time based on the task requirement and obstacles in the environment. In addition, it must be noted that the affine span condition of the leaders in Theorem 1 is for the nominal formation. The leaders do not need to satisfy this condition when the formation maneuvers. For example, as shown in the simulation result, the leaders may become collinear and hence do not affinely span $\mathbb{R}^2$. Finally, in the simulation, the acceleration feedback is delayed by 0.001 second. It is observed in the simulation

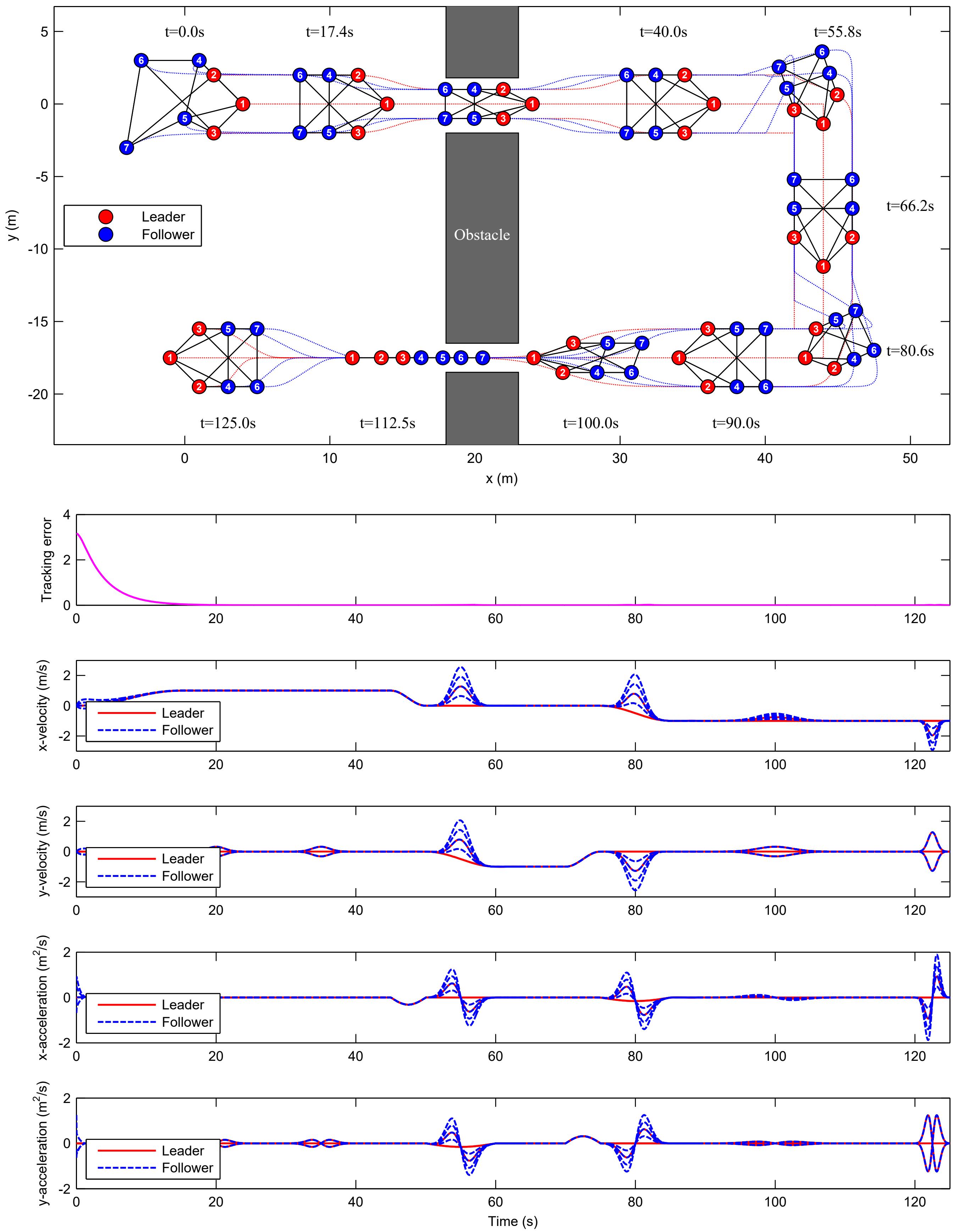


Fig. 4: A simulation example to illustrate control law (25) for the double-integrator agent dynamics. The control gains are chosen as $k_p = 0.5$ and $k_v = 2$. The simulation animation can be found at https://youtu.be/HyCn8r7LBZw.

that larger delays would result in larger tracking errors though the tracking errors are always bounded.

The second simulation example as shown in Fig. 5 demonstrates the control law in (31) for unicycle agents subject to velocity saturation constraints. In this example, the leaders are stationary. The Lyapunov function converges monotonically to zero. Note that the relative positions of the leaders are different from those in the nominal formation. As a result, the final formation is an affine transformation of the nominal formation. It is shown that the collinearity and parallel lines are preserved in the final formation though the shape of the final formation is distorted.

For the sake of simplicity, undirected lines are used to represent the interactions among the agents in the above simulation results. However, it must be noted that the interaction between a follower and a leader is directional instead of bidirectional (or undirected) because the leaders do not need to receive the followers' information.

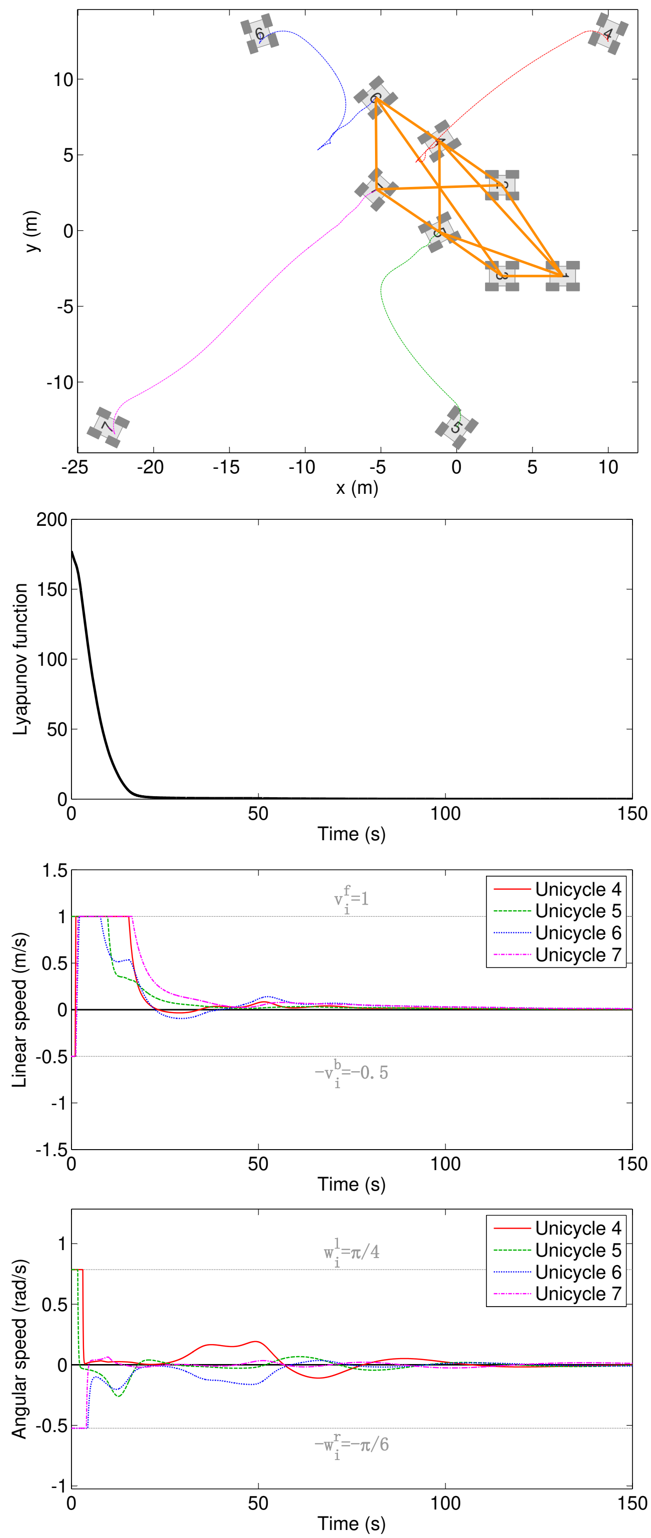


Fig. 5: A simulation example to illustrate control law (31) for the unicycle model with velocity saturation constraints.

VIII. Conclusions

This paper proposed a new approach based on stress matrices to achieve formation maneuver control in arbitrary dimensions. Distributed control laws for single-integrator, double-integrator, and unicycle agent models have been proposed and proved to be globally stable. The proposed control laws can track any target formation that is a time-varying affine transformation of a nominal formation. As a result, the centroid, orientation, scales in different directions, and other geometric parameters of the formation can all be changed continuously. The control laws do not require a common global orientation if the relative measurements can be measured in each agent's local reference frame.

Stress matrices can be viewed as generalized graph Laplacian matrices with negative or zero edge weights. The linear affine formation control laws proposed in this paper have similar expressions as consensus protocols or containment control laws [32], [33] (i.e., consensus protocols with multiple leaders). The work presented in this paper demonstrated that with negative edge weights, the consensus-type control laws may exhibit many new interesting features. Consensus problems over networks with negative weights have received growing research attention in recent years [34], [35]. There are several important topics for future research. For example, the results presented in this paper may be generalized by considering more complicated agent dynamics, motion constraints, and directed underlying graphs.

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