On Optimizing Multi-Sequence Functionals for Competitive Analysis

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Abstract

The efficiency of an online motion planning algorithm often is measured by a constant competitive factor C. Competitivity means, that the cost of an C-competitive online strategy with incomplete information is only C times worse than the optimal offline solution with full information. If a strategy is represented by an infinite sequence $X = f_1, f_2, \ldots$, the problem of finding a strategy with minimal C often results in minimizing functionals F_k in X. There are two main paradigm for finding an optimal sequence $f_1, f_2, \ldots$ that minimizes F_k for all k. Namely, optimality of the exponential function and equality approach. If the strategy has to be defined by more than one interacting sequence both approaches may fail. We show a simple motion planning example with two interacting sequences and present its solution.

1 Introduction

Search games, i. e. games where two players, a searcher and a hider, compete with each other, are studied in many variations in the last 60 years since the first work by Koopman in 1946. For example, Bellman [3] introduced the search for an immobile hider located on the real line with a known probability distribution, Gal [5] and independently Baeza-Yates et. al. [2] solve this problem for a uniformly distributed location of the hider. The book by Gal [5] and the reissue by Alpern and Gal [1] gives a comprehensive overview on results on search games.

The length of the searcher's trajectory is often used as payoff of a search game. To get a finite value for the game, we use the competitive framework, that is, we compare the length of the searcher's trajectory to the shortest distance to the hider. Gal [5] calls this a normalized cost function. More precisely, we call a search strategy *competitive* with a factor C, if $|\pi| \leq C \cdot |\pi_{\text{opt}}|$ holds for every location of the hider, where $|\pi|$ denotes the length of the searcher's path and $|\pi_{\text{opt}}|$ the shortest path. The competitive framework was introduced by Sleator and Tarjan [15], and used in many settings since then, see for example the survey by Fiat and Woeginger [4] or, for the field of online robot motion planning, see the surveys [12, 8].

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In most settings, a search strategy, X, can be given by a sequence of values $f_1, f_2, f_3, \ldots$ denoting e. g. the exploration depth in the i-th iteration step, and the competitive factor can be given by a functional $F_k(X)$, where k denotes the number of iteration steps. Since we want to minimize the costs, we have to find a strategy that minimizes F_k. There are two commonly used methods to find such a strategy. The first is, to show that the functionals F_k fulfill certain conditions—see below. Gal [5] showed, that a strategy with $f_i = a^i$ minimizes the F_k's, and we have just to find an appropriate a using simple analysis. Another method is, to show that there is a strategy X' that achieves the optimal competitive factor, C, not only asymptotically, but *exactely in every step.* With this we establish a closed form for X'. Both methods, however, work well for strategies that can be given by *one* sequence f_i. If we have search games that rely on more than one parameter, the theorem by Gal may not be applicable and the equality approach may fail.

We introduce a variation of the search for a goal on an infinite line: the goal is located on two rays emanating from the searcher's origin, with an angle γ between the rays. The searcher can move in the free space between the two rays, and finds the goal as soon as it reaches the goals position or a position behind the goal, seen from the origin. A reasonable strategy to solve this problem can be described by *two* sequences, α_i and β_i, see Figure 1.

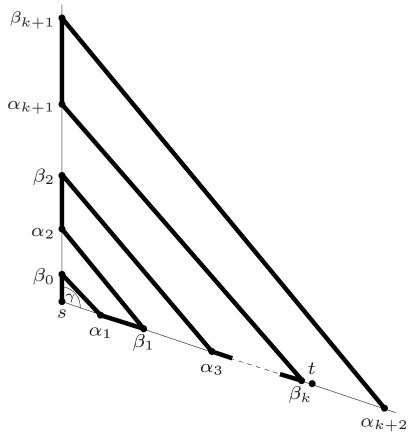

Figure 1: Representation of a strategy with two sequences.

2 Exponential function and equality approach

A strategy for searching an unknown goal on an infinite ray can be represented by an infinite sequence $F = f_1, f_2, \ldots$ of positive values. The agent moves f_1 steps to the left of the start, returns to the goal, moves f_2 steps to the right of the start point, returns to the goal and so on. The worst-case for the competitive factor occurs, if we miss the goal at step k by an ϵ, return to the start, move f_{k+1} to the other direction, return to the start and find the goal in the $(k+2)$-th step at distance $f_k + \epsilon$. Thus, the worst-case factor is given by $\frac{2\sum_{i=1}^{k+1} f_i + f_k}{f_k} = 1 + 2\frac{\sum_{i=1}^{k+1} f_i}{f_k}$, and it suffices to find a sequence $X = f_1, f_2, \ldots$ that minimizes $F_k(f_1, f_2, \ldots) := \frac{\sum_{i=1}^{k+1} f_i}{f_k}$ for all k. Note, that we additionally have to take care for the first movement. We assume that the goal is at least one step away from the start which gives the additional inequality $2f_1 \leq C \cdot 1$.

Let $F(X) := \lim_{k\to\infty} F_k(X)$ more precisely, we are searching for a strategy X with

$$\inf_Y \sup_k F_k(Y) = C \text{ and } \sup_k F_k(X) = C\,.$$

Two main approaches for solving the given problem are discussed in the literature. We briefly repeat the main ideas.

Optimality of the exponential function: The functional F_k is continous and unimodal. Unimodality is defined by $F_k(A \cdot X) = F_k(X)$ and $F_k(X+Y) \leq \max\{F_k(X), F_k(Y)\}$ for every constant A and two sequences X and Y. Unimodality means that scalar multiplication and the addition of two sequences does not increase the value of the functional.

If F_k additionally fulfills some other reasonable properties, it was shown by [1, 5, 14] that an exponential function minimizes F_k, or more precisely

$$\sup_k F_k(X) \geq \inf_a \sup_k F_k(A_a)$$

where $A_a = a^0, a^1, a^2, \ldots$ and $a > 0$. Altogether, the problem of searching a point on a line is solved by

$$\inf_a \frac{\sum_{i=1}^{k+1} a^{i-1}}{a^{k-1}} = \frac{2^2}{2-1} = 4\,.$$

Equality approach: On the other hand some authors [7, 10, 11, 9, 13] suggest to adjust an optimal strategy $X = f_1, f_2, \ldots$ with $F_k(X) \leq C$ to an optimal strategy $X' = f_1', f_2', \ldots$ with $F_k(X') = C$ where C is the (probably unknown) best achievable factor. It can be shown that for the 2-ray search problem such a strategy X' exists. The main reason is that $F_k = \frac{\sum_{i=1}^{k} f_i}{f_k}$ increases in f_k and F_l decrease in f_k for all $l \neq k$. Therefore we can inductively adjust a given optimal strategy X. How will we find the optimal strategy? One will try to retrieve a recurrence for the values of X' from the equation $F_k(X') = C = F_{k+1}(X')$

For the 2-ray search problem we assume that $X = f_1, f_2, \ldots$ achieves equality in every step. Thus, we have $\sum_{i=1}^{k+2} f_i = Cf_{k+1}$ and $\sum_{i=1}^{k+1} f_i = Cf_k$. Subtracting both sides gives the recurrence $f_{k+2} = C(f_{k+1} - f_k)$ for $k = 1, 2, \ldots$ Obtaining positive solutions for recurrences can be solved by analytic means, see [6]. It can be shown that for $C < 4$ there is no positive sequence that fulfills the given recurrence $f_{k+2} = C(f_{k+1} - f_k)$. Additionally, for $f_i := (i+1)2^i$ we have $f_{k+2} = (k+3)2^{k+1} = 4(f_{k+1} - f_k) = (3k+4)2^{k+2} - (k+1)2^{k+2}$ and $C = 4$ is optimal.

Altogether, we have two optimal solutions stemming from different paradigm. In the following we will combine both paradigm in order to solve more sophisticated functionals.

3 A simple online problem

We discuss a variant of the 2-ray search problem. We are searching for a target on 2-rays r_1 and r_2, emanating from a single source s and building an angle γ, Figure 1. It is allowed to move in the plane from one ray to the other. A target t at distance $|st|$ on ray r_i can be detected, if we visit a point p on r_i with $|pt| \geq |st|$. Therefore a search strategy need not visit all points on the ray. We denote the problem as the 2γ-ray-scan problem. The distance to the goal and the goal's ray is not known in advance. The best offline strategy moves directly along the corresponding ray to the goal.

[2, 1] dicuss a similar variant without looking back, the goal is detected only if the goal is visited. This variant can be described by a single sequence and was solved in [1] with an exponential function.

A strategy for the 2γ-ray-scan problem can be represented as follows. We start with ray r_1 and move along it for a while up to distance β_0. Then we will move on a straight line to a point at distance α_1 on ray r_2 and move along r_2 for a while until leaving the ray at distance β_1 and so on. Altogeher, we have a sequence of leave points and a sequence of hit points and we denote every hit point by its distance α_i and every leave point by its distance β_i, see 1. For convenience, we set $\alpha_0 = 0$. A reasonable strategy fulfills $\beta_{i-2} \leq \alpha_i \leq \beta_i$.

The worst-case for the competitive factor occurs, if we miss the goal by an ϵ on the first ray, move to the second ray and find the goal after returning back to the first ray. Setting $\alpha_0 = 0$ we have to minimize the following functional

$$G_k([(\alpha_0, \alpha_1, \ldots), (\beta_0, \beta_1, \ldots)] :=$$

$$\frac{\sum_{i=0}^{k+1} \beta_i - \alpha_i + \sqrt{\alpha_{i+1}^2 - 2\alpha_{i+1}\beta_i \cos\gamma + \beta_i^2}}{\beta_k}\,.$$

One can proof the optimality of a exponential function for β_i by showing the prerequisites in [5, 1, 14].

Theorem 1 *If there exists a strategy* $X = [(\alpha_0, \alpha_1, \ldots), (\beta_0, \beta_1, \ldots)]$ *such that*

$$\inf_Y \sup_k G_k(Y) = C \text{ and } \sup_k G_k(X) = C,$$

then there is always a solution $Z = [(\alpha_0, \alpha_1, \ldots), (1, \beta^1, \beta^2, \ldots)]$ *so that*

$$\inf_Y \sup_k G_k(Y) = C \text{ and } \sup_k G_k(Z) = C\,.$$

Proof. We show that the unimodality condition holds, the other prerequisites are easily fulfilled. Scalar multiplication is simply satisfied.

It suffices to show that $\sqrt{a_{i+1}^2 - 2a_{i+1}b_i \cos\gamma + b_i^2} + \sqrt{c_{i+1}^2 - 2c_{i+1}d_i \cos\gamma + d_i^2}$ is bigger than $\sqrt{(a_{i+1}+c_{i+1})^2 - 2(a_{i+1}+c_{i+1})(b_i+d_i)\cos\gamma + + (b_i+d_i)^2}$ which is the triangle inequality for the vectors $((a_{i+1}-b_i)\cos(\gamma/2), (a_{i+1}+b_i)\sin(\gamma/2))$ and $((c_{i+1}-d_i)\cos(\gamma/2), (c_{i+1}+d_i)\sin(\gamma/2))$.

Now, let $G_k([(a_0,\ldots),(b_0,\ldots)]) \le D$ and $G_k([(c_0,\ldots),(d_0,\ldots)]) \le D$ then we have $\sum_{i=0}^{k+1} b_i - a_i + \sqrt{a_{i+1}^2 - 2a_{i+1}b_i\cos\gamma + b_i^2} + + \sum_{i=0}^{k+1} c_i - d_i + \sqrt{c_{i+1}^2 - 2c_{i+1}d_i\cos\gamma + d_i^2} \le D(b_k + d_k)$ and the left-hand side of the inequality is greater than or equal to $\sum_{i=0}^{k+1}(b_i+d_i) - (a_i + c_i)\sqrt{(a_{i+1}+c_{i+1})^2 - 2(a_{i+1}+c_{i+1})(b_i+d_i)\cos\gamma + + (b_i+d_i)^2}$ which completes the proof. □

Unfortunately, this result will not give us a strategy because there is a second sequence. We still have to optimize

$G_k([(\alpha_0, \alpha_1, \ldots), \beta]) :=$

$$\frac{\sum_{i=0}^{k+1} \beta^i - \alpha_i + \sqrt{\alpha_{i+1}^2 - 2\alpha_{i+1}\beta^i\cos\gamma + \beta_i^2}}{\beta^k}$$

On the other hand if we can show that there is a strategy with $G_k([(\alpha_0, \alpha_1, \ldots), (\beta_0, \beta_1, \ldots)]) = C$ for all $k > l$, the subtraction of two equations

$$\sum_{i=0}^{k} \beta_i - \alpha_i + \sqrt{\alpha_{i+1}^2 - 2\alpha_{i+1}\beta_i\cos\gamma + \beta_i^2} = C\beta_{k-1}$$

$$\text{and } \sum_{i=0}^{k+1} \beta_i - \alpha_i + \sqrt{\alpha_{i+1}^2 - 2\alpha_{i+1}\beta_i\cos\gamma + \beta_i^2} = C\beta_k$$

results in a recurrence

$$\beta_{k+1} - \alpha_{k+1} + \sqrt{\alpha_{k+2}^2 - 2\alpha_{k+2}\beta_{k+1}\cos\gamma + \beta_{k+1}^2} = C(\beta_k - \beta_{k-1})\,.$$

Unfortunately, this is a non-linear recurrence and cannot be solved easily.

4 Combining two paradigms

We suggest to combine both approaches. First, we show that at least for $\gamma \le \pi/2$ there is indeed a strategy $[(\alpha_0, \alpha_1, \ldots), (\beta_0, \beta_1, \ldots)]$ so that $\sum_{i=0}^{k+1} \beta_i - \alpha_i + \sqrt{\alpha_{i+1}^2 - 2\alpha_{i+1}\beta_i\cos\gamma + \beta_i^2} = C\beta_k$ for all $k \ge 1$.

Then we make use of the subtraction idea above and obtain

$$C = \left(\frac{1}{\beta_k - \beta_{k-1}}\right)\left(\beta_{k+1} - \alpha_{k+1} + \sqrt{\alpha_{k+2}^2 - \alpha_{k+2}\beta_{k+1}2\cos\gamma + \beta_{k+1}^2}\right) \quad (1)$$

which again gives a functional but without a sum in the denominator. Now, we solve this functionals by showing that the prerequisites of the exponential solution is again fulfilled.

Lemma 2 *For* $\gamma \ge \frac{\pi}{2}$ *there is always an optimal solution* $X = [(\alpha_0, \alpha_1, \ldots), (\beta_0, \beta_1, \ldots)]$ *that achieves* $\sum_{i=0}^{k+1} \beta_i - \alpha_i + \sqrt{\alpha_{i+1}^2 - 2\alpha_{i+1}\beta_i\cos\gamma + \beta_i^2} = C\beta_k$ *for all* $k \ge 0$.

Proof. We show the property by induction. We adjust a given strategy $[(\alpha_0, \alpha_1, \ldots), (\beta_0, \beta_1, \ldots)]$ to a strategy $[(\alpha_0, \alpha_1, \ldots), (\beta_0', \beta_1', \ldots)]$ such that all β_i' decrease but are still positive.

For $k = 0$ we may have $\sum_{i=0}^{2} \beta_i - \alpha_i + \sqrt{\alpha_{i+1}^2 - 2\alpha_{i+1}\beta_i\cos\gamma + \beta_i^2} < C\beta_0$ for the optimal C. Thus, we decrease β_0 to $\beta_0' := \beta_0 - \epsilon$ until we obtain equality. The distance $\sqrt{\alpha_1^2 - 2\alpha_1\beta_0\cos\gamma + \beta_0^2}$ decreases for $\gamma \ge \frac{\pi}{2}$ and $G_k([(\alpha_0, \alpha_1, \ldots), (\beta_0', \beta_1, \ldots)])$ gets smaller for all $k > 1$. β_0' is positive since we have to subsume the distance from β_1 to $\alpha_2 > \beta_0$.

Now let us assume that the property holds for all $l \le k-1$ and let $G_k([(\alpha_0, \alpha_1, \ldots), (\beta_0', \beta_1', \ldots)]) < C$. We again can decrease β_k to $\beta_k' := \beta_k - \epsilon$ until we have equality. For $\alpha_k < \beta_k$ we decrease the denominator of $G_l([(\alpha_0, \alpha_1, \ldots), (\beta_0', \beta_1', \ldots)])$ for all $l \ge k$ and for $k-1$ but not for $l \le k-2$. Note, that $\sqrt{\alpha_{k+1}^2 - 2\alpha_{k+1}\beta_k\cos(\gamma) + \beta_k^2}$ decreases for $\gamma \ge \frac{\pi}{2}$. If $\alpha_k = \beta_k'$ is reached, we additionally decrease the denominator for $l = k-2$. Note, that this is true, since for $\gamma \ge \frac{\pi}{2}$ the distances $\sqrt{\alpha_{k+1}^2 - 2\alpha_{k+1}\beta_k'\cos\gamma + \beta_k'^2}$ and $\sqrt{\beta_k'^2 - 2\beta_k'\beta_{k-1}\cos\gamma + \beta_{k-1}^2}$ decrease.

Finally we will achieve equality for $G_k([(\alpha_0, \alpha_1, \ldots), (\beta_0', \beta_1', \ldots)])$ with β_k'. β_k' is positive, since we have to subsume the distance β_{k+1} to α_{k+2}. Unfortunately, $G_l([(\alpha_0, \alpha_1, \ldots), (\beta_0', \beta_1', \ldots)]) < C$ may hold for $l = k-2$ and $l = k-1$. By induction hypothesis, we adjust the strategy again such that $G_l([(\alpha_0, \alpha_1, \ldots), (\beta_0', \beta_1', \ldots)]) = C$ for all $l \le k-1$. Now it again may happen that $G_k([(\alpha_0, \alpha_1, \ldots), (\beta_0', \beta_1', \ldots)]) < C$ holds.

We repeat the above process, since the vector $(\beta'_0, \beta'_1, \ldots, \beta'_k)$ always decrease by construction but still remain positive, the vector $(\beta'_0, \beta'_1, \ldots, \beta'_k)$ finally has to run into a positive limit that fulfills equality. $\square$

Note, that for the equality strategy we still have $\beta_{k-2} \leq \alpha_k \leq \beta_k$. The functional (1) for $k \geq 1$ fulfills the criterion for the exponential function which can be proven by the same arguments as in the proof of Theorem 1.

Lemma 3 *Functional (1) can be optimized by an exponential function, that is, there is an optimal strategy with factor* $\left(\frac{1}{\beta^k - \beta^{k-1}}\right)\left(\beta^{k+1} - \alpha_{k+1} + \right.$ $\left. + \sqrt{\alpha_{k+2}^2 - 2\alpha_{k+2}\beta^{k+1}\cos\gamma + \beta_{k+1}^2}\,\right)$.

Now we can set $\alpha_i = C_i \beta^i$ for $C_i \in [1, \frac{1}{\beta^2}]$ and it remains to optimize

$$\frac{\beta^2 \left(1 - C_i + \sqrt{(C_{i+1}\beta)^2 - 2C_{i+1}\beta\cos\gamma + 1}\right)}{\beta - 1}$$

for β and C_i and C_{i+1}. In the next iteration step C_{i+1} takes over the role of C_i. Therefore the best we can do is that we fix C_i by a constant D which means that we have to optimize

$$\frac{\beta^2 \left(1 - D + \sqrt{(D\beta)^2 - 2D\beta\cos\gamma + 1}\right)}{\beta - 1}$$

for β and $D \in [1, 1/\beta^2]$.

Theorem 4 *The strategy for the 2γ-ray-scan problem for $\gamma \geq \pi/2$ can be achieved by minimizing*

$$f(\beta, \gamma, D) := \frac{\beta^2 \left(1 - D + \sqrt{(D\beta)^2 - 2D\beta\cos\gamma + 1}\right)}{\beta - 1}$$

over β and $D \in [1, 1/\beta^2]$.

It can be shown that there is a reasonable $D(\gamma, \beta)$ that minimizes $f(\beta, \gamma, D)$. If $D(\gamma, \beta)$ is inside $[1, 1/\beta^2]$ then we have to compare the minima over β for $D = 1$, $D = 1/\beta^2$ and $D = D(\gamma, \beta)$. $D = 1/\beta^2$ represents a strategy without any gaps. $D = 1$ represents a strategy that does not slip along the rays and $D = D(\gamma, \beta)$ represents something in between.

For example for $\gamma = \pi/2$ we can show that

$$f(\gamma, \beta, D(\gamma, \beta)) = \frac{\beta^2 \left(1 - \frac{1}{\sqrt{-1+\beta^2}\beta} + \sqrt{\frac{1}{-1+\beta^2} + 1}\right)}{\beta - 1}$$

which is minimal for $\beta = 1.839\ldots$ and $D = 0.352\ldots \in [1, 0.295\ldots]$. The optimal competitive factor is given by $7.413\ldots$ wheras the optimal factor for $D = 1/\beta^2$ is $7.472\ldots$

For $\gamma = \pi$ we obtain the *normal* doubling strategy, which is represented by $\beta = 2$ and $D = 1/4$.

5 Conclusion

We have shown that for solving a double sequence functional it is useful to combine the methods of equality approach and the optimality of the exponential function. With the help of multi-sequence functionals more sophisticated strategies can be represented. We think that the presented method can be extended to functionals with more than two sequences and that it should be possible to iterate the combination process more than once.

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