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Approximate observability of infinite dimensional bilinear systems using a Volterra series expansion

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Abstract

In this paper, the approximate observability of a class of infinite dimensional bilinear systems is investigated. The observability conditions are discussed based on a Volterra series representation of this class of systems. A testable observability criterion is derived for the case where the infinitesimal generator is self-adjoint or Riesz-spectral operator. The theoretical results are illustrated by examples.

1 Introduction

System theoretic properties, such as stability, controllability and observability for infinite dimensional systems have been extensively studied over the past four decades. There is a well established systems theory for infinite dimensional linear dynamical systems (Russell [19], Curtain and Zwart [9], and Zuazua [27] and the references therein). Roughly speaking, the controllability problem consists in driving the state of the system (the solution of the controlled equation under consideration) to a prescribed final target state (exactly or in some approximate way) in finite time. The observability problem concerns the problem of reconstructing the full trajectory from measured outputs. For infinite dimensional linear systems, controllability and observability form a duality relationship considering the linear system and its dual system so that these two properties can be investigated under the same framework. In this paper, we are mainly concerned about the observability of infinite dimensional bilinear systems. It is worth pointing out that observability is one of the most fundamental properties, along with controllability and stability, in control systems theory. If sensors are arranged such that a system is observable, in principle, we may uniquely reconstruct the full trajectory or the initial states of the system without detecting them. Observability indicates how one can arrange sensors to determine states with a smaller number of sensors than the number of states, and plays an important role in both finite and infinite dimensional control theory. Thus, observability is an important criterion when implementing sensors and designing state observers for physical systems.

Unlike finite-dimensional systems, there are various definitions of observability for infinite dimensional systems which in the finite dimensional case coincide. Most observability is defined in terms of the distinguishability of a pair of initial states and two important concepts are exact and approximate observability (Curtain and Zwart [9]). The definition of observability can also depend on the length of the time interval or be independent of any specific time interval (Curtain and Zwart [9]). Generally, for infinite dimensional linear systems, the control input is irrelevant to the observability whilst for nonlinear systems, observability is normally input dependent. In this paper, we concentrate on systems governed by bilinear partial differential equations and consider the approximate observability which is input dependent. There are a few results about the observability and observers of infinite dimensional bilinear systems (Belikov [2], Xu [25], Bounit and Hammouri [6], and Gauthier, Xu, and Bounabat [11]). The difference between these earlier studies and our approach is that we will use a Volterra series representation of this class of infinite dimensional bilinear systems in the analysis of observability.

The Volterra series (Volterra [23], Rugh [18]) is a functional series expansion which is generally used to represent the input/output relationship of a nonlinear system with mild nonlinearities. It has been successfully applied in the analysis of finite dimensional nonlinear control systems in the time and frequency domains (Schetzen [22], Sansen [21], Billings and Tsang [4],[5], Peyton-Jones and Billings [17], Billings and Peyton-Jones [3] and the references therein). One of the most important problems in the Volterra series approach is the existence and convergence of such series, which has been addressed by several authors for finite dimensional nonlinear systems. d'Alessandro, Isidori, and Ruberti [10] studied the Volterra series representation for bilinear systems. Linear-analytic systems have been studied by Brockett [8] and Lesiak and Krener [14]. Other relevant theoretical results also exist (Sandberg [20], Boyd and Chua [7], Banks [1], and Peng and Lang [16]). More recently, the Volterra series expansion for infinite dimensional nonlinear systems has been investigated. Guo, et al [12] proved the existence and convergence of a Volterra series representation for the mild solutions of a class of infinite dimensional nonlinear systems. In Helie and Laroche [13] it was shown that a series expansion of a class of infinite dimensional bilinear systems, nonlinear in the state and affine in the input, can be obtained. Based on these previous results, in this paper, the approximate observability of a class of infinite dimensional bilinear systems is investigated using the Volterra series representation.

The paper is organised as follows. In Section 2 a formal Volterra series representation is derived for the solution of the underlying infinite dimensional bilinear systems. Section 3 gives the definition of the observability we are investigating and the observability conditions are discussed based on a Volterra series representation of this class of systems. A testable observability criterion is derived for the case where the infinitesimal generator is self-adjoint or the Riesz-spectral operator. Examples are presented in Section 4. Finally, conclusions are drawn in Section 5.

2 Preliminary

Consider the following infinite dimensional bilinear system with input u and output y

$$\begin{aligned} \dot{z}(t) &= Az(t) + D(z(t), u(t)), z(0) = z_0, t \geq 0 \\ y(t) &= Cz(t) \end{aligned} \tag{1}$$

where $x(t) \in Z, u(t) \in U$, and $y(t) \in Y$, Z, U, and Y are Hilbert spaces. It is assumed that A is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$ of bounded linear operators on Z. This means that there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\|S(t)\|_{\mathcal{L}(Z)} \leq M \exp(\omega t), t \geq 0 \tag{2}$$

It also implies that A is closed and densely defined in Z. Assume that C is a bounded linear operator from Z to Y, D is a bounded bilinear operator from $Z \times U$ to Z, with norm

$$\|D\|_{\mathcal{L}(Z\times U,Z)} = \sup_{\substack{z\in Z,u\in U\\ \|z\|_Z=\|u\|_U=1}} \|D(z,u)\|_Z \tag{3}$$

and

$$\|D(z,u)\|_Z \leq L\|z\|_Z\|u\|_U \tag{4}$$

where $L > 0$ is a positive constant. In this paper, we consider the admissible controls are essentially bounded, i.e., $u \in U_a \subset L^\infty([0,\infty);U)$. Following the standard definition (e.g. Pazy [15]), for a given $T > 0$, the mild solution $z \in C([0,T];Z)$ of (1) is defined as the solution of the following integral equation

$$z(t) = S(t)z_0 + \int_0^t S(t-\tau_1)D(z(\tau_1),u(\tau_1))d\tau_1, t \in [0,T] \tag{5}$$

The existence and uniqueness of the mild solution of system (1) over $[0,T]$ can be readily shown by using the Banach fixed point theorem with the standard arguments (e.g. Theorem 6.1.2 Pazy [15] and Theorem 2.1 Zhang and Joo [26]).

Lemma 1 For every $z_0 \in Z$ and $u \in U_a$, the bilinear system (1) has a unique mild solution $z \in C([0,T];Z)$, Moreover $z_0 \to z$ is Lipschitz from Z to $C([0,T];Z)$.

Note that the bounded bilinear operator $D : Z \times U \to Z$ induces a bounded linear operator D_1 from Z to $\mathcal{L}(U,Z)$ (the set of bounded linear operators from U to Z) as

$$D_1zu = D(z,u) \tag{6}$$

or a bounded linear operator D_2 from U to $\mathcal{L}(Z)$ (the set of bounded linear operators from Z to Z) as

$$D_2uz = D(z,u) \tag{7}$$

It follows that we can rewrite (5) as

$$
\begin{aligned}
z(t) &= S(t)z_0 + \int_0^t S(t-\tau_1)D_1 z(\tau_1)u(\tau_1)d\tau_1 \\
&= S(t)z_0 + \int_0^t S(t-\tau_1)D_2 u(\tau_1)z(\tau_1)d\tau_1
\end{aligned} \tag{8}
$$

Actually, the unique mild solution of (1) can also be expressed as

$$
\begin{aligned}
z(t) &= S_u(t,0)z_0 \\
&= S(t)z_0 + \int_0^t S(t-\tau_1)D_2 u(\tau_1)S_u(\tau_1,0)z_0 d\tau_1
\end{aligned} \tag{9}
$$

where $S_u(t,s)$ is the mild evolution operator generated by the operator $A + D_2(u), u \in U_a$ (Definition 3.2.4 and Theorem 3.2.5, Curtain and Zwart [9]).

There are several ways to derive the Volterra series representation such as the standard Picard iteration or the regular perturbation approach. A simple way to understand this is to substitute for $z(\tau)$ in (8) using an expression of this same form,

$$
\begin{aligned}
z(t) &= S(t)z_0 + \int_0^t S(t-\tau_1)D_1[S(\tau_1)z_0 \\
&\quad + \int_0^{\tau_1} S(\tau_1-\tau_2)D_1 z(\tau_2)u(\tau_2)d\tau_2]u(\tau_1)d\tau_1 \\
&= S(t)z_0 + \int_0^t S(t-\tau_1)D_1 S(\tau_1)z_0 u(\tau_1)d\tau_1 \\
&\quad + \int_0^t \int_0^{\tau_1} S(t-\tau_1)D_1 S(\tau_1-\tau_2)D_1 z(\tau_2)u(\tau_2)u(\tau_1)d\tau_2 d\tau_1
\end{aligned} \tag{10}
$$

Substituting for $z(\tau_2)$ in (10) using an expression of the form (8), and continuing in this manner yields, after $N-1$ steps,

$$
\begin{aligned}
z(t) &= S(t)z_0 + \sum_{n=1}^{N-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} S(t-\tau_1)D_1 \cdots D_1 S(\tau_{n-1}-\tau_n) \\
&\qquad D_1 S(\tau_n) z_0 u(\tau_n) u(\tau_{n-1}) \cdots u(\tau_1) d\tau_n \cdots d\tau_1 \\
&\quad + \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{N-1}} S(t-\tau_1) D_1 S(\tau_1-\tau_2) D_1 \cdots D_1 S(\tau_{N-1}-\tau_N) \\
&\qquad D_1 z(\tau_N) u(\tau_N) u(\tau_{N-1}) \cdots u(\tau_2) u(\tau_1) d\tau_N \cdots d\tau_1
\end{aligned} \tag{11}
$$

where we denote $\tau_0 = t$. The last term in (11) should approach to 0 in a uniform way on any finite time interval $[0, T]$ if the above iteration process converges (Recall that the iteration is absolutely and uniformly convergent on $[0, T]$ in this case provided $\|u\|_{U_a} \leq \varepsilon$ and T is sufficiently small). It follows the the standard Volterra series representation of the system (1) is given by

$$
\begin{aligned}
z(t) &= \sum_{n=0}^{\infty} z_n(t) \\
&= h_0(t) + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} h_n(t, \tau_1, \cdots, \tau_n) u(\tau_n) \cdots u(\tau_1) d\tau_n \cdots d\tau_1
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
h_0(t) &= S(t)z_0 \\
h_n(t, \tau_1, \cdots, \tau_n) &= S(t-\tau_1) D_1 S(\tau_1-\tau_2) \cdots D_1 S(\tau_{n-1}-\tau_n) D_1 S(\tau_n) z_0
\end{aligned} \tag{13}
$$

for $t \geq \tau_1 \geq \tau_2 \geq \cdots \geq \tau_n \geq 0$ and $h_n = 0$ otherwise.

The convergence of such series has been shown to be normally convergent with a convergence radius ρ, which depends on the magnitudes of the input $u \in L^\infty(\mathbb{T}; U)$ and the homogeneous solution $S(\cdot)z_0 \in L^\infty(\mathbb{T}; Z)$ in [13], where $\mathbb{T} = [0, T]$ or $\mathbb{R}_+$. Helie and Laroche [13] gave an explicit expression of the bound as

$$\Phi(\|u\|_{L^\infty(\mathbb{T},U)} + \frac{1}{\omega}\|S(\cdot)x_0\|_{L^\infty(\mathbb{T},Z)}) \tag{14}$$

where $\Phi(s)$ is analytic on the open disk with radius ρ. The result in [13] provides potentials in many applications such as the optimization of parameterized stabilizing controllers through the maximization of the convergence parameter ρ.

Alternatively, we can show the Volterra series (12) is uniformly convergent on $[0,T], T > 0$ as an infinite series of functions from $[0,T]$ to Z using Weierstrass M-test. From (2) and (13), for any $t \in [0,T]$

$$\|z_0(t)\|_Z = \|h_0(t)\|_Z \leq \|S(t)\|_{\mathcal{L}(Z)}\|z_0\|_Z \leq M\exp(\omega t)\|z_0\|_Z \leq M\exp(\omega T)\|z_0\|_Z \tag{15}$$

and for $n \geq 1$ and for any $\tau_1, \cdots, \tau_n \in [0,T]$

$$\begin{aligned}
&\|z_n(t)\|_Z = \|\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} h_n(t,\tau_1,\cdots,\tau_n)u(\tau_n)\cdots u(\tau_1)d\tau_n\cdots d\tau_1\|_Z \\
&\leq \int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} \|h_n(t,\tau_1,\cdots,\tau_n)u(\tau_n)\cdots u(\tau_1)\|_Z d\tau_n\cdots d\tau_1 \\
&\leq \|z_0\|_Z\|u\|_{U_a}^n\|D_1\|_{\mathcal{L}(Z,\mathcal{L}(U,Z))}^n M^{n+1}\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}}\exp(\omega t)d\tau_n\cdots d\tau_1 \\
&= M\exp(\omega t)\|z_0\|_Z\|u\|_{U_a}^n\|D_1\|_{\mathcal{L}(Z,\mathcal{L}(U,Z))}^n M^n t^n/n! \\
&= M\exp(\omega T)\|z_0\|_Z\|u\|_{U_a}^n\|D_1\|_{\mathcal{L}(Z,\mathcal{L}(U,Z))}^n M^n T^n/n!
\end{aligned} \tag{16}$$

This means that the formally generated Volterra series (12) is dominated by a series

$$\begin{aligned}
&M\|z_0\|_Z\exp(\omega T)\textstyle\sum_{n=0}^\infty \|u\|_{U_a}^n\|D_1\|_{\mathcal{L}(Z,\mathcal{L}(U,Z))}^n M^n T^n/n! \\
&= M\|z_0\|_Z\exp((\omega + \|u\|_{U_a}\|D_1\|_{\mathcal{L}(Z,\mathcal{L}(U,Z))}M)T) < \infty
\end{aligned} \tag{17}$$

which is convergent on $[0,T]$. This indicates that the obtained Volterra series is uniformly convergent on $[0,T]$ for any $0 < T < \infty$, which gives the following result

Lemma 2 Let $u \in U_a \subset L^\infty([0,T],U)$, the Volterra series (12) derived from the bilinear system (1) is uniformly convergent on $[0,T]$.

It is easy to see that if the C_0-semigroup $S(t), t \geq 0$ is exponentially stable, that is, there exist constant $M > 0$ and $\alpha > 0$ such that $\|(S(t)\| \leq M\exp(-\alpha t)$, then the Volterra series (12) converges on $[0,\infty)$.

For the rest of the paper, we assume that $u \in U_a \subset L^\infty([0,T],U)$ and $z_0 \in Z$ and the Volterra series representation (12) converges uniformly to the mild solution (5) on $[0,T]$.

3 Main result

There are many definitions of observability of infinite dimensional linear and bilinear systems(Williamson [24], Belikov [2], Curtain and Zwart [9], Bounit and Hammouri [6]). Informally, all the observability problems are concerned with the ability of systems to reconstruct the full trajectory from measured outputs. Let $y(t,z_0,u)$ denote the output of the system (1) at time t, when the input $u(s)$ is used during $s \in [0,T]$, given that the system passed through the state z_0 at time $t=0$. From (1), (9), and (12) we rewrite it using $D_2 : U \to \mathcal{L}(Z)$ to obtain

$$\begin{aligned} y(t,z_0,u) &= Cz(t) \\ &= CS_u(t,0)z_0 \\ &= CS(t)z_0 + \sum_{n=1}^{\infty}\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} CS(t-\tau_1)D_2(u(\tau_1))S(\tau_1-\tau_2) \\ &\quad \cdots D_2(u(\tau_{n-1}))S(\tau_{n-1}-\tau_n)D_2(u(\tau_n))S(\tau_n)z_0 d\tau_n\cdots d\tau_1 \end{aligned} \tag{18}$$

It follows that $y(\cdot,z_0,u) \in C([0,T];Y)$ so that y is always in $L^2([0,T];Y)$ (we use this space to derive some adjoint operators), and the series representation (18) of y is uniformly convergent on $[0,T]$. The observability problem is to determine z_0, given $y(t,z_0,u), t \in [0,T]$ and $u \in U_a$. First, we consider the observability with respect to a given input, which leads to the following definition

Definition 1 Given an input $u \in U_a$, the bilinear system (1) is approximately observable with respect to u on $[0,T]$ if $CS_u(t,0)z = 0, z \in Z$ for all $t \in [0,T]$ implies $z = 0$.

Given an input $u \in U_a$, we consider the observability gramian $\mathcal{L}_u$ of system (1) on $[0, T]$ as

$$\mathcal{L}_u = (\mathcal{C}_u)^* \mathcal{C}_u \tag{19}$$

where $\mathcal{C}_u$ is the bounded linear map from Z to $L^2([0, T], Y)$ defined by

$$\mathcal{C}_u z = CS_u(\cdot, 0)z, z \in Z \tag{20}$$

and $*$ denotes its adjoint.

Theorem 1 The following assertions are equivalent:

(a) The bilinear system (1) is approximately observable with respect to u on $[0, T]$,

(b) $\mathcal{L}_u > 0$,

(c) $\mathbf{ker}\mathcal{C}_u = \{0\}$.

Furthermore, in terms of the Volterra series representation (18), the operator $\mathcal{C}_u \in \mathcal{L}(Z, L^2([0, T], Y))$ can be expressed by using C_0-semigroup $S(t)$ and the operator D as

$$\begin{aligned} \mathcal{C}_u z &= CS(\cdot)z + \sum_{n=1}^{\infty} \int_0^{\cdot} \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} CS(\cdot - \tau_1) D_2(u(\tau_1)) S(\tau_1 - \tau_2) \\ &\quad \cdots D_2(u(\tau_{n-1})) S(\tau_{n-1} - \tau_n) D_2(u(\tau_n)) S(\tau_n) z \, d\tau_n \cdots d\tau_1 \end{aligned} \tag{21}$$

and its adjoint operator is given by

$$\begin{aligned} (\mathcal{C}_u)^* y &= \int_0^T S^*(t) C^* y(t) dt \\ &\quad + \sum_{n=1}^{\infty} \int_0^T \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} S^*(\tau_n) D_2(u(\tau_n))^* S^*(\tau_{n-1} - \tau_n) \\ &\quad \cdots D_2(u(\tau_2))^* S^*(\tau_1 - \tau_2) D_2(u(\tau_1))^* S^*(t - \tau_1) C^* y(t) \, d\tau_n \cdots d\tau_1 dt \end{aligned} \tag{22}$$

and the series here is uniformly convergent on $[0,T]$.

Proof. (a) and (c) are equivalently because $\mathcal{C}_u$ is bounded linear operator from Z to $L^2([0,T],Y)$. Note that

$$< \mathcal{L}_u z, z >=< (\mathcal{C}_u)^* \mathcal{C}_u z, z >=< \mathcal{C}_u z, \mathcal{C}_u z >= \|\mathcal{C}_u z\|^2 \tag{23}$$

So $\mathcal{L}_u > 0$ if and only if $\mathbf{ker}\mathcal{C}_u = \{0\}$.

Furthermore, following Theorem 2.2.6 of Curtain and Zwart [9], $S^*(t), t \in [0,T]$ is a C_0-semigroup with infinitesimal generator A^* on Z. From the definition of the adjoint, the adjoint operators of the individual operator terms in the series representation (21) of $\mathcal{C}_u \in \mathcal{L}(Z, L^2([0,T],Y))$ are given by

$$\begin{aligned}
&\int_0^T S^*(t)C^*y(t)dt \\
&\int_0^T \int_0^t S^*(\tau_1)D(u(\tau_1))^*S^*(t-\tau_1)C^*y(t)dt \\
&\vdots \\
&\int_0^T \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} S^*(\tau_n)D_2(u(\tau_n))^*S^*(\tau_{n-1}-\tau_n)\cdots \\
&\qquad D_2(u(\tau_2))^*S^*(\tau_1-\tau_2)D_2(u(\tau_1))^*S^*(t-\tau_1)C^*y(t)d\tau_n\cdots d\tau_1 dt \\
&\vdots
\end{aligned} \tag{24}$$

Consider the sequence of partial sums

$$\begin{aligned}
&P_m^* y = \int_0^T S^*(t)C^*y(t)dt \\
&+\sum_{n=1}^m \int_0^T \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} S^*(\tau_n)D_2(u(\tau_n))^*S^*(\tau_{n-1}-\tau_n) \\
&\cdots D_2(u(\tau_2))^*S^*(\tau_1-\tau_2)D_2(u(\tau_1))^*S^*(t-\tau_1)C^*y(t)d\tau_n\cdots d\tau_1 dt
\end{aligned} \tag{25}$$

They are the adjoint operator of the sequence of partial sums of $\mathcal{C}_u$

$$P_m z = CS(t)z + \sum_{n=1}^{m} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} CS(\cdot - \tau_1) D_2(u(\tau_1)) S(\tau_1 - \tau_2)$$
$$\cdots D_2(u(\tau_{n-1})) S(\tau_{n-1} - \tau_n) D_2(u(\tau_n)) S(\tau_n) z d\tau_n \cdots d\tau_1 \tag{26}$$

Note that

$$\|(\mathcal{C}_u)^* y - P_m^* y\| = \|\mathcal{C}_u z - P_m z\|, y \in L^2([0,T], Y), z \in Z \tag{27}$$

Since $\mathcal{C}_u$ is uniformly convergent on $[0,T]$, $(\mathcal{C}_u)^*$ is uniformly convergent as the series given in (22) on $[0,T]$ as well. ■

Remark It follows from the explicit expressions (21) and (22) that $\mathcal{L}_u$ can be calculated as

$$< \mathcal{L}_u z, z > = < z, W_{0,0} z > + 2 \sum_{n=1}^{\infty} < z, W_{0,n} z > + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} < z, W_{m,n} z > \tag{28}$$

where

$$W_{0,0} z = \int_0^T S^*(t) C^* CS(t) z dt \tag{29}$$

$$W_{0,n} z = \sum_{n=1}^{\infty} \int_0^T \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} S^*(t) C^* CS(t - \tau_1) D_2(u(\tau_1)) S(\tau_1 - \tau_2)$$
$$\cdots D_2(u(\tau_{n-1})) S(\tau_{n-1} - \tau_n) D_2(u(\tau_n)) S(\tau_n) z d\tau_n \cdots d\tau_1 dt \tag{30}$$

and for $m, n > 1$,

$$W_{m,n} z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^T \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{m-1}} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} S^*(\tau_m)$$
$$D_2(u(\tau_m))^* S^*(\tau_{m-1} - \tau_m) \cdots D_2(u(\tau_2))^* S^*(\tau_1 - \tau_2) D_2(u(\tau_1))^*$$
$$S(t - \tau_1)^* C^* CS(t - t_1) D_2(u(t_1)) S(t_1 - t_2) \cdots D_2(u(t_{n-1}))$$
$$S(t_{n-1} - t_n) D_2(u(t_n)) S(t_n) z dt_n \cdots dt_1 d\tau_m \cdots d\tau_1 dt \tag{31}$$

The derived expression for the observability gramian (28) is difficult to be used to test for the observability. However, in practice a finite order truncated Volterra series is often sufficient to represent the original system. In this case, the observability gramian (28) can be directly used to check the observability.

The following definition is concerned about the observability with respect to all inputs.

Definition 2: An initial states $z_0 \in Z$ of the system (1) is said to be indistinguishable from 0 on $[0, T]$ if the response $y(t, z_0, u), t \in [0, T]$ with $z(0) = z_0$ is identical to the response with $z(0) = 0$ for all $u \in U_a$. The bilinear system (1) is said to be approximately observable if there are no indistinguishable states.

All of the indistinguishable states from 0 on $[0, T]$ form a subspace $\mathcal{N}$ of Z, that is

$$\begin{aligned} \mathcal{N} &= \{z \in Z | y(t, z, u) = 0, \text{ for all } t \in [0, T] \text{ and for all } u \in U_a\} \\ &= \bigcap_{u \in U_a} \mathbf{ker} \mathcal{C}_u \end{aligned} \tag{32}$$

and the system (1) is approximately observable iff $\mathcal{N} = \{0\}$.

In what follows, an approximate observability criterion will be derived for the bilinear system (1) with scalar input u and output y. Assume that D and C are defined by

$$D(z, u) = \alpha u z, \text{ where } z \in Z, \alpha \neq 0, u \in R \tag{33}$$

and

$$Cz =< c, z >, z, c \neq 0 \in Z \tag{34}$$

Firstly, we will consider the case where the linear operator A is self-adjoint or Hermitian. Later we will consider the more general case where the linear operator A in (1) is a Riesz-spectral operator. Now let A be self-adjoint defined by

$$Az = \sum_{n=1}^{\infty} \lambda_n < z, \phi_n > \phi_n \tag{35}$$

where

- $\{\lambda_n, n \geq 1\}$ are eigenvalues such that

$$\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n > \cdots \tag{36}$$

- the corresponding eigenvectors $\{\phi_n, n \geq 1\}$ form an orthonormal basis

$$< \phi_n, \phi_m >= \delta_{nm} \tag{37}$$

in the Hilbert space Z, where δ_{nm} is the Kronecker delta.

It follows that the C_0 semigroup $S(t), t \geq 0$ can be described by using these eigenvalues and eigenfunctions

$$S(t)z = \sum_{n=1}^{\infty} e^{\lambda_n t} < z, \phi_n > \phi_n \tag{38}$$

Theorem 2 Consider the bilinear system (1), where A is self-adjoint defined as above. D and C are given in (33) and (34). Then the system (1) is approximately observable if and only if $< c, \phi_n > \neq 0$ for all $n \geq 1$.

Proof. For any $z_0 \in D(A)$, the assumption (33) and following the property of semigroups yields

$$\begin{aligned} z(t) &= S(t)z_0 + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} S(t-\tau_1)S(\tau_1-\tau_2)\cdots S(\tau_n)z_0 \\ &\quad \alpha u(\tau_1)\alpha u(\tau_2)\cdots \alpha u(\tau_n) d\tau_n \cdots d\tau_1 \\ &= S(t)z_0 + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} S(t)z_0 \alpha^n u(\tau_1)u(\tau_2)\cdots u(\tau_n) d\tau_n \cdots d\tau_1 \end{aligned} \tag{39}$$

and substituting (38) into (39) yields

$$\begin{aligned} z(t) &= \sum_{m=1}^{\infty} e^{\lambda_m t} < z_0, \phi_m > \phi_m + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \alpha^n u(\tau_1)u(\tau_2)\cdots \\ &\quad u(\tau_n) d\tau_n \cdots d\tau_1 \sum_{m=1}^{\infty} e^{\lambda_m t} < z_0, \phi_m > \phi_m, t \in [0, T] \end{aligned} \tag{40}$$

It follows that the output of the system is given by

$$\begin{aligned} y(t) &= < c, z(t) > \\ &= (1 + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \alpha^n u(\tau_1) u(\tau_2) \cdots u(\tau_n) d\tau_n \cdots d\tau_1) \\ & \quad \sum_{m=1}^{\infty} e^{\lambda_m t} < z_0, \phi_m >< c, \phi_m >, t \in [0, T] \end{aligned} \tag{41}$$

For $u \in U_a$, the series $1+\sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \alpha^n u(\tau_1) u(\tau_2) \cdots u(\tau_n) d\tau_n \cdots d\tau_1$ is absolutely and uniformly convergent on $[0, T]$ because

$$|\int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \alpha^n u(\tau_1) u(\tau_2) \cdots u(\tau_n) d\tau_n \cdots d\tau_1| \leq |\alpha|^n \|u\|_{U_a}^n T^n / n! \tag{42}$$

so that we can have that

$$y(t) = \gamma_u(t) \sum_{m=1}^{\infty} e^{\lambda_m t} < z_0, \phi_m >< c, \phi_m >, t \in [0, T] \tag{43}$$

It follows that if there exists a $l \geq 1$ such that $< c, \phi_l >= 0$, then we can always find a $z_0 \in Z$ satisfying $< z_0, \phi_m >= 0$ for all $m \geq 1$ but l, say $z_0 = \phi_l$ such that $y(t) = 0, t \in [0, T]$. This indicates that the system (1) is approximately observable if and only if $< c, \phi_m >\neq 0$ for all $m \geq 1$. ■

Now, assume that the Riesz-spectral operator A has simple eigenvalues $\{\lambda_n, n = \pm 1, \pm 2, \cdots\}$ and corresponding eigenvectors $\{\phi_n, n = \pm 1, \pm 2, \cdots\}$. Let $\{\psi_n, n = \pm 1, \pm 2, \cdots\}$ be the eigenvectors of A^*, the adjoint of A, such that $< \phi_n, \psi_m >= \delta_{nm}$. It follows that the operator A has the representation, as the infinitesimal generator of a C_0-semigroup $S(t)$,

$$Az = \sum_{n=-\infty}^{\infty} \lambda_n < z, \psi_n > \phi_n \tag{44}$$

and

$$S(t)z = \sum_{n=-\infty}^{\infty} e^{\lambda_n t} < z, \psi_n > \phi_n \tag{45}$$

Theorem 3 Consider the bilinear system (1), where A is a Riesz-spectral operator defined above. D and C are given in (33) and (34). Then the system (1) is approximately observable if and only if $< c, \phi_n > \neq 0$ for all $n = \pm 1, \pm 2, \cdots$.

The proof is similar to that of Theorem 2, thus omitted.

4 Examples

Example 1 Consider the following reaction-diffusion process

$$\begin{aligned} &\frac{\partial}{\partial t} z(x,t) = \frac{\partial^2}{\partial x^2} z(x,t) + z(x,t)u(t) \\ &z(x,0) = z_0(x), t \geq 0, x \in [0,l], l > 0 \end{aligned} \tag{46}$$

with Dirichlet boundary conditions

$$z(0,t) = z(l,t) = 0 \tag{47}$$

or with periodic boundary conditions

$$\frac{\partial}{\partial x} z(0,t) = \frac{\partial}{\partial x} z(l,t), z(0,t) = z(l,t), \tag{48}$$

The output is given by an observation at a fixed point x_o

$$y(t) = z(x_o, t) \tag{49}$$

or by an observation around a fixed point x_o

$$y(t) = \int_0^l c(x) z(x,t) dx \tag{50}$$

where $c(x)$ is given by

$$c(x) = \frac{1}{2\varepsilon} I_{[x_o - \varepsilon, x_o + \varepsilon]}(x) \tag{51}$$

where

$$I_{[x_o - \varepsilon, x_o + \varepsilon]}(x) = \begin{cases} 1 & x_o - \varepsilon \leq x \leq x_o + \varepsilon \\ 0 & otherwise \end{cases} \tag{52}$$

where it is assumed that $\varepsilon > 0$. Let $Z = L_2(0,1)$ and A be the linear operator in Z defined by $D(A) = \{z, z', z'' \in Z : z(0) = z(l) = 0\}$ or $\{z, z', z'' \in Z : z_x(0) = z_x(l), z(0) = z(l)\}$ as $Az = \nu z''$ for $z \in D(A)$. So it is easy to see that (46) can be formulated as (1) with $Z = L_2(0,1)$, $U = Y = C$, $D(z,u) := zu$, $D \in \mathcal{L}(Z \times C, Z)$, and $Cz := z(x_0)$ or

$$Cz := \int_0^l c(x)z(x)dx \tag{53}$$

$C \in \mathcal{L}(Z, C)$. Note that A here is a self-adjoint operator. For Dirichlet boundary condition (47), it is readily seen that

$$\lambda_n = -\nu(2n\pi/l)^2 \tag{54}$$

and

$$\phi_n(x) = \sqrt{\frac{1}{l}} \sin \frac{2n\pi x}{l} \tag{55}$$

for all $n \geq 1$ are the eigenvalues and eigenvectors, respectively. For the mixed boundary condition (48), we have the following eigenvalues and eigenvectors

$$\lambda_n = -\nu(2n\pi/l)^2 \tag{56}$$

and

$$\phi_n(x) = \sqrt{\frac{1}{l}} (\sin \frac{2n\pi x}{l} + \cos \frac{2n\pi x}{l}) \tag{57}$$

for all $n \geq 1$. For the output $y(t) = Cz(x,t) = \int_0^l c(x)z(x,t)dx$ in (50), the observability condition given in Theorem 3 for Dirichlet boundary condition (47) is actually

$$\frac{\sqrt{l}}{2n\pi\varepsilon} \sin \frac{2n\pi x_0}{l} \sin \frac{2n\pi\varepsilon}{l} \neq 0, n \geq 1 \tag{58}$$

Note that if we take the limit as $\varepsilon \to 0$, we obtain the observability condition for the case where $y(t) = z(x_0, t)$ as

$$\sin \frac{2n\pi x_0}{l} \neq 0, n \geq 1 \tag{59}$$

For the periodic boundary conditions (48) and the output (50), the observability condition will be

$$\frac{\sqrt{l}}{2n\pi\varepsilon}(\sin\frac{2n\pi x_0}{l}+\cos\frac{2n\pi x_0}{l})\sin\frac{2n\pi\varepsilon}{l}\neq 0, n\geq 1 \tag{60}$$

and for the output (49), it is

$$\sin\frac{2n\pi x_0}{l}+\cos\frac{2n\pi x_0}{l}\neq 0, n\geq 1 \tag{61}$$

Example 2 Consider the following one-dimensional wave equation

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}w(x,t)=\frac{\partial^2}{\partial x^2}w(x,t)+u(t)\frac{\partial}{\partial t}w(x,t)\\&w(x,0)=w_0(x), t\geq 0, x\in(0,1), u\in R\end{aligned} \tag{62}$$

with Dirichlet boundary conditions

$$w(0,t)=w(1,t)=0, t\geq 0 \tag{63}$$

The output is given by

$$y(t)=\int_0^l c(x)w(x,t)dx \tag{64}$$

where $c(x)$ is given by (51).

System (62) can be represented as the first order differential equation in $Z=H_0^1(0,1)\times L^2(0,1)$:

$$\dot{z}(t)=Az(t)+u(t)Dz(t) \tag{65}$$

where

$$z(t)=\begin{pmatrix} w(t)\\ \dot{w}(t)\end{pmatrix}, A=\begin{pmatrix} 0 & I\\ A_0 & 0\end{pmatrix} \tag{66}$$

and

$$A_0=\partial^2/\partial x^2 \text{ and } D=\begin{pmatrix} 0 & 0\\ 0 & I\end{pmatrix} \tag{67}$$

here $D(A_0)=H_0^1(0,L)\cap H^2(0,L)$. The operator A is skew-adjoint ($A^*=-A$) with $D(A)=D(A_0)\times D(A_0^{1/2})$ and generates an isometric C_0 semigroup on H. The system (62) can then be formulated as a bilinear system as follows

$$
\begin{aligned}
\dot{z}(t) &= Az(t) + u(t)Dz(t) \\
y(t) &= Cz(t), Cz(t) = \int_0^l c(x)z_1(x,t)dx
\end{aligned} \tag{68}
$$

The operator A has the eigenvalues $\{\lambda_n = jn\pi, n = \pm 1, \pm 2, \cdots\}$ and the eigenvectors

$$
\phi_n(x) = \frac{1}{\lambda_n} \begin{pmatrix} \sin(n\pi x) \\ \lambda_n \sin(n\pi x) \end{pmatrix}, n = \pm 1, \pm 2, \cdots \tag{69}
$$

which forms a Riesz basis. It follows that the observability condition given in Theorem 3 is actually

$$
\frac{\sin(n\pi x_0)\sin(n\pi\varepsilon)}{n\pi\varepsilon} \neq 0, n = \pm 1, \pm 2, \cdots \tag{70}
$$

Note that if we take the limit as $\varepsilon \to 0$, we obtain the observability condition for the case where $y(t) = z_1(x_0, t)$ as

$$
\sin(n\pi x_0) \neq 0, n = \pm 1, \pm 2, \cdots \tag{71}
$$

5 Conclusions

A formal Volterra series representation of the solution of a class of infinite dimensional bilinear systems has been derived which provides a new way to investigate the observability of these systems. Two types of approximate observability have been discussed, that is the approximate observability with respect to a given input and with respect to all admissible inputs. It has been found that the observability gramian for the approximately observability with respect to a given input can be explicitly expressed in terms of the C_0-semigroup of the operator A, the bounded bilinear operator D, and the property of the given input. A testable observability criterion for the case where the infinitesimal generator is self-adjoint or Riesz-spectral operator has also been given as a practical test. The importance of the results in this paper lies in that they provide a new way to study the system properties of infinite dimensional nonlinear systems.

Another closely related question is the design of an observer by using the Volterra series representation for infinite-dimensional systems. Comparing with the classic methods, one of the advantages of using a Volterra series representation is that it provides a parametric input-output representation for the underlying nonlinear systems. Moreover, Volterra series provides a new way to investigate infinite-dimensional nonlinear systems in frequency domain, which could have important implication in the observer design for nonlinear systems.

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