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MULTIPLICATIVE PERTURBATION ANALYSIS FOR THE GENERALIZED CHOLESKY BLOCK DOWNDATING PROBLEM

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Abstract. This article is devoted to the multiplicative perturbation analysis of the generalized Cholesky block downdating problem. The strong rigorous multiplicative perturbation bounds are first presented by bringing together the modified matrix-vector equation approach with the technique of Lyapunov majorant function and the Banach fixed point theorem. Then, the weak rigorous multiplicative bounds are developed by using the matrix-equation approach. Numerical results demonstrate that these bounds are constantly tighter than the additive perturbation bounds.

1. Introduction

Suppose that $A \in \mathbb{R}_m^{m\times m}$ is symmetric positive definite, $B \in \mathbb{R}_n^{n\times m}$, and $C \in \mathbb{R}^{n\times n}$ are symmetric positive and semi-definite, then the symmetric quasi-definite matrix $K \in \mathbb{R}^{(m+n)\times(m+n)}$ can be expressed as

$$K = \begin{bmatrix} A & B^{\mathrm{T}} \\ B & -C \end{bmatrix}. \tag{1.1}$$

The matrix K always has the generalized Cholesky factorization

$$K = LJ_{m+n}L^{\mathrm{T}}, \tag{1.2}$$

where

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}, \quad J_{m+n} = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix},$$

$L_{11} \in \mathbb{R}_m^{m\times m}$, $L_{22} \in \mathbb{R}_n^{n\times n}$ are lower triangular and $L_{21} \in \mathbb{R}_n^{n\times m}$. From (1.2), it can be simply verify that

$$A = L_{11}L_{11}^{\mathrm{T}}, \quad B = L_{21}L_{11}^{\mathrm{T}}, \quad C + L_{21}L_{21}^{\mathrm{T}} = L_{22}L_{22}^{\mathrm{T}}.$$

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If the diagonal elements of the lower triangular matrices, L_{11} and L_{22} are positive, the factorization (1.2) is unique and L is known as the generalized Cholesky factor [1].

In this paper, we consider the generalized Cholesky block downdating problem (GCBD)

$$LJ_{m+n}L^{\mathrm{T}} - YY^{\mathrm{T}} = VJ_{m+n}V^{\mathrm{T}}. \tag{1.3}$$

Given that K is the same as in (1.1) and $Y = \left(Y_m^{\mathrm{T}}, Y_n^{\mathrm{T}}\right) \in \mathbb{R}^{m+n\times k}$, find a lower triangular matrix

$$V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)},$$

where $V_{11} \in \mathbb{R}_m^{m\times m}$, $V_{22} \in \mathbb{R}_n^{n\times n}$ are lower triangular with positive elements and $V_{21} \in \mathbb{R}_n^{n\times m}$. From [2, Corollary 1], it is simply to show that when $\left\|L^{-1}Y\right\|_2 < 1$ the GCBD problem (1.3) is always exists and the matrix V is known as the GCBD factor. Moreover, in this case

$$A - Y_mY_m^{\mathrm{T}} = V_{11}V_{11}^{\mathrm{T}}, \quad B - Y_nY_m^{\mathrm{T}} = V_{21}V_{11}^{\mathrm{T}}, \quad C + V_{21}V_{21}^{\mathrm{T}} + Y_nY_n^{\mathrm{T}} = V_{22}V_{22}^{\mathrm{T}}.$$

This problem is reduced to the Cholesky block downdating problem if we choose $K = A$, i.e., B and C are nonexistent. The Cholesky block downdating problem has acquired remarkable consideration, and its special case, i.e., single downdating $(Y \in \mathbb{R}^{m\times 1})$, has been extensively studied in the literature; see [3, 4, 5, 6, 7, 8, 9, 10] for details. Additive perturbation and multiplicative perturbation are the two distinctive types of perturbation models of matrix factorizations. The additive perturbation analysis for the GCBD problem has been considered in [11, 12].

Obviously, an additive perturbation can be obtained from the multiplicative perturbation, but the derived additive perturbation bounds will destroy the unique structures of multiplicative perturbations and lose their tendency [13]. Since the matrix scaling technique is often employed to provide better-conditioned problems [13], the multiplicative perturbation has received significant attention, and has some elements of interest compared with the additive perturbation; see [13, 14, 15, 16, 17] and references therein. Until now, there has been no work on the multiplicative perturbation bounds for the GCBD problem. So, it is interesting to introduce the multiplicative perturbation bounds for the GCBD problem.

Particularly, for (1.3), we assume two types of multiplicative perturbation matrices, $W = 1_{m+n} + N$ on L: (1) $W = I_{m+n} + N$ is a general matrix; (2) $W = I_{m+n} + N$ is a lower triangular matrix. The rest of the paper is organized as follows: In Section 3, we will develop the strong rigorous multiplicative perturbation bounds by using the modified matrix-vector equation approach [13, 18, 19, 20], the Lyapunov majorant function (e.g., [11, Chapter 5]), and the Banach fixed point theorem (e.g., [11, Appendix 5]). Moreover, we will use the matrix-equation approach [21] to derive the weak rigorous multiplicative bounds in Section 4. These bounds will be less expensive to compute as compared to the strong, rigorous multiplicative perturbation bounds. Section 2 provides some useful notation and preliminary knowledge. Finally, we provide some numerical experiments to verify the theoretical results, and we show the numerical comparison between the multiplicative rigorous perturbation bounds and the additive rigorous perturbation bounds [11, 12] for the GCBD problem in Section 5.

2. Preliminaries

Some notations can be endorsed in [21] to make the presentation apparent. We still illustrate them here to make easier for readers.

The Frobenius norm and spectral norm for a given matrix $Z = (z_{ij}) \in \mathbb{R}^{m\times n}$ are denoted by $\|Z\|_F$ and $\|Z\|_2$, respectively. The following inequalities hold for these two matrix norms; see [22] for details.

$$\|QRS\|_2 \leqslant \|Q\|_2 \|R\|_2 \|S\|_2, \quad \|QRS\|_F \leqslant \|Q\|_2 \|R\|_F \|S\|_2, \tag{2.1}$$

whenever the matrix product QRS is well-defined.

For any matrix $Z = [z_1, z_2, \dots, z_n] = (z_{ij}) \in \mathbb{R}^{n\times n}$, denote the vector of the last i elements of z_j by $z_j^{(i)}$ and define

$$\mathrm{lvec}(Z) := \begin{bmatrix} z_1^{(n)} \\ z_2^{(n-1)} \\ \vdots \\ z_n^{(1)} \end{bmatrix} \in \mathbb{R}^{v_1}, \quad \mathrm{vec}(Z) := \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{R}^{n^2}, \quad \mathrm{slt}(Z) := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ z_{21} & 0 & 0 & \cdots & 0 \\ z_{31} & z_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{n,n-1} & 0 \end{bmatrix},$$

$$\mathrm{low}(Z) := \begin{bmatrix} \frac{1}{2}z_{11} & 0 & \cdots & 0 \\ z_{21} & \frac{1}{2}z_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & \frac{1}{2}z_{nn} \end{bmatrix}, \quad \mathrm{lt}(Z) := \begin{bmatrix} z_{11} & 0 & \cdots & 0 \\ z_{21} & z_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix},$$

$\mathrm{sut}(Z) = \mathrm{slt}(Z^{\mathrm{T}})^{\mathrm{T}}$, and $\mathrm{diag}(Z) = \mathrm{diag}(z_{11}, z_{22}, \dots, z_{nn})$, where $v_1 = \frac{n(n+1)}{2}$. Using the structures of these operators, we have

$$\mathrm{lvec}(Z) = \aleph_{\mathrm{lvec}}\mathrm{vec}(Z), \quad \mathrm{vec}(\mathrm{lt}(Z)) = \aleph_{\mathrm{lt}}\mathrm{vec}(Z), \quad \mathrm{vec}(\mathrm{low}(Z)) = \aleph_{\mathrm{low}}\mathrm{vec}(Z), \tag{2.2}$$

where

$$\aleph_{\mathrm{lvec}} = \mathrm{diag}(G_1, G_2, \cdots, G_n) \in \mathbb{R}^{v_1\times n^2}, \quad G_i = \left[0_{n-(i-1)\times(i-1)}, I_{n-(i-1)}\right] \in \mathbb{R}^{n-(i-1)\times n},$$
$$\aleph_{\mathrm{lt}} = \mathrm{diag}\left(\hat{G}_1, \hat{G}_2, \cdots, \hat{G}_n\right) \in \mathbb{R}^{n^2\times n^2}, \quad \hat{G}_i = \mathrm{diag}\left(0_{(i-1)\times(i-1)}, I_{n-(i-1)}\right) \in \mathbb{R}^{n\times n},$$
$$\aleph_{\mathrm{low}} = \mathrm{diag}\left(\tilde{G}_1, \tilde{G}_2, \cdots, \tilde{G}_n\right) \in \mathbb{R}^{n^2\times n^2}, \quad \tilde{G}_i = \mathrm{diag}\left(0_{(i-1)\times(i-1)}, 1/2, I_{n-i}\right) \in \mathbb{R}^{n\times n}.$$

Moreover,

$$\aleph_{\mathrm{lvec}}\aleph_{\mathrm{lvec}}^T = I_{v_1}, \quad \aleph_{\mathrm{lvec}}^T\aleph_{\mathrm{lvec}} = \aleph_{\mathrm{lt}}. \tag{2.3}$$

Let $\mathrm{lvec}^\dagger : \mathbb{R}^{v_1} \to \mathbb{R}^{n\times n}$ be the right inverse of the operator ‘lvec’ such that $\mathrm{lvec}\cdot\mathrm{lvec}^\dagger = 1_{v_1\times v_1}$ and $\mathrm{lvec}^\dagger\cdot\mathrm{lvec} = \mathrm{lt}$. Then the matrix of the operator ’lvec’ is $\aleph_{\mathrm{lvec}}^T$. That is, $\mathrm{lvec}^\dagger(Z) = \aleph_{\mathrm{lvec}}^T\mathrm{vec}(Z)$.

Let $\mathbb{D}_n \in \mathbb{R}^{n\times n}$ be the set of diagonal matrices with positive diagonal elements. Then, for any $D_n = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathbb{D}_n$, it follows that

$$\mathrm{low}(ZD_n) = \mathrm{low}(Z)D_n, \quad \mathrm{low}(D_nZ) = D_n\mathrm{low}(Z). \tag{2.4}$$

Moreover, from [23, Lemma 5.1], we have

$$\left\|\mathrm{low}\,(Z) + D_n\mathrm{low}\left(Z^T\right) D_n^{-1}\right\|_F \leqslant \sqrt{1+\varsigma_{D_n}^2}\|Z\|_F, \tag{2.5}$$

where $\varsigma_{D_n} = \max\limits_{1\leqslant i<j\leqslant n}\{\sigma_j/\sigma_i\}$. From [21], we have

$$\|\mathrm{low}(Z)\|_F \leqslant \|Z\|_F. \tag{2.6}$$

If Z is symmetric, then

$$\|\mathrm{low}(Z)\|_F \leqslant \frac{1}{\sqrt{2}}\|Z\|_F. \tag{2.7}$$

Also, we have

$$\left\|\mathrm{low}\left(Z+Z^{\mathrm{T}}\right)\right\|_F \leqslant \sqrt{2}\,\|Z\|_F. \tag{2.8}$$

The Kronecker product between $Z = (Z_{ij}) \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{p\times q}$ is defined as $Z\otimes X = [z_{ij}X] \in \mathbb{R}^{mp\times nq}$. Some useful results of the Kronecker product are listed below [24]:

$$\mathrm{vec}\,(ZCX) = (X^T \otimes Z)\mathrm{vec}\,(C), \tag{2.9}$$

$$\Pi\,\mathrm{vec}(Z) = \mathrm{vec}\left(Z^{\mathrm{T}}\right), \tag{2.10}$$

$$\|X\otimes Z\|_2 = \|X\|_2\|Z\|_2, \tag{2.11}$$

$$(X\otimes Z)(B\otimes C) = (XB\otimes ZC), \tag{2.12}$$

$$(X\otimes Z)^{-1} = X^{-1}\otimes Z^{-1},\ \text{if } X \text{ and } Z \text{ are nonsingular}, \tag{2.13}$$

where B and C are of suitable orders.

3. Strong rigorous multiplicative perturbation bounds

Let us consider that the matrices L, Y and V in (1.3) are perturbed as

$$L \to LW, \quad Y \to YU, \quad V \to \Delta V,$$

where $W = I_{m+n} + N \in \mathbb{R}^{(m+n)\times(m+n)}$, $U = I_{m+n} + M \in \mathbb{R}^{(m+n)\times(m+n)}$ and $\Delta V \in \mathbb{R}^{(m+n)\times(m+n)}$ is a lower triangular matrix with positive diagonal elements. Therefore, the perturbed form of (1.3) is

$$\begin{aligned}(V+\Delta V)J_{m+n}(V+\Delta V)^{\mathrm{T}} = {}& (I_{m+n}+N)\,LJ_{m+n}L^{\mathrm{T}}\,(I_{m+n}+N)^{\mathrm{T}} \\ & - (I_{m+n}+M)\,YY^{\mathrm{T}}\,(I_{m+n}+M)^{\mathrm{T}}.\end{aligned} \tag{3.1}$$

Extending (3.1) and using (1.3), we have

$$VJ_{m+n}(\Delta V)^{\mathrm{T}}+(\Delta V)J_{m+n}V^{\mathrm{T}}=N\left(LJ_{m+n}L^{\mathrm{T}}\right)+\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}-M\left(YY^{\mathrm{T}}\right)-\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}$$
$$+N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}-M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}-\Delta VJ_{m+n}(\Delta V)^{\mathrm{T}}. \tag{3.2}$$

Premultiplying (3.2) by V^{-1} and right postmultiplying it by $V^{-\mathrm{T}}$ lead to

$$\begin{aligned}&J_{m+n}(\Delta V)^{\mathrm{T}}V^{-\mathrm{T}}+V^{-1}(\Delta V)J_{m+n}\\ =&V^{-1}\left(N\left(LJ_{m+n}L^{\mathrm{T}}\right)+\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}\right)V^{-\mathrm{T}}-V^{-1}\left(M\left(YY^{\mathrm{T}}\right)+\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}\right)V^{-\mathrm{T}}\\ &+V^{-1}\left(N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}-M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}-\Delta VJ_{m+n}(\Delta V)^{\mathrm{T}}\right)V^{-\mathrm{T}}.\end{aligned} \tag{3.3}$$

As performed in [8, 9, 11, 14], from (3.3), we have

$$\begin{aligned}V^{-1}\Delta VJ_{m+n}=&\operatorname{low}\left(V^{-1}\left(N\left(LJ_{m+n}L^{\mathrm{T}}\right)+\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\\ &-\operatorname{low}\left(V^{-1}\left(M\left(YY^{\mathrm{T}}\right)+\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\\ &+\operatorname{low}\left(V^{-1}\left(N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}-M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}-\Delta VJ_{m+n}(\Delta V)^{\mathrm{T}}\right)V^{-\mathrm{T}}\right).\end{aligned} \tag{3.4}$$

Applying the operator ‘vec’ to (3.4), and noting (2.2), (2.9) and (2.10), we get

$$\begin{aligned}&\left(J_{m+n}^{\mathrm{T}}\otimes V^{-1}\right)\operatorname{vec}(\Delta V)\\ =&\aleph_{\mathrm{low}}\left(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes V^{-1}\right)+\left(V^{-1}\otimes V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\right)\operatorname{vec}(N)\\ &-\aleph_{\mathrm{low}}\left(\left(V^{-1}(YY^{\mathrm{T}})^{\mathrm{T}}\otimes V^{-1}\right)+\left(V^{-1}\otimes V^{-1}\left(YY^{\mathrm{T}}\right)\right)\Pi\right)\operatorname{vec}(M)\\ &+\aleph_{\mathrm{low}}\left(V^{-1}\otimes V^{-1}\right)\operatorname{vec}\left(N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}-M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}-\Delta VJ_{m+n}(\Delta V)^{\mathrm{T}}\right).\end{aligned}$$

As performed in [8, 9, 11, 14], we can obtain

$$\begin{aligned}\operatorname{vec}(\Delta V)=&\left(J_{m+n}^{-\mathrm{T}}\otimes V\right)\aleph_{\mathrm{low}}\left(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes V^{-1}\right)\right.\\ &\left.+\left(V^{-1}\otimes V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\right)\operatorname{vec}(N)\\ &-\left(J_{m+n}^{-\mathrm{T}}\otimes V\right)\aleph_{\mathrm{low}}\left(\left(V^{-1}(YY^{\mathrm{T}})^{\mathrm{T}}\otimes V^{-1}\right)+\left(V^{-1}\otimes V^{-1}\left(YY^{\mathrm{T}}\right)\right)\Pi\right)\operatorname{vec}(M)\\ &+\left(J_{m+n}^{-\mathrm{T}}\otimes V\right)\aleph_{\mathrm{low}}\left(V^{-1}\otimes V^{-1}\right)\\ &\times\operatorname{vec}\left(N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}-M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}-\Delta VJ_{m+n}(\Delta V)^{\mathrm{T}}\right),\end{aligned} \tag{3.5}$$

and show that (3.5) is equivalent to

$$\begin{aligned}\operatorname{lvec}(\Delta V)=&\aleph_{\mathrm{lvec}}\left(J_{m+n}^{-\mathrm{T}}\otimes V\right)\aleph_{\mathrm{low}}\left(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes V^{-1}\right)\right.\\ &\left.+\left(V^{-1}\otimes V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\right)\operatorname{vec}(N)\\ &-\aleph_{\mathrm{lvec}}\left(J_{m+n}^{-\mathrm{T}}\otimes V\right)\aleph_{\mathrm{low}}\left(\left(V^{-1}(YY^{\mathrm{T}})^{\mathrm{T}}\otimes V^{-1}\right)\right.\\ &\left.+\left(V^{-1}\otimes V^{-1}\left(YY^{\mathrm{T}}\right)\right)\Pi\right)\operatorname{vec}(M)\\ &+\aleph_{\mathrm{lvec}}\left(J_{m+n}^{-\mathrm{T}}\otimes V\right)\aleph_{\mathrm{low}}\left(V^{-1}\otimes V^{-1}\right)\operatorname{vec}\left(N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}\right.\\ &\left.-M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}-\Delta VJ_{m+n}(\Delta V)^{\mathrm{T}}\right).\end{aligned} \tag{3.6}$$

As a matter of convenience, suppose

$$\begin{aligned} Q_{LM} &= \aleph_{\text{lvec}}\left(J_{m+n}^{-\text{T}} \otimes V\right) \aleph_{\text{low}}\left(\left(V^{-1}(LJ_{m+n}L^{\text{T}}) \otimes V^{-1}\right) + \left(V^{-1} \otimes V^{-1}(LJ_{m+n}L^{\text{T}})\right)\Pi\right) \\ Q_Y &= \aleph_{\text{lvec}}\left(J_{m+n}^{-\text{T}} \otimes V\right) \aleph_{\text{low}}\left(\left(V^{-1}(YY^{\text{T}})^{\text{T}} \otimes V^{-1}\right) + \left(V^{-1} \otimes V^{-1}\left(YY^{\text{T}}\right)\right)\Pi\right) \\ R_V &= .\aleph_{\text{lvec}}\left(J_{m+n}^{-\text{T}} \otimes V\right) \aleph_{\text{low}}\left(V^{-1} \otimes V^{-1}\right). \end{aligned} \tag{3.7}$$

Then, (3.6) becomes

$$\begin{aligned} \text{lvec}(\Delta V) = &\left(Q_{LM}\,\text{vec}(N) - Q_Y\,\text{vec}(M)\right. \\ &\left.+R_V\,\text{vec}\left(N\left(LJ_{m+n}L^{\text{T}}\right)N^{\text{T}} - M\left(YY^{\text{T}}\right)M^{\text{T}} - \Delta V J_{m+n}(\Delta V)^{\text{T}}\right)\right). \end{aligned} \tag{3.8}$$

Thus, applying the operator 'lvec †' to (3.8) yields

$$\begin{aligned} \Delta V = &\ \text{lvec}^{\dagger}\left(Q_{LM}\,\text{vec}(N) - Q_Y\,\text{vec}(M)\right. \\ &\left.+R_V\,\text{vec}\left(N\left(LJ_{m+n}L^{\text{T}}\right)N^{\text{T}} - M\left(YY^{\text{T}}\right)M^{\text{T}} - \Delta V J_{m+n}(\Delta V)^{\text{T}}\right)\right). \end{aligned} \tag{3.9}$$

We will derive the rigorous multiplicative perturbation bound for ΔV using the method of the Lyapunov majorant function and the Banach fixed point theorem based on the operator equation (3.10) as shown in [8, 9, 11, 14]. The equation (3.9) can be written as an operator equation for ΔV:

$$\begin{aligned} \Delta V &= \Phi(\Delta V, N, M) \\ &= \text{lvec}^{\dagger}\left(Q_{LM}\,\text{vec}(N) - Q_Y\,\text{vec}(M)\right. \\ &\quad\left.+R_V\,\text{vec}\left(N\left(LJ_{m+n}L^{\text{T}}\right)N^{\text{T}} - M\left(YY^{\text{T}}\right)M^{\text{T}} - \Delta V J_{m+n}(\Delta V)^{\text{T}}\right)\right). \end{aligned} \tag{3.10}$$

Suppose that $H \in \mathbb{R}^{(m+n)\times(m+n)}$ is lower triangular with positive diagonal elements and has the same structure as that of ΔV, $\|H\|_F \leqslant \eta$ for some $\eta \geqslant 0$, $\|N\|_F = \sigma_1$ and $\|M\|_F = \sigma_2$. Then it follows from the definition of the operator 'lvec†' and (2.1) that

$$\|\Phi(H,N,M)\|_F \leqslant \|Q_{LM}\|_2\,\sigma_1 + \|Q_Y\|_2\,\sigma_2 + \|R_V\|_2\left(\|L\|_2^2\sigma_1^2 + \|Y\|_2^2\sigma_2^2 + \eta^2\right). \tag{3.11}$$

Using (3.11), we get the Lyapunov majorant function of the operator equation (3.10)

$$q(\eta,\sigma_1,\sigma_2) = \|Q_{LM}\|_2\,\sigma_1 + \|Q_Y\|_2\,\sigma_2 + \|R_V\|_2\left(\|L\|_2^2\sigma_1^2 + \|Y\|_2^2\sigma_2^2 + \eta^2\right)$$

and the Lyapunov majorant equation

$$\begin{aligned} &q(\eta,\sigma_1,\sigma_2) = \eta, \quad \text{i.e.} \\ &\|Q_{LM}\|_2\,\sigma_1 + \|Q_Y\|_2\,\sigma_2 + \|R_V\|_2\left(\|L\|_2^2\sigma_1^2 + \|Y\|_2^2\sigma_2^2 + \eta^2\right) = \eta. \end{aligned} \tag{3.12}$$

Suppose that σ_1,

$$\begin{aligned} \sigma_2 \in \Omega = \Big\{\sigma_1,\sigma_2 \geqslant 0 : 1 - 4\,\|R_V\|_2\,\big(\,&\|Q_{LM}\|_2\,\sigma_1 + \|Q_Y\|_2\,\sigma_2 \\ &+ \|R_V\|_2\left(\|L\|_2^2\sigma_1^2 + \|Y\|_2^2\sigma_2^2\right)\big) \geqslant 0\Big\}. \end{aligned}$$

Then, the equation (3.12) has two nonnegative roots: $\eta_1(\sigma_1,\sigma_2) \leqslant \eta_2(\sigma_1,\sigma_2)$ with

$$\begin{aligned}\eta_1(\sigma_1,\sigma_2) &= f(\sigma_1,\sigma_2)\\ &= \frac{2\left(\|Q_{LM}\|_2\,\sigma_1+\|Q_Y\|_2\,\sigma_2+\|R_V\|_2\left(\|L\|_2^2\sigma_1^2+\|Y\|_2^2\sigma_2^2\right)\right)}{1+\sqrt{1-4\|R_V\|_2\left(\|Q_{LM}\|_2\,\sigma_1+\|Q_Y\|_2\,\sigma_2+\|R_V\|_2\left(\|L\|_2^2\sigma_1^2+\|Y\|_2^2\sigma_2^2\right)\right)}}.\end{aligned}$$

Let the set $A(\sigma_1,\sigma_2)$ be $A(\sigma_1,\sigma_2)=\{H\in\mathbb{R}^{(m+n)\times(m+n)}: H$ has the same structure as that of ΔV and $\|H\|_F\leqslant\eta\}$, which is closed and convex. Furthermore, we can simply verify that the operator $\Phi(\cdot,N,M)$ maps the set $A(\sigma_1,\sigma_2)$ into itself for $H,\tilde{H}\in A(\sigma_1,\sigma_2)$,

$$\|\Phi(H,N,M)-\Phi(\tilde{H},N,M)\|_F\leqslant q'_\eta\left(f(\sigma_1,\sigma_2),\sigma_1,\sigma_2\right)\|H-\tilde{H}\|_F.$$

Since the derivative of the function $q(\eta,\sigma_1,\sigma_2)$ relative to η at $f(\sigma_1,\sigma_2)$ satisfies

$$\begin{aligned}&q'_\eta\left(f(\sigma_1,\sigma_2),\sigma_1,\sigma_2\right)\\ &=1-\sqrt{1-4\|R_V\|_2\left(\|Q_{LM}\|_2\,\sigma_1+\|Q_Y\|_2\,\sigma_2+\|R_V\|_2\left(\|L\|_2^2\sigma_1^2+\|Y\|_2^2\sigma_2^2\right)\right)}<1,\end{aligned}$$

when $\sigma_1,\sigma_2\in\Omega_1=\{\sigma_1,\sigma_2\geqslant 0: 1-4\|R_V\|_2(\|Q_{LM}\|_2\sigma_1+\|Q_Y\|_2\sigma_2+\|R_V\|_2(\|L\|_2^2\sigma_1^2+\|Y\|_2^2\sigma_2^2))>0\}$. Then the operator $\Phi(\cdot,N,M)$ is contractive on the set $A(\sigma_1,\sigma_2)$ for $\sigma_1,\sigma_2\in\Omega_1$. Thus, from the Banach fixed point theorem, we have that the operator Equation (3.10), i.e. the matrix equation (3.2), has a unique solution in the set $A(\sigma_1,\sigma_2)$. As a result, $\|\Delta V\|_F\leqslant f(\sigma_1,\sigma_2)$ for $\sigma_1,\sigma_2\in\Omega_1$. We summarize these results in the following theorem.

THEOREM 3.1. *Given a lower triangular matrix $L\in\mathbb{R}^{(m+n)\times(m+n)}$ with positive diagonal elements and a matrix $Y\in\mathbb{R}^{(m+n)\times k}$ such that the generalized Cholesky factorization $VJ_{m+n}V^{\mathrm{T}}=LJ_{m+n}L^{\mathrm{T}}-YY^{\mathrm{T}}$ holds. Suppose $W=I_{m+n}+N\in\mathbb{R}^{(m+n)\times(m+n)}$ and $U=I_{m+n}+M\in\mathbb{R}^{(m+n)\times(m+n)}$. If*

$$\|R_V\|_2\left(\|Q_{LM}\|_2\|N\|_F+\|Q_Y\|_2\|M\|_F+\|R_V\|_2\left(\|L\|_2^2\|N\|_F^2+\|Y\|_2^2\|M\|_F^2\right)\right)<\frac{1}{4}, \tag{3.13}$$

then the following generalized Cholesky factorization holds:

$$\begin{aligned}&(V+\Delta V)J_{m+n}(V+\Delta V)^{\mathrm{T}}\\ &=LW(LW)^{\mathrm{T}}-YU(YU)^{\mathrm{T}},\\ &=(I_{m+n}+N)LJ_{m+n}L^{\mathrm{T}}(I_{m+n}+N)^{\mathrm{T}}-(I_{m+n}+M)YY^{\mathrm{T}}(I_{m+n}+M)^{\mathrm{T}},\end{aligned}$$

and

$$\begin{aligned}&\|\Delta V\|_F\\ &\leqslant\frac{2\left(\|Q_{LM}\|_2\|N\|_F+\|Q_Y\|_2\|M\|_F+\|R_V\|_2\left(\|L\|_2^2\|N\|_M^2+\|Y\|_2^2\|M\|_F^2\right)\right)}{1+\sqrt{1-4\|R_V\|_2\left(\|Q_{LM}\|_2\|N\|_F+\|Q_Y\|_2\|M\|_F+\|R_V\|_2\left(\|L\|_2^2\|N\|_F^2+\|Y\|_2^2\|M\|_F^2\right)\right)}} \qquad (3.14)\\ &\leqslant 2\left(\|Q_{LM}\|_2\|N\|_F+\|Q_Y\|_2\|M\|_F+\|R_V\|_2\left(\|L\|_2^2\|N\|_F^2+\|Y\|_2^2\|M\|_F^2\right)\right). \qquad (3.15)\end{aligned}$$

Proof. It is absolutely not difficult to check that the condition (3.13) is the same as the one in Ω_1. So, (3.14) and thus (3.15) hold. □

REMARK 3.2. The resulting first order multiplicative perturbation bound can be obtained from (3.15) by neglecting high-order terms

$$\begin{aligned}\|\Delta V\|_F &\leqslant \|Q_{LM}\|_2\|N\|_F + \|Q_Y\|_2\|M\|_F \\ &= \|Q_{LM}\|_2\|W - I_{m+n}\|_F + \|Q_Y\|_2\|U - I_{m+n}\|_F. \end{aligned} \tag{3.16}$$

We used $W = I_{m+n} + N$ as a general matrix in the previous analysis. Next, we choose $W = I_{m+n} + N$ as a lower triangular matrix. Thus, from (3.10), (2.2) and (2.3), it follows that

$$\begin{aligned}\mathrm{lvec}(\Delta V) = &\big(Q_{LT}\,\mathrm{lvec}\,(N) - Q_Y\,\mathrm{vec}(\Delta Y) \\ &+ R_V\,\mathrm{vec}\big(N(LJ_{m+n}L^{\mathrm{T}})N^{\mathrm{T}} - M(YY^{\mathrm{T}})M^{\mathrm{T}} - \Delta V J_{m+n}(\Delta V)^{\mathrm{T}}\big)\big). \end{aligned} \tag{3.17}$$

where

$$\begin{aligned}Q_{LT} = \aleph_{\mathrm{lvec}}\big(J_{m+n}^{-\mathrm{T}} \otimes V\big)\,\aleph_{\mathrm{low}}\big(\big(V^{-1}(LJ_{m+n}L^{\mathrm{T}}) \otimes V^{-1}\big) \\ + \big(V^{-1} \otimes V^{-1}(LJ_{m+n}L^{\mathrm{T}})\big)\,\Pi\big)\,\aleph_{\mathrm{lvec}}^{\mathrm{T}}. \end{aligned} \tag{3.18}$$

Based on the results of Theorem 3.1, we have the following theorem.

THEOREM 3.3. *Given a lower triangular matrix $L \in \mathbb{R}^{(m+n)\times(m+n)}$ with positive diagonal elements and a matrix $Y \in \mathbb{R}^{(m+n)\times k}$ such that the generalized Cholesky factorization $VJ_{m+n}V^{\mathrm{T}} = LJ_{m+n}L^{\mathrm{T}} - YY^{\mathrm{T}}$ holds. Suppose $W = I_{m+n} + N \in \mathbb{R}^{(m+n)\times(m+n)}$ and $U = I_{m+n} + M \in \mathbb{R}^{(m+n)\times(m+n)}$. If*

$$\|R_V\|_2\big(\|Q_{LT}\|_2\|N\|_F + \|Q_Y\|_2\|M\|_F + \|R_V\|_2\big(\|L\|_2^2\|N\|_F^2 + \|Y\|_2^2\|M\|_F^2\big)\big) < \frac{1}{4}, \tag{3.19}$$

then the following generalized Cholesky factorization holds:

$$\begin{aligned}&(V + \Delta V)J_{m+n}(V + \Delta V)^{\mathrm{T}} \\ &= LW(LW)^{\mathrm{T}} - YU(YU)^{\mathrm{T}}, \\ &= (I_{m+n} + N)LJ_{m+n}L^{\mathrm{T}}(I_{m+n} + N)^{\mathrm{T}} - (I_{m+n} + M)YY^{\mathrm{T}}(I_{m+n} + M)^{\mathrm{T}}, \end{aligned}$$

and

$$\|\Delta V\|_F \leqslant \frac{2\big(\|Q_{LT}\|_2\|N\|_F + \|Q_Y\|_2\|\Delta Y\|_F + \|R_V\|_2\big(\|L\|_2^2\|N\|_F^2 + \|Y\|_2^2\|M\|_F^2\big)\big)}{1 + \sqrt{1 - 4\|R_V\|_2\big(\|Q_{LT}\|_2\|N\|_F + \|Q_Y\|_2\|M\|_F + \|R_V\|_2\big(\|L\|_2^2\|N\|_F^2 + \|Y\|_2^2\|M\|_F^2\big)\big)}}, \tag{3.20}$$

$$\leqslant 2\big(\|Q_{LT}\|_2\|N\|_F + \|Q_Y\|_2\|M\|_F + \|R_V\|_2\big(\|L\|_2^2\|N\|_F^2 + \|Y\|_2^2\|M\|_F^2\big)\big). \tag{3.21}$$

REMARK 3.4. By ignoring the high-order terms, we can investigate a first-order multiplicative perturbation bound from (3.21):

$$\begin{aligned}\|\Delta V\|_F &\leqslant \|Q_{LT}\|_2\,\|N\|_F + \|Q_Y\|_2\,\|M\|_F \\ &= \|Q_{LT}\|_2\,\|W - I_{m+n}\|_F + \|Q_Y\|_2\,\|U - I_{m+n}\|_F\,. \end{aligned} \tag{3.22}$$

4. Weak rigorous multiplicative perturbation bounds

To obtain the explicit expression of weak rigorous multiplicative perturbation bounds for the GCBD problem, we will adopt the matrix-equation approach originated by Chang [21] in this section.

THEOREM 4.1. *Given a lower triangular matrix $L \in \mathbb{R}^{(m+n)\times(m+n)}$ with positive diagonal elements and a matrix $Y \in \mathbb{R}^{(m+n)\times k}$ such that the generalized Cholesky factorization $VJ_{m+n}V^{\mathrm{T}} = LJ_{m+n}L^{\mathrm{T}} - YY^{\mathrm{T}}$ holds. Suppose $S = \left(S_m^T, S_n^T\right)^T \in \mathbb{R}^{(m+n)\times k}$ and $H \in \mathbb{R}^{(m+n)\times(m\times n)}$ be lower triangular of the following form*

$$H = \begin{bmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{bmatrix},$$

where $H_{11} \in \mathbb{R}^{m\times m}$ and $H_{22} \in \mathbb{R}^{n\times n}$ are lower triangular, $H_{21} \in \mathbb{R}^{n\times m}$. Define $N = \varepsilon H$ and $M = \varepsilon S$ for some $\varepsilon \geqslant 0$. If

$$\|N\|_2 < 1, \quad \frac{\left\|L^{-1}\right\|_2 (\|Y\|_2 + \|YM\|_2)}{1 - \|N\|_2} < 1, \tag{4.1}$$

then $(L(I_{m+n}+N))J_{m+n}(L(I_{m+n}+N))^{\mathrm{T}} - (Y(I_{m+n}+M))(Y(I_{m+n}+M))^{\mathrm{T}}$ has the generalized Cholesky factorization

$$\begin{aligned}&(V+\Delta V)J_{m+n}(V+\Delta V)^{\mathrm{T}} \\ &= (I_{m+n}+N)LJ_{m+n}L^{\mathrm{T}}(I_{m+n}+N)^{\mathrm{T}} - (I_{m+n}+M)YY^{\mathrm{T}}(I_{m+n}+M)^{\mathrm{T}}.\end{aligned} \tag{4.2}$$

Proof. For any $|t| \leqslant \varepsilon$ from the first condition of (4.1), it follows that

$$\|tH\|_2 \leqslant \|\varepsilon H\|_2 = \|N\|_2 < 1. \tag{4.3}$$

Then $L(I_{m+n}+tH)$ is nonsingular. Thus, we have

$$\begin{aligned}(L(I_{m+n}+tH))J_{m+n}(L(I_{m+n}+tH))^{\mathrm{T}} - (Y(I_{m+n}+tS))(Y(I_{m+n}+tS))^{\mathrm{T}} \\ = (L(I_{m+n}+tH))\left(J_{m+n} - ZZ^T\right)(L(I_{m+n}+tH))^{\mathrm{T}},\end{aligned}$$

where $Z = (L(I_{m+n}+tH))^{-1}(Y(I_{m+n}+tS)) = (I_{m+n}+tH)^{-1}L^{-1}(Y(I_{m+n}+tS))$. From (4.3) and noting (2.1), we get

$$\left\|(I_{m+n}+tH)^{-1}\right\|_2 \leqslant \frac{1}{1-\|tH\|_2}.$$

Furthermore,

$$\|Z\|_2 \leqslant \frac{\left\|L^{-1}\right\|_2 (\|Y\|_2 + \|tSY\|_2)}{1 - \|tH\|_2} \leqslant \mu < 1. \tag{4.4}$$

Using Zeyl's theorem on eigenvalues of Nermitian matrices (e.g., [25], pp. 181), for the i-th eigenvalue of $J_{m+n} - ZZ^T$,

$$\lambda_i(J_{m+n}) + \lambda_{\min}\left(-ZZ^T\right) \leqslant \lambda_i\left(J_{m+n} - ZZ^T\right) \leqslant \lambda_i(J_{m+n}) + \lambda_{\max}\left(-ZZ^T\right).$$

Note that (4.4) implies $\lambda_{\min}\left(-ZZ^T\right) \geqslant -\mu^2$. Assume the eigenvalues of a matrix be ordered in non-decreasing order. Thus, for $0 < i \leqslant n$,

$$-1 - \mu^2 \leqslant \lambda_i\left(J_{m+n} - ZZ^T\right) \leqslant -1,$$

and for $n < i \leqslant m+n$,

$$1 - \mu^2 \leqslant \lambda_i\left(J_{m+n} - ZZ^T\right) \leqslant 1.$$

That is, $J_{m+n} - ZZ^T$ is nonsingular and has m positive eigenvalues and n negative eigenvalues, so is $(L(I_{m+n}+tH))J_{m+n}(L(I_{m+n}+tH))^{\mathrm{T}} - (Y(I_{m+n}+tS))(Y(I_{m+n}+tS))^{\mathrm{T}}$. In addition, we obtain

$$(L(I_{m+n}+tH))J_{m+n}(L(I_{m+n}+tH))^{\mathrm{T}} - (Y(I_{m+n}+tS))(Y(I_{m+n}+tS))^{\mathrm{T}} = \begin{bmatrix} G_{11} & G_{21}^T \\ G_{21} & -G_{22} \end{bmatrix},$$

where

$$\begin{aligned} G_{11} &= (L_{11}(I_m + tH_{11}))(L_{11}(I_m + tH_{11}))^{\mathrm{T}} - (Y_m(I_m + tS_m))(Y_m(I_m + tS_m))^{\mathrm{T}} \\ G_{22} &= (L_{22}(I_n + tH_{22}))(L_{22}(I_n + tH_{22}))^{\mathrm{T}} - (L_{21}(I_n + tH_{21}))(L_{21}(I_n + tH_{21}))^{\mathrm{T}} \\ &\quad + (Y_n(I_n + tS_n))(Y_n(I_n + tS_n))^{\mathrm{T}}. \end{aligned}$$

Note that

$$tH = \begin{bmatrix} tH_{11} & 0 \\ * & * \end{bmatrix}. \tag{4.5}$$

Then $\|tH_{11}\|_2 \leqslant \|tH\|_2 < 1$, which implies that $(L_{11}(I_m + tH_{11}))$ is nonsingular. Thus, G_{11} can be rewritten as

$$G_{11} = (L_{11}(I_m + tH_{11}))\left(I_m - Z_m Z_m^T\right)(L_{11}(I_m + tH_{11}))^{\mathrm{T}},$$

where $Z_m = (L_{11}(I_m + tH_{11}))^{-1}(Y_m(I_m + tS_m))$. Similar to (4.5), we can note that Z_m is the principal submatrix with order m of Z. Then, using (4.4), $\|Z_m\|_2 \leqslant \|Z\|_2 \leqslant \mu < 1$. Accordingly, $I_m - Z_m Z_m^T$ is symmetric positive definite, so is G_{11}. Furthermore, G_{22} is clearly symmetric positive semi-definite. Thus, for each $|t| \leqslant \varepsilon$, $(L(I_{m+n} + tH))J_{m+n}$ $(L(I_{m+n} + tH))^{\mathrm{T}} - (Y(I_{m+n} + tS))(Y(I_{m+n} + tS))^{\mathrm{T}}$ has the generalized Cholesky factorization

$$\begin{aligned} V(t)J_{m+n}V^T(t) &= (L(I_{m+n} + tH))J_{m+n}(L(I_{m+n} + tH))^{\mathrm{T}} \\ &\quad - (Y(I_{m+n} + tS))(Y(I_{m+n} + tS))^{\mathrm{T}}. \end{aligned}$$

Note that $V(0) = V$ and $V(\varepsilon) = V + \Delta V$. Then (4.2) holds. □

THEOREM 4.2. *Given a lower triangular matrix $L \in \mathbb{R}^{(m+n)\times(m+n)}$ with positive diagonal elements and a matrix $Y \in \mathbb{R}^{(m+n)\times k}$ such that the generalized Cholesky factorization $VJ_{m+n}V^{\mathrm{T}} = LJ_{m+n}L^{\mathrm{T}} - YY^{\mathrm{T}}$ holds. Suppose $W = I_{m+n} + N \in \mathbb{R}^{(m+n)\times(m+n)}$ and $U = I_{m+n} + M \in \mathbb{R}^{(m+n)\times(m+n)}$. If*

$$\|N\|_2 < 1, \quad \frac{\left\|L^{-1}\right\|_2 (\|Y\|_2 + \|YM\|_2)}{1-\|N\|_2} < 1, \tag{4.6}$$

and

$$\Big(\left\|\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_2 \left\|V^{-1}N\right\|_F + \left\|\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_2 \left\|V^{-1}M\right\|_F + \left\|V^{-1}NL\right\|_F^2 + \left\|V^{-1}MY\right\|_F^2 \Big) < \frac{1}{2}, \tag{4.7}$$

then the generalized Cholesky factorization (4.2) always exists and

$$\begin{aligned}\|\Delta V\|_F \leqslant &(2+\sqrt{2}) \inf_{D_{m+n}\in\mathbb{D}_{m+n}} \left(\sqrt{1+\zeta_D^2}\kappa\left(VD_{m+n}^{-1}\right)\right) \left\|V^{-1}\right\|_2 \left(\|L\|_2^2\|N\|_F + \|Y\|_2^2\|M\|_F\right)\\ &+ (2+\sqrt{2}) \inf_{D_{m+n}\in\mathbb{D}_{m+n}} \left(\kappa\left(VD_{m+n}^{-1}\right)\right) \left\|V^{-1}\right\|_2 \left(\|L\|_2^2\|N\|_F^2 + \|Y\|_2^2\|M\|_F^2\right).\end{aligned} \tag{4.8}$$

Proof. Let $W(t) = I_{m+n} + tN$ and $U(t) = I_{m+n} + tM$, for any $|t| \leqslant \varepsilon$. Using (1.3) leads to

$$\begin{aligned}&(L(I_{m+n}+tN))J_{m+n}(L(I_{m+n}+tN))^{\mathrm{T}} - (Y(I_{m+n}+tM))(Y(I_{m+n}+tM))^{\mathrm{T}}\\ &= (L(I_{m+n}+tN))\left(J_{m+n} - (L(I_{m+n}+tN))^{-1}\left((Y(I_{m+n}+tM))(Y(I_{m+n}+tM))^{\mathrm{T}}\right)\right.\\ &\quad \left.\times (L(I_{m+n}+tN))^{-\mathrm{T}}\right)(L(I_{m+n}+tN))^{\mathrm{T}},\end{aligned}$$

and

$$\left\|(L(I_{m+n}+tN))^{-1}(Y(I_{m+n}+tM))\right\|_2 \leqslant \frac{\left\|L^{-1}\right\|_2 (\|Y\|_2 + \|YM\|_2)}{1-\|N\|_2},$$

and considering the results of Theorem 4.1, it follows that (4.6) holds,

$$(L(I_{m+n}+tN))J_{m+n}(L(I_{m+n}+tN))^{\mathrm{T}} - (Y(I_{m+n}+tM))(Y(I_{m+n}+tM))^{\mathrm{T}}$$

is positive definite and has the unique generalized Cholesky factorization, i.e.

$$\begin{aligned}&(V+\Delta V(t))J_{m+n}(V+\Delta V(t))^{\mathrm{T}}\\ &= (L(I_{m+n}+tN))J_{m+n}(L(I_{m+n}+tN))^{\mathrm{T}} - (Y(I_{m+n}+tM))(Y(I_{m+n}+tM))^{\mathrm{T}},\end{aligned} \tag{4.9}$$

which, with $\Delta V(0) = 0$ and $\Delta V(\varepsilon) = \Delta V$, implies (4.2). Using (4.9) and (1.3) leads to

$$\begin{aligned}&\left(tN\left(LJ_{m+n}L^{\mathrm{T}}\right) + t\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}} + t^2N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}\right) - \left(tM\left(YY^{\mathrm{T}}\right)\right.\\ &\quad \left. + tM\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}} + t^2M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}\right)\\ &= VJ_{m+n}(\Delta V(t))^{\mathrm{T}} + \Delta V(t)J_{m+n}V^{\mathrm{T}} + \Delta V(t)J_{m+n}(\Delta V(t))^{\mathrm{T}}.\end{aligned}$$

Pre-multiplying the above equation by V^{-1} and post-multiplying by $V^{-\mathrm{T}}$ lead to

$$\begin{aligned}
&J_{m+n}(\Delta V(t))^{\mathrm{T}}V^{-\mathrm{T}}+V^{-1}\Delta V(t)J_{m+n}\\
&=tV^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}+tV^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}+t^2V^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\\
&\quad-\left(tV^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}}+tV^{-1}\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}+t^2V^{-1}M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right)\\
&\quad-V^{-1}\Delta V(t)J_{m+n}(\Delta V(t))^{\mathrm{T}}V^{-\mathrm{T}}.
\end{aligned}\tag{4.10}$$

Since $V^{-1}\Delta V(t)$ is a lower triangular matrix. Using the symbol 'low' and noting (4.10), we get

$$\begin{aligned}
V^{-1}\Delta V(t)J_{m+n}=\;&\mathrm{low}\left(tV^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}+tV^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right)\\
&+\mathrm{low}\left(t^2V^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right)\\
&-\mathrm{low}\left(tV^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}}+tV^{-1}\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right)\\
&-\mathrm{low}\left(t^2V^{-1}M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right)\\
&-\mathrm{low}\left(V^{-1}\Delta V(t)J_{m+n}(\Delta V(t))^{\mathrm{T}}V^{-\mathrm{T}}\right).
\end{aligned}\tag{4.11}$$

Applying the Frobenius norm to (4.11) and considering (2.1), (2.7) and (2.8) yields

$$\begin{aligned}
&\left\|V^{-1}\Delta V(t)J_{m+n}\right\|_F\\
\leqslant&\sqrt{2}\left(\left\|tV^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_F+\left\|tV^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_F\right)\\
&+\frac{1}{\sqrt{2}}\left\|\left(V^{-1}\Delta V(t)J_{m+n}(\Delta V(t))^{\mathrm{T}}V^{-\mathrm{T}}\right)\right\|_F\\
&+\frac{1}{\sqrt{2}}\left(\left\|t^2V^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right\|_F+\left\|t^2V^{-1}M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right\|_F\right)\\
\leqslant&\sqrt{2}\left(\left\|tV^{-1}N\right\|_F\left\|\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_2+\left\|tV^{-1}M\right\|_2\left\|\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_F\right)\\
&+\frac{1}{\sqrt{2}}\left(\left\|V^{-1}\Delta V(t)\right\|_F^2+\left\|tV^{-1}NL\right\|_F^2+\left\|tV^{-1}MY\right\|_F^2\right).
\end{aligned}\tag{4.12}$$

Assume $p(t)=\left\|V^{-1}\Delta V(t)J_{m+n}\right\|_F$ and

$$\begin{aligned}
q(t)=\;&2\left(\left\|tV^{-1}N\right\|_F\left\|\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_2+\left\|tV^{-1}M\right\|_2\left\|\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_F\right)\\
&+\left(\left\|tV^{-1}NL\right\|_F^2+\left\|tV^{-1}MY\right\|_F^2\right).
\end{aligned}$$

Then, the above equation can be rewritten as

$$p(t)^2-\sqrt{2}p(t)+q(t)\geqslant 0.$$

Equation (4.7), clearly shows that $q(t) < 1/2$, for $|t| \leqslant \varepsilon$. Therefore, $p(t) \leqslant p_1(t)$ or $p(t) \geqslant p_2(t)$, where

$$p_1(t) = \frac{1}{\sqrt{2}}(1 - \sqrt{1-2q(t)}) < p_2(t) = \frac{1}{\sqrt{2}}(1 + \sqrt{1-2q(t)}). \tag{4.13}$$

Note that $p(t)$ is continuous and $p(0) = 0 = p_1(0) < p_2(0) = \sqrt{2}$. Then $p(t) \leqslant p_1(t)$, for any $|t| \leqslant \varepsilon$. Consequently, $p(\varepsilon) \leqslant p_1(\varepsilon)$, i.e.

$$\left\|V^{-1}\Delta V(t)J_{m+n}\right\|_F \leqslant \frac{1}{\sqrt{2}}(1 - \sqrt{1-2q(t)}) < \frac{1}{\sqrt{2}}. \tag{4.14}$$

Putting $t = \varepsilon$ in (4.11)

$$\begin{aligned} V^{-1}\Delta V J_{m+n} = &\operatorname{low}\left(V^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}} + V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right)\\ &+\operatorname{low}\left(V^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right) - \operatorname{low}\left(V^{-1}M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right)\\ &-\operatorname{low}\left(V^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}} + V^{-1}\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right)\\ &-\operatorname{low}\left(V^{-1}\Delta V J_{m+n}(\Delta V)^{\mathrm{T}}V^{-\mathrm{T}}\right). \end{aligned} \tag{4.15}$$

Right multiplying by $D_{m+n} \in \mathbb{D}_{m+n}$, and noting $V = D_{m+n}\check{V}$ and (2.4) yield

$$\begin{aligned} \check{V}^{-1}\Delta V J_{m+n} = &\operatorname{low}\left(\check{V}^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}} + D_{m+n}(\check{V}^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}})^{\mathrm{T}}D_{m+n}^{-1}\right)\\ &+\operatorname{low}\left(\check{V}^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right) - \operatorname{low}\left(\check{V}^{-1}M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right)\\ &-\operatorname{low}\left(\check{V}^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}} + D_{m+n}(\check{V}^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}})^{\mathrm{T}}D_{m+n}^{-1}\right)\\ &-\operatorname{low}\left(\check{V}^{-1}\Delta V J_{m+n}(\Delta V)^{\mathrm{T}}V^{-\mathrm{T}}\right). \end{aligned} \tag{4.16}$$

Implementing the Frobenius norm on (4.16) and utilizing (2.5), (2.6), and (2.1) lead to

$$\begin{aligned} \left\|\check{V}^{-1}\Delta V J_{m+n}\right\|_F \leqslant &\sqrt{1+\zeta_{D_{m+n}}^2}\Big(\left\|\check{V}^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_F\\ &+\left\|\check{V}^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}}\right\|_F\Big)\\ &+\|\check{V}^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\|_F + \left\|\check{V}^{-1}M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right\|_F\\ &+\left\|\check{V}^{-1}\Delta V J_{m+n}(\Delta V)^{\mathrm{T}}V^{-\mathrm{T}}\right\|_F\\ \leqslant &\sqrt{1+\zeta_{D_{m+n}}^2}\left\|\check{V}^{-1}\right\|_2\left\|V^{-1}\right\|_2\left(\|L\|_2^2\|N\|_F + \|Y\|_2^2\|M\|_F\right)\\ &+\left\|\check{V}^{-1}\Delta V\right\|_2\left\|V^{-1}\Delta V\right\|_F\\ &+\left\|\check{V}^{-1}\right\|_2\left\|V^{-1}\right\|_2\left(\|L\|_2^2\|N\|_F^2 + \|Y\|_2^2\|M\|_F^2\right). \end{aligned} \tag{4.17}$$

Considering (4.14), (2.1), (4.6) and (4.7), we obtain

$$\begin{aligned}\left\|\check{V}^{-1}\Delta V J_{m+n}\right\|_F \leqslant &(2+\sqrt{2})\sqrt{1+\zeta_D^2}\left\|\check{V}^{-1}\right\|_2\left\|V^{-1}\right\|_2\left(\|L\|_2^2\|N\|_F+\|Y\|_2^2\|M\|_F\right)\\ &+(2+\sqrt{2})\left\|\check{V}^{-1}\right\|_2\left\|V^{-1}\right\|_2\left(\|L\|_2^2\|N\|_F^2+\|Y\|_2^2\|M\|_F^2\right), \quad (4.18)\end{aligned}$$

which, along with the fact that $\|J_{m+n}\|_2=1$ and the supporting proof

$$\|\Delta V\|_F \leqslant \left\|\check{V}\check{V}^{-1}\Delta V J_{m+n}\right\|_F \leqslant \|\check{V}\|_2\left\|\check{V}^{-1}\Delta V\right\|_F\|J_{m+n}\|_2, \quad \text{by (2.1)} \tag{4.19}$$

shows the bound (4.8). □

To check the efficiency of bounds, let us prove that the bound (3.15) is significantly sharper than (4.8). Taking into consideration [26, Corollary 3.4], for any $D_{m+n}\in\mathbb{D}_{m+n}$ and $X\in\mathbb{R}^{(m+n)\times(m+n)}$, using (2.11) and (2.12), we obtain

$$\begin{aligned}\|Q_{LM}\|_2 =&\Big\|\aleph_{\text{lvec}}\left(J_{m+n}^{-\mathrm{T}}\otimes V\right)\left(I_{m+n}\otimes D_{m+n}^{-1}\right)\left(I_{m+n}\otimes D_{m+n}\right)\\ &\times\aleph_{\text{low}}\Big(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes V^{-1}\right)+\left(V^{-1}\otimes V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\Big)\Big\|_2\\ =&\Big\|\aleph_{\text{lvec}}\left(J_{m+n}\otimes VD_{m+n}^{-1}\right)\aleph_{\text{low}}\Big(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\right)\\ &+\left(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\Big)\Big\|_2 \quad \text{by (2.12)}\\ \leqslant&\left\|VD_{m+n}^{-1}\right\|_2\Big\|\aleph_{\text{low}}\Big(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\right)\\ &+\left(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\Big)\Big\|_2 \quad \text{by (2.11)}\\ =&\left\|VD_{m+n}^{-1}\right\|_2\max_{\|\operatorname{vec}(X)\|_2=1}\Big\|\aleph_{\text{low}}\Big(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\right)\\ &+\left(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\Big)\operatorname{vec}(X)\Big\|_2. \quad (4.20)\end{aligned}$$

Taking into account (2.9), (2.2), (2.4), (2.5) and (2.1) yields

$$\begin{aligned}&\max_{\|\operatorname{vec}(X)\|_2=1}\Big\|\aleph_{\text{low}}\Big(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\right)\\ &+\left(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\Big)\operatorname{vec}(X)\Big\|_2\\ =&\max_{\|\operatorname{vec}(X)\|_2=1}\Big\|\aleph_{\text{low}}\operatorname{vec}\Big(D_{m+n}V^{-1}X(LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}\\ &+D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})X^{\mathrm{T}}V^{-\mathrm{T}}\Big)\Big\|_2 \quad \text{by (2.9)}\end{aligned}$$

$$
\begin{aligned}
&= \max_{\|\mathrm{vec}(X)\|_2=1} \Big\| \mathrm{vec}\left(\mathrm{low}\left(D_{m+n}V^{-1}X(LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}\right.\right. \\
&\quad \left.\left. + D_{m+n}\left(D_{m+n}V^{-1}X(LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}\right)^{\mathrm{T}} D_{m+n}^{-1}\right)\right)\Big\|_2 \quad \text{by (2.2)} \\
&= \max_{\|X\|_F=1} \Big\| \mathrm{low}\left(D_{m+n}V^{-1}X(LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}\right) \\
&\quad + D_{m+n}\mathrm{low}\left(D_{m+n}V^{-1}X(LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}\right)^{\mathrm{T}} D_{m+n}^{-1}\Big\|_F \quad \text{by (2.4)} \\
&\leqslant \max_{\|X\|_F=1} \sqrt{1+\zeta_{D_{m+n}}^2}\, \|D_{m+n}V^{-1}X(LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}\|_F \quad \text{by (2.5)} \\
&\leqslant \sqrt{1+\zeta_{D_{m+n}}^2}\, \|D_{m+n}V^{-1}\|_2 \left\|(LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}\right\|_2. \quad \text{by (2.1)}
\end{aligned} \tag{4.21}
$$

Hence, putting (4.21) into (4.20) gives

$$
\|Q_{LM}\|_2 \leqslant \left(\inf_{D_{m+n}\in\mathbb{D}_{m+n}} \sqrt{1+\zeta_{D_{m+n}}^2}\,\kappa\left(VD_{m+n}^{-1}\right)\right) \|L\|_2^2 \|\|V^{-1}\|_2. \tag{4.22}
$$

Therefore, we can demonstrate that,

$$
\|Q_Y\|_2 \leqslant \left(\inf_{D_{m+n}\in\mathbb{D}_{m+n}} \sqrt{1+\zeta_{D_{m+n}}^2}\,\kappa\left(VD_{m+n}^{-1}\right)\right) \|Y\|_2^2 \|\|V^{-1}\|_2, \tag{4.23}
$$

$$
\|R_V\|_2 \leqslant \left(\inf_{D_{m+n}\in\mathbb{D}_{m+n}} \kappa\left(VD_{m+n}^{-1}\right)\right) \left\|V^{-1}\right\|_2 ., \tag{4.24}
$$

(4.24) together with the fact $\kappa\left(VD_{m+n}^{-1}\right) \geqslant 1$ and (4.22) indicates that the bound (3.15) is absolutely tighter than (4.8).

REMARK 4.3. By ignoring the high-order terms, we can obtain a first-order multiplicative bound from (4.8):

$$
\|\Delta V\|_F \leqslant \inf_{D_{m+n}\in\mathbb{D}_{m+n}} \left(\sqrt{1+\zeta_{D_{m+n}}^2}\,\kappa\left(VD_{m+n}^{-1}\right)\right) \left\|V^{-1}\right\|_2 \left(\|L\|_2^2\|N\|_F + \|Y\|_2^2\|M\|_F\right). \tag{4.25}
$$

Obviously, from (4.22) and (4.23), we can check that the bound in Remark 3.2 is always tighter than the (4.25).

THEOREM 4.4. *With the same assumptions as in Theorem 4.1, we have*

$$
\begin{aligned}
\|\Delta V\|_F \leqslant& (2+\sqrt{2}) \inf_{D_{m+n}\in\mathbb{D}_{m+n}} \left(\sqrt{1+\zeta_{D_{m+n}}^2}\,\kappa\left(VD_{m+n}^{-1}\right)\right) \\
&\times \left(\left\|\mathrm{sut}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2 \|N\|_F + \left\|V^{-1}\right\|_2 \|Y\|_2^2\|M\|_F \right) \\
&+ (2+\sqrt{2}) \inf_{D_{m+n}\in\mathbb{D}_{m+n}} \left(\kappa\left(VD_{m+n}^{-1}\right)\right) \\
&\times \left(\left\|\mathrm{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2 \|N\|_F + \left\|V^{-1}\right\|_2 \left(\|L\|_2^2\|N\|_F^2 + \|Y\|_2^2\|M\|_F^2\right)\right).
\end{aligned} \tag{4.26}
$$

Proof. Applying the symbols 'sut', 'diag' and 'slt', $\operatorname{low}\left(V^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}+V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right)$ can be rewritten as

$$\begin{aligned}&\operatorname{low}\left(V^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}+V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right) &(4.27)\\ =&V^{-1}N\operatorname{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)+\operatorname{low}\Big(V^{-1}N\operatorname{sut}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\\ &+\operatorname{slt}\left(V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)\right)N^{\mathrm{T}}V^{-\mathrm{T}}\Big). &(4.28)\end{aligned}$$

Substituting (4.27) into (4.15) and then pre-multiplying it by $D_{m+n}\in\mathbb{D}_{m+n}$ and using $V=D_{m+n}\check{V}$, we have

$$\begin{aligned}&\check{V}^{-1}\Delta VJ_{m+n}\\ =&V^{-1}N\operatorname{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\\ &+\operatorname{low}\left(V^{-1}N\operatorname{sut}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)+D_{m+n}\left(\operatorname{slt}\left(V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right)D_{m+n}^{-1}\right)\\ &+\operatorname{low}\left(\check{V}^{-1}N\left(LJ_{m+n}L^{\mathrm{T}}\right)N^{\mathrm{T}}V^{-\mathrm{T}}\right)-\operatorname{low}\left(\check{V}^{-1}M\left(YY^{\mathrm{T}}\right)M^{\mathrm{T}}V^{-\mathrm{T}}\right)\\ &-\operatorname{low}\left(\check{V}^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}}+D_{m+n}(\check{V}^{-1}M\left(YY^{\mathrm{T}}\right)V^{-\mathrm{T}})^{\mathrm{T}}D_{m+n}^{-1}\right)\\ &-\operatorname{low}\left(\check{V}^{-1}\Delta VJ_{m+n}(\Delta V)^{\mathrm{T}}V^{-\mathrm{T}}\right). \qquad (4.29)\end{aligned}$$

Hence, we get the bound (4.26) by applying the Frobenius norm to (4.29) and utilizing (2.5),(2.6), (4.14), and (4.19).

Moving forward, we will show that the bound (3.21) is tighter than (4.26). By using (3.18), we get

$$\begin{aligned}\|Q_{LT}\|_2\leqslant&\Big\|\left(J_{m+n}^{-\mathrm{T}}\otimes V\right)\left(I_{m+n}\otimes D_{m+n}^{-1}\right)\left(I_{m+n}\otimes D_{m+n}\right)\aleph_{\mathrm{low}}\Big(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes V^{-1}\right)\\ &+\left(V^{-1}\otimes V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\Big)\aleph_{\mathrm{lvec}}^{\mathrm{T}}\Big\|\\ =&\Big\|\left(J_{m+n}^{-\mathrm{T}}\otimes VD_{m+n}^{-1}\right)\aleph_{\mathrm{low}}\Big(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\right)\\ &+\left(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\Big)\aleph_{\mathrm{lvec}}^{\mathrm{T}}\Big\| \quad \text{by (2.12)}\\ \leqslant&\left\|VD_{m+n}^{-1}\right\|_2\max_{\|\mathrm{lvec}(X)\|_2=1}\Big\|\aleph_{\mathrm{low}}\Big(\left(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\right)\\ &+\left(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\right)\Pi\Big)\aleph_{\mathrm{lvec}}^{\mathrm{T}}\mathrm{lvec}(X)\Big\|_2. \qquad (4.30)\end{aligned}$$

We assume X is a lower triangular matrix. Obviously, we only take the lower triangular part of X. As a result of using (2.2), (2.3), and (2.9), we get

$$\begin{aligned}
&\max_{\|\text{lvec}(X)\|_2=1}\Big\|\aleph_{\text{low}}\Big(\big(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\big) \\
&\qquad + \big(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\big)\Pi\Big)\aleph_{\text{lvec}}^{\mathrm{T}}\aleph_{\text{lvec}}\operatorname{vec}(X)\Big\|_2 \\
&= \max_{\|\text{lvec}(X)\|_2=1}\Big\|\aleph_{\text{low}}\Big(\big(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\big) \\
&\qquad + \big(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\big)\Pi\Big)\aleph_{\text{lt}}\operatorname{vec}(X)\Big\|_2 \quad \text{by (2.3)} \\
&= \max_{\|\text{lvec}(X)\|_2=1}\Big\|\aleph_{\text{low}}\Big(\big(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\big) \\
&\qquad + \big(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\big)\Pi\Big)\operatorname{vec}(\text{lt}(X))\Big\|_2 \quad \text{by (2.2)} \\
&= \max_{\|\text{lvec}(X)\|_2=1}\Big\|\aleph_{\text{low}}\Big(\big(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\otimes D_{m+n}V^{-1}\big) \\
&\qquad + \big(V^{-1}\otimes D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})\big)\Pi\Big)\operatorname{vec}(X)\Big\|_2 \\
&= \max_{\|\text{lvec}(X)\|_2=1}\Big\|\aleph_{\text{low}}\operatorname{vec}\Big(D_{m+n}V^{-1}X\big(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\big)^{\mathrm{T}} \\
&\qquad + D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})X^{\mathrm{T}}V^{-\mathrm{T}}\Big)\Big\|_2 \quad \text{by (2.9)} \\
&= \max_{\|\text{lvec}(X)\|_2=1}\Big\|\operatorname{vec}\Big(\text{low}\Big(D_{m+n}V^{-1}X\big(V^{-1}(LJ_{m+n}L^{\mathrm{T}})\big)^{\mathrm{T}} \\
&\qquad + D_{m+n}V^{-1}(LJ_{m+n}L^{\mathrm{T}})X^{\mathrm{T}}V^{-\mathrm{T}}\Big)\Big)\Big\|_2 \quad \text{by (2.2)} \\
&= \max_{\|\text{lvec}(X)\|_2=1}\Big\|\operatorname{vec}\Big(\text{low}\Big(D_{m+n}V^{-1}X\Big(\text{sut}\big(V^{-1}\big(LJ_{m+n}L^{\mathrm{T}}\big)\big)^{\mathrm{T}} \\
&\qquad + D_{m+n}\Big(\text{slt}\big(V^{-1}\big(LJ_{m+n}L^{\mathrm{T}}\big)\big) + \text{diag}\big(V^{-1}\big(LJ_{m+n}L^{\mathrm{T}}\big)\big)^{\mathrm{T}}\Big) \\
&\qquad + \text{diag}\big(V^{-1}\big(LJ_{m+n}L^{\mathrm{T}}\big)\big)\Big)X^{\mathrm{T}}V^{-\mathrm{T}}\Big)\Big)\Big\|_2.
\end{aligned} \tag{4.31}$$

For any lower triangular matrix K, we have

$$\text{low}(K) + \text{low}\left(K^{\mathrm{T}}\right) = K.$$

Therefore, if we set $K \equiv D_{m+n}V^{-1}X\,\text{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)$, then

$$\begin{aligned}
&D_{m+n}V^{-1}X\,\text{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right) \\
&= \text{low}\Big(D_{m+n}V^{-1}X\,\text{diag}\left(V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)\right)^{\mathrm{T}} + D_{m+n}\,\text{diag}\left(V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)\right)X^{\mathrm{T}}V^{-\mathrm{T}}\Big),
\end{aligned}$$

which together with (4.31) and (4.30) implies

$$\begin{aligned}\|Q_{LT}\|_2 \leqslant & \left\|VD_{m+n}^{-1}\right\|_2 \max_{\|\mathrm{lvec}(X)\|_2=1} \left\| \mathrm{vec}\left(D_{m+n}V^{-1}X\,\mathrm{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right.\right. \\ & + \mathrm{low}\left(D_{m+n}V^{-1}X\left(\mathrm{sut}\left(V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)\right)\right)^{\mathrm{T}}\right) \\ & \left.\left. + D_{m+n}\left(\mathrm{slt}\left(V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)\right)\right)X^{\mathrm{T}}V^{-\mathrm{T}}\right)\right\|_2 \\ \leqslant & \left\|VD_{m+n}^{-1}\right\|_2 \max_{\|X\|_F=1} \left\|D_{m+n}V^{-1}X\,\mathrm{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_F \\ & + \left\|VD_{m+n}^{-1}\right\|_2 \max_{\|X\|_F=1} \left\| \mathrm{low}\left(D_{m+n}V^{-1}X\left(\mathrm{sut}\left(V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)\right)\right)^{\mathrm{T}}\right)\right. \\ & \left. + D_{m+n}\left(\mathrm{slt}\left(V^{-1}\left(LJ_{m+n}L^{\mathrm{T}}\right)\right)\right)X^{\mathrm{T}}V^{-\mathrm{T}}\right\|_F .\end{aligned}$$

With the help of (2.5), we get

$$\begin{aligned}\|Q_{LT}\|_2 \leqslant & \kappa\left(VD_{m+n}^{-1}\right)\left\|\mathrm{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2 \\ & + \max_{\|X\|_F=1}\left\|\mathrm{low}\left(D_{m+n}V^{-1}X(\mathrm{sut}((LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}))\right)\right. \\ & \left. + D_{m+n}\left(\mathrm{low}\,(D_{m+n}V^{-1}X(\mathrm{sut}((LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}))\right)^{\mathrm{T}} D_{m+n}^{-1}\right\|_F \\ \leqslant & \kappa\left(VD_{m+n}^{-1}\right)\left\|\mathrm{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2 \\ & + \sqrt{1+\zeta_{D_{m+n}}^2}\left\|VD_{m+n}^{-1}\right\|_2 \max_{\|X\|_F=1}\left\|\left(D_{m+n}V^{-1}X(\mathrm{sut}((LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}))\right)\right\|_F \\ \leqslant & \kappa\left(VD_{m+n}^{-1}\right)\left\|\mathrm{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2 \\ & + \sqrt{1+\zeta_{D_{m+n}}^2}\,\kappa\left(VD_{m+n}^{-1}\right)\left\|\left(\mathrm{sut}(LJ_{m+n}L^{\mathrm{T}})V^{-\mathrm{T}}\right)\right\|_2 .\end{aligned} \tag{4.32}$$

This result together with the fact $\kappa\left(VD_{m+n}^{-1}\right) \geqslant 1$ and (4.23) illustrates that the bound (3.21) is significantly smaller than (4.26). □

REMARK 4.5. By neglecting the high-order terms, we can derive a first-order multiplicative bound from (4.26):

$$\begin{aligned}\|\Delta V\|_F \leqslant & \inf_{D_{m+n}\in\mathbb{D}_{m+n}}\left(\sqrt{1+\zeta_{D_{m+n}}^2}\,\kappa\left(VD_{m+n}^{-1}\right)\right)\left(\left\|\mathrm{sut}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2\|N\|_F\right. \\ & \left. + \left\|V^{-1}\right\|_2\|Y\|_2^2\|M\|_F\right) \\ & + \inf_{D_{m+n}\in\mathbb{D}_{m+n}}\left(\kappa\left(VD_{m+n}^{-1}\right)\right)\left(\left\|\mathrm{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2\|N\|_F\right).\end{aligned} \tag{4.33}$$

From (4.32) and (4.23), it is easy to check that the bound in Remark 3.4 is tighter than (4.33), when $W = I_{m+n} + N$ is assumed to be a lower triangular matrix.

5. Numerical results

In this section, we provide three numerical examples to illustrate the results derived in Sections 3–4. We use an algorithm from [1] to obtain the generalized Cholesky factors L and V in (1.3). All numerical experiments are performed by using Matlab 2018a.

EXAMPLE 5.1. In first example, we compare the strong and weak multiplicative perturbation bounds. Let $A = [a_{ij}] = P_m + I_m \in \mathbb{R}^{m\times m}$ where $P_m = [p_{ij}]$ is the Pascal matrix, $(i.e., p_{1j} = p_{i1} = 1, p_{ij} = p_{i(j-1)} + p_{(i-1)j})$, $B = [b_{ij}] = 0.7 * [\max(i,j)] \in \mathbb{R}^{n\times m}$ and $C = (c_{ij}) \in \mathbb{R}^{n\times n}$ is the Lehmar matrix, $(i.e., c_{ij} = i/j$ for $j \geqslant i)$, $J_{m+n} = \operatorname{diag}(I_m, -I_n)$ and $Y = P\Lambda P^{\mathrm{T}}$, where $P \in \mathbb{R}^{(m+n)\times(m+n)}$ is an orthogonal matrix taken from the QR factorization, $\Lambda = \operatorname{diag}(d,d,\ldots,d) \in \mathbb{R}^{(m+n)\times(m+n)}$ with $d = 0.02$. From [11], the scaled matrix D_{m+n} is denoted as follows: suppose $\zeta_1 = \sqrt{\sum_{j=1}^{m+n} f_{1j}^2}$, $\zeta_i = \sqrt{\sum_{j=i}^{m+n} f_{ij}^2}$ if $\sqrt{\sum_{j=1}^{m+n} f_{ij}^2} \leqslant \zeta_{i-1}$ otherwise $\zeta_i = \zeta_{i-1}$, for $i = 2,\ldots,m+n$, $F = (f_{ij}) = V^{\mathrm{T}}$.

Table 1: *Results for various values of n, m and k*

n,m,k	5,5,10	10,10,20	13,12,25	18,12,30	20,20,40	40,10,50
$\\|Q_{LM}\\|_2$	4.1503	13.0342	26.3812	39.3403	52.5054	87.4032
$t_{\\|Q_{LM}\\|_2}$	0.011385	0.014349	0.039375	0.497504	2.567943	9.750072
γ_{11}	32.9031	190.1043	530.2156	6.5214e+02	2.5475e+03	5.6403e+03
$t_{\gamma_{11}}$	0.014312	0.015310	0.016385	0.018066	0.015910	0.019883
$\\|Q_Y\\|_2$	0.1832	0.2072	0.3702	0.5361	1.0945	1.7304
$t_{\\|Q_Y\\|_2}$	0.014291	0.021533	0.028610	0.169868	3.543310	13.918751
γ_{12}	0.4732	3.1903	9.0704	39.1204	67.0123	124.2617
$t_{\gamma_{12}}$	0.011736	0.014821	0.015570	0.016765	0.015092	0.019160
$\\|Q_{LT}\\|_2$	2.0145	3.1026	3.5017	3.7102	3.9143	4.9813
$t_{\\|Q_{LT}\\|_2}$	0.015054	0.019183	0.029361	0.182677	0.577617	3.769436
γ_{13}	8.0537	21.7241	33.2653	59.4293	83.3427	162.3076
$t_{\gamma_{13}}$	0.015381	0.017907	0.017463	0.015684	0.018650	0.019702
$\\|R_V\\|_2$	11.3821	26.4376	51.0436	132.3054	174.3076	294.2643
$t_{\\|R_V\\|_2}$	0.016202	0.015268	0.022487	0.014396	0.015743	0.017608
γ_{14}	23.0543	146.0354	397.0548	3.4256e+02	7.5473e+02	3.7354e+03
$t_{\gamma_{14}}$	0.014631	0.017183	0.016836	0.016179	0.015437	0.019326
$\\|R_Y\\|_2$	0.2537	0.3741	0.7451	1.5803	3.5071	4.9644
$t_{\\|R_Y\\|_2}$	0.015494	0.015406	0.013707	0.016474	0.014801	0.015409
γ_{15}	0.4322	1.3605	6.8065	31.7402	59.4223	95.7733
$t_{\gamma_{15}}$	0.014988	0.015179	0.014046	0.019076	0.016228	0.019694

Moreover, in Table 1, we denote

$$\gamma_{11} = \sqrt{1+\zeta_{D_{m+n}}^2}\kappa\left(VD_{m+n}^{-1}\right)\|L\|_2^2\left\|V^{-1}\right\|_2, \quad \gamma_{12} = \sqrt{1+\zeta_{D_{m+n}}^2}\kappa\left(VD_{m+n}^{-1}\right)\|Y\|_2^2\left\|V^{-1}\right\|_2,$$

$$\gamma_{13} = \kappa\left(VD_{m+n}^{-1}\right)\left(\left\|\operatorname{diag}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2 + \sqrt{1+\zeta_{D_{m+n}}^2}\left\|\operatorname{slt}\left(\left(LJ_{m+n}L^{\mathrm{T}}\right)V^{-\mathrm{T}}\right)\right\|_2\right),$$

$$\gamma_{14} = \kappa\left(VD_{m+n}^{-1}\right)\|L\|_2^2\left\|V^{-1}\right\|_2, \quad \gamma_{15} = \kappa\left(VD_{m+n}^{-1}\right)\|Y\|_2^2\left\|V^{-1}\right\|_2,$$

and $t_{(.)}$ denote the time cost in seconds for computing the bounds (3.15), (3.21), (4.8), (4.26).

From Table 1, we can see that the strong rigorous multiplicative perturbation bounds (3.15) and (3.22), the rows marked by $\|Q_{LM}\|_2$, $\|Q_Y\|_2$, $\|Q_{LT}\|_2$, $\|R_V\|_2\|L\|_2^2$ and $\|R_V\|_2\|Y\|_2^2$ are always tighter than the weak rigorous multiplicative perturbation bounds (4.8) and (4.26), the rows marked by $\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}$ and γ_{15}. In addition, we can also observe that it is indeed more expensive to estimate the bounds (3.15) and (3.22); compare the rows marked by $t_{\gamma_{11}}$, $t_{\gamma_{12}}$, $t_{\gamma_{13}}$, $t_{\gamma_{14}}$ and $t_{\gamma_{15}}$.

EXAMPLE 5.2. The test matrix K_1 and K_2 are set to be

$$K_1 = \begin{bmatrix} 70 & 70 & 70 & 3 & 3 \\ 70 & 90 & 100 & 7 & 9 \\ 70 & 100 & 200 & 13 & 17 \\ 3 & 7 & 13 & -20 & -56 \\ 3 & 9 & 17 & -56 & -80 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 2.4957 & -0.9922 & 0.9008 & -1.0195 & -0.3655 \\ -0.9922 & 2.4957 & -0.9922 & 0.9008 & -1.0195 \\ 0.9008 & -0.9922 & 2.4957 & -0.9922 & 0.9008 \\ -1.0195 & 0.9008 & -0.9922 & -2.4957 & 0.9922 \\ -0.3655 & -1.0195 & 0.9008 & 0.9922 & -2.4957 \end{bmatrix},$$

here $Y = \mu\left[0.240\ -0.899\ 0.899\ 1.560\ -2.390\right]^T$ and $J_{3+2} = \text{diag}(I_3, -I_2)$.

Table 2: *Results for various values of* μ

ω	Test matrices	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
$\|Q_{LM}\|_2$	K_1	1.2903	1.2903	1.2903	1.2904	1.2904	1.2904
	K_2	1.5214	1.5214	1.5214	1.5215	1.5215	1.5215
γ_{11}	K_1	1.2643e+02	1.2643e+02	1.2872e+02	1.7642e+02	1.7642e+02	1.7642e+02
	K_2	1.4453e+02	1.4453e+02	1.4467e+02	1.9362e+02	1.9362e+02	1.9362e+02
$\|Q_Y\|_2$	K_1	2.2632	2.2580	2.2531	0.0232	3.3227e-04	4.2827e-06
	K_2	2.5214	2.5112	2.5034	0.3204	5.3097e-04	7.3437e-06
γ_{12}	K_1	3.9126	3.9042	2.9959	1.4420	4.8028e-02	4.7028e-04
	K_2	5.8632	5.8436	5.6209	1.8062	9.9201e-02	8.7403e-03
$\|Q_{LT}\|_2$	K_1	1.1357	1.1357	1.1357	1.1357	1.1357	1.1357
	K_2	1.4792	1.4792	1.4792	1.4792	1.4792	1.4792
γ_{13}	K_1	76.1082	76.3042	76.6201	77.3321	77.4508	77.6902
	K_2	84.1376	84.4201	84.4853	85.0675	85.3065	85.3457
$\|R_V\|_2$	K_1	7.3214e+01	7.3214e+01	7.5326e+01	7.5422e+01	7.6402e+01	7.6402e+01
	K_2	9.3053e+01	9.3053e+01	9.4467e+01	9.4761e+01	9.9502e+01	9.9502e+01
γ_{14}	K_1	1.1643e+02	1.1643e+02	1.1872e+02	1.6642e+02	1.6642e+02	1.6642e+02
	K_2	1.3453e+02	1.3453e+02	1.3467e+02	1.8362e+02	1.8362e+02	1.8362e+02
$\|R_Y\|_2$	K_1	0.7861	0.7835	0.6328	0.3116	1.2092e-03	1.0453e-05
	K_2	0.9861	0.9835	0.8328	0.5116	1.1654e-02	1.1921e-04
γ_{15}	K_1	3.6126	3.6042	2.9059	1.0420	3.2028e-02	3.2028e-04
	K_2	5.5632	5.5436	5.3209	1.2062	5.9201e-02	5.7403e-03

Table 2 provides the numerical results for various values of μ's, as follows:

$$\mu_1 = 1.004015006005433\text{e}-2, \quad \mu_2 = 1.003021021209640\text{e}-2,$$
$$\mu_3 = 9.036225416303058\text{e}-3, \quad \mu_4 = \mu_3\text{e}-1, \quad \mu_5 = \mu_3\text{e}-3, \quad \mu_6 = \mu_3\text{e}-5.$$

From Table 2, we can find that the strong multiplicative perturbation bounds (3.15) and (3.21) are always less than the weak multiplicative perturbation bounds (4.8) and (4.26), where $\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}$ and γ_{15} are given in the previous example. Furthermore, Tables 1 and 2 show that when the multiplicative perturbation matrix $W = I_{m+n} + N$ is set to be a lower triangular matrix, the strong and weak multiplicative perturbation bounds for the GCBD problem are smaller than the corresponding ones when the matrix $W = I_{m+n} + N$ is set to be a general matrix.

EXAMPLE 5.3. Let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -e & 1 & 0 & 0 & 0 & 0 & 0 \\ -e & -e & 1 & 0 & 0 & 0 & 0 \\ -e & -e & -e & 1 & 0 & 0 & 0 \\ -e & -e & -e & -e & 1 & 0 & 0 \\ -e & -e & -e & -e & -e & 1 & 0 \\ -e & -e & -e & -e & -e & -e & 1 \end{bmatrix} \operatorname{diag}\left(1, \delta, \delta^2, \delta^3, \delta^4, \delta^5, \delta^6\right),$$

$$Y = \mu \begin{bmatrix} 0.240 \\ -0.899 \\ 0.899 \\ 1.360 \\ -2.190 \\ 1.560 \\ 2.301 \end{bmatrix}, \quad J_{4+3} = \operatorname{diag}(I_4, -I_3),$$

where $e = 0.98$ and $\delta = \sqrt{1-e^2}$. From [11, 12], we have the following strong and weak additive rigorous perturbation bounds respectively:

$$\begin{aligned} G_{LG} &= \aleph_{\text{lvec}} \left(J_{m+n}^{-T} \otimes V\right) \aleph_{\text{low}} \left(\left(V^{-1} \otimes V^{-1} L J_{m+n}\right) \Pi + \left(V^{-1} L J_{m+n}^{T} \otimes V^{-1}\right)\right), \\ G_Y &= \aleph_{\text{lvec}} \left(J_{m+n}^{-T} \otimes V\right) \aleph_{\text{low}} \left(\left(V^{-1} \otimes V^{-1} Y\right) \Pi + \left(V^{-1} Y \otimes V^{-1}\right)\right), \\ H_V &= \aleph_{\text{lvec}} \left(J_{m+n}^{-T} \otimes V\right) \aleph_{\text{low}} \left(V^{-1} \otimes V^{-1}\right), \\ G_{LT} &= \left(\aleph_{\text{lvec}} \left(J_{m+n}^{-T} \otimes V\right) \aleph_{\text{low}} \left(\left(V^{-1} \otimes V^{-1} L J_{m+n}\right) \Pi + \left(V^{-1} L J_{m+n}^{\mathrm{T}} \otimes V^{-1}\right)\right)\right) \aleph_{\text{lvec}}^{\mathrm{T}}, \end{aligned}$$

and

$$\begin{aligned} \beta_{11} &= \sqrt{1+\zeta_{D_{m+n}}^2}\,\kappa\left(V D_{m+n}^{-1}\right) \left\|V^{-1} L\right\|_2, \quad \beta_{12} = \sqrt{1+\zeta_{D_{m+n}}^2}\,\kappa\left(V D_{m+n}^{-1}\right) \left\|V^{-1} Y\right\|_2, \\ \beta_{13} &= \kappa\left(V D_{m+n}^{-1}\right) \left(\left\|\operatorname{diag}\left(\mathrm{V}^{-1} L\right)\right\|_2 + \sqrt{1+\zeta_{D_{m+n}}^2} \left\|\operatorname{slt}\left(V^{-1} L\right)\right\|_2\right), \\ \beta_{14} &= \kappa\left(V D_{m+n}^{-1}\right) \left\|V^{-1}\right\|_2. \end{aligned}$$

In this example, we compare the additive and multiplicative rigorous perturbation bounds for different values of μ's, which are given above. From Table 3, we can clearly observe that the strong and weak multiplicative bounds $\|Q_{LM}\|_2$, $\|Q_Y\|_2$, $\|Q_{LT}\|_2$, $\|R_V\|_2$, and γ_{11}, γ_{12}, γ_{13}, γ_{14} are tighter than the strong and weak additive perturbation

Table 3: *Comparison of additive and multiplicative rigorous perturbation bounds*

ω	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
$\\|Q_{LM}\\|_2$	2.2997	2.2996	2.2983	2.2981	2.2981	2.2981
$\\|G_{LG}\\|_2$	12.6871	12.6871	12.6871	12.6871	12.6871	12.6871
$\\|Q_Y\\|_2$	21.4490	21.0468	9.0424	0.0592	5.4072e-06	5.9131e-10
$\\|G_Y\\|_2$	43.2302	43.6391	26.3015	1.8391	6.3261e-02	7.2604e-03
$\\|Q_{LT}\\|_2$	1.3628	1.3628	1.3628	1.3628	1.3628	1.3628
$\\|G_{LT}\\|_2$	2.5943	2.5943	2.5943	2.5943	2.5943	2.5943
$\\|R_V\\|_2$	46.3491	46.3491	48.3607	48.3607	57.5203	57.7287
$\\|H_V\\|_2$	89.6343	89.6343	92.7343	92.9807	98.8408	98.8626
γ_{11}	8.0674e+02	8.2665e+02	8.5674e+02	8.5832e+02	8.7609e+02	8.9612e+02
β_{11}	1.4553e+03	1.4597e+03	1.6138e+03	1.6234e+03	1.9721e+03	1.9860e+03
γ_{12}	46.5435	45.6645	19.4237	1.1456	1.2537e-03	1.2537e-06
β_{12}	1.4806e+03	1.4540e+03	686.5176	44.3787	0.4431	0.1434
γ_{13}	35.0578	35.0389	35.0104	17.1122	17.1307	17.1307
β_{13}	106.3003	104.3310	47.1390	37.2814	37.1612	37.1612
γ_{14}	5.3607e+02	5.3607e+02	5.4241e+02	5.4241e+02	5.7437e+02	5.8327e+02
β_{14}	1.2062e+03	1.2062e+03	1.2938e+03	1.2938e+03	1.6072e+03	1.6849e+03

bounds $\|G_{LG}\|_2, \|G_Y\|_2, \|G_{LT}\|_2$, $\|H_V\|_2$, and β_{11}, β_{12}, β_{13}, β_{14} respectively. The multiplicative perturbation bounds are useful to estimate the tighter bounds since the matrix is significantly ill-conditioned.

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