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Some convergence results using K iteration process in $CAT(0)$ spaces

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Abstract

In this paper, some strong and Δ-convergence results are proved for Suzuki generalized nonexpansive mappings in the setting of *CAT*(0) spaces using the *K* iteration process. We also give an example to show the efficiency of the *K* iteration process. Our results are the extension, improvement and generalization of many well-known results in the literature of fixed point theory in *CAT*(0) spaces.

MSC: Primary 47H09; secondary 47H10

Keywords: Suzuki generalized nonexpansive mapping; *CAT*(0) space; *K* iteration process; Δ-convergence; Strong convergence

1 Introduction

The well-known Banach contraction theorem uses the Picard iteration process for approximation of fixed point. Numerical computation of fixed points for suitable classes of contractive mappings, on appropriate geometric framework, is an active research area nowadays [1–3]. Many iterative processes have been developed to approximate fixed points of different type of mappings. Some of the well-known iterative processes are those of Mann [4], Ishikawa [5], Agarwal [6], Noor [7], Abbas [8], SP [9], S* [10], CR [11], Normal-S [12], Picard Mann [13], Picard-S [14], Thakur New [15] and so on. These processes have a wide rang of applications to general variational inequalities or equilibrium problems as well as to split feasibility problems [16–19]. Recently, Hussain et al. [20] introduced a new three-step iteration process known as the K iteration process and proved that it is converging fast as compared to all above-mentioned iteration processes. They use a uniformly convex Banach space as a ground space and prove strong and weak convergence theorems. On the other hand, we know that every Banach space is a $CAT(0)$ space.

Motivated by the above, in this paper, first we develop an example of Suzuki generalized nonexpansive mappings which is not nonexpansive. We compare the speed of convergence of the K iteration process with the leading two steps S-iteration process and leading three steps Picard-S-iteration process. Finally, we prove some strong and Δ-convergence theorems for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ spaces.

2 Preliminaries

For details as regards $CAT(0)$ spaces please see [21]. Some results are recalled here for $CAT(0)$ space X.

Lemma 2.1 ([7]) *For $x, y \in X$ and let $\xi \in [0,1]$, there exists a unique point $s \in [x,y]$ where $[x,y]$ is the line segment joining x and y, such that*

$$d(x,s) = \xi d(x,y) \quad \text{and} \quad d(y,s) = (1-\xi)d(x,y). \tag{1}$$

The notation $((1-\xi)x \oplus \xi y)$ is used for the unique point s satisfying (1).

Lemma 2.2 ([13, Lemma 2.4]) *For $x, y, z \in X$ and $\xi \in [0,1]$, we have*

$$d\big(z, \xi x \oplus (1-\xi)y\big) \leq \xi d(z,x) + (1-\xi)d(z,y). \tag{2}$$

Let C be a nonempty closed convex subset of a $CAT(0)$ space X, and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r\big(x, \{x_n\}\big) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r\big(C, \{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in C\big\},$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A\big(C, \{x_n\}\big) = \big\{x \in C : r\big(x, \{x_n\}\big) = r\big(C, \{x_n\}\big)\big\}.$$

Just like in uniformly convex Banach spaces, it is well known that $A(C, \{x_n\})$ consists of exactly one point in a complete $CAT(0)$ space.

Definition 2.3 In $CAT(0)$ space X, a sequence $\{x_n\}$ is said to be Δ-convergent to $s \in X$ if s is the unique asymptotic center of $\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = s$ and call s the $\Delta\text{-}\lim$ of $\{x_n\}$.

A point p is called a fixed point of a mapping T if $T(p) = p$, and $F(T)$ represents the set of all fixed points of the mapping T. Let C be a nonempty subset of a $CAT(0)$ space X.

A mapping $T : C \to C$ is called a contraction if there exists $\xi \in (0,1)$ such that

$$d(Tx, Ty) \leq \xi d(x,y)$$

for all $x, y \in C$.

A mapping $T : C \to C$ is called nonexpansive if

$$d(Tx, Ty) \leq d(x,y)$$

for all $x, y \in C$.

In 2008, Suzuki [22] introduced a new condition on a mapping, called condition (C), which is weaker than nonexpansiveness. A mapping $T : C \to C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2}d(x, Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx, Ty) \leq d(x,y). \tag{3}$$

The mapping satisfying condition (C) is called a Suzuki generalized nonexpansive mapping. The following is an example of a Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 2.4 Define a mapping $T : [0,1] \to [0,1]$ by

$$Tx = \begin{cases} 1-x & \text{if } x \in [0, \frac{1}{10}), \\ \frac{x+1}{2} & \text{if } x \in [\frac{1}{10}, 1]. \end{cases}$$

We need to prove that T is a Suzuki generalized nonexpansive mapping but not nonexpansive.

If $x = \frac{1}{11}$, $y = \frac{1}{10}$ we see that

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| \\ &= \left|1 - \frac{1}{11} - \frac{11}{20}\right| \\ &= \frac{79}{220} \\ &> \frac{1}{110} \\ &= d(x, y). \end{aligned}$$

Hence T is not a nonexpansive mapping.

To verify that T is a Suzuki generalized nonexpansive mapping, consider the following cases:

Case I: Let $x \in [0, \frac{1}{10})$, then $\frac{1}{2}d(x, Tx) = \frac{1-2x}{2} \in (\frac{2}{5}, \frac{1}{2}]$. For $\frac{1}{2}d(x, Tx) \leq d(x, y)$ we must have $\frac{1-2x}{2} \leq y - x$, i.e., $\frac{1}{2} \leq y$, hence $y \in [\frac{1}{2}, 1]$. We have

$$d(Tx, Ty) = \left|\frac{y+1}{2} - (1-x)\right| = \left|\frac{y+2x-1}{2}\right| < \frac{1}{10}$$

and

$$d(x;y) = |x-y| > \left|\frac{1}{10} - \frac{1}{2}\right| = \frac{2}{5}.$$

Hence $\frac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq d(x, y)$.

Case II: Let $x \in [\frac{1}{10}, 1]$, then $\frac{1}{2}d(x, Tx) = \frac{1}{2}|\frac{x+1}{2} - x| = \frac{1-x}{4} \in [0, \frac{9}{40}]$. For $\frac{1}{2}d(x, Tx) \leq d(x, y)$ we must have $\frac{1-x}{4} \leq |y - x|$, which gives two possibilities:

(a). Let $x < y$, then $\frac{1-x}{4} \leq y - x \Longrightarrow y \geq \frac{1+3x}{4} \Longrightarrow y \in [\frac{13}{40}, 1] \subset [\frac{1}{10}, 1]$. So

$$d(Tx, Ty) = \left|\frac{x+1}{2} - \frac{y+1}{2}\right| = \frac{1}{2}d(x, y) \leq d(x, y).$$

Hence $\frac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq d(x, y)$.

(b). Let $x > y$, then $\frac{1-x}{4} \leq x - y \Longrightarrow y \leq x - \frac{1-x}{4} = \frac{5x-1}{4} \Longrightarrow y \in [-\frac{1}{8}, 1]$. Since $y \in [0, 1]$, so $y \leq \frac{5x-1}{4} \Longrightarrow x \in [\frac{1}{5}, 1]$. So the case is $x \in [\frac{1}{5}, 1]$ and $y \in [0, 1]$.

Now $x \in [\frac{1}{5}, 1]$ and $y \in [\frac{1}{10}, 1]$ is already included in (a). So let $x \in [\frac{1}{5}, 1]$ and $y \in [0, \frac{1}{10})$, then

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{x+1}{2} - (1-y) \right| \\ &= \left| \frac{x+2y-1}{2} \right|. \end{aligned}$$

For convenience, first we consider $x \in [\frac{1}{5}, \frac{7}{8}]$ and $y \in [0, \frac{1}{10})$, then $d(Tx, Ty) \le \frac{3}{80}$ and $d(x,y) > \frac{1}{10}$. Hence $d(Tx, Ty) \le d(x,y)$.

Next consider $x \in [\frac{7}{8}, 1]$ and $y \in [0, \frac{1}{10})$, then $d(Tx, Ty) \le \frac{1}{10}$ and $d(x,y) > \frac{72}{80}$. Hence $d(Tx, Ty) \le d(x,y)$. So $\frac{1}{2}d(x, Tx) \le d(x,y) \Longrightarrow d(Tx, Ty) \le d(x,y)$.

Hence T is a Suzuki generalized nonexpansive mapping.

We now list some basic results.

Proposition 2.5 *Let C be a nonempty subset of a $CAT(0)$ space X and $T : C \to C$ be any mapping. Then*:

(i) [22, Proposition 1] *If T is nonexpansive then T is a Suzuki generalized nonexpansive mapping.*
(ii) [22, Proposition 2] *If T is a Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*
(iii) [22, Lemma 7] *If T is a Suzuki generalized nonexpansive mapping, then*

$$d(x, Ty) \le 3d(Tx, x) + d(x,y)$$

for all $x, y \in C$.

Lemma 2.6 ([22, Theorem 5]) *Let C be a weakly compact convex subset of a $CAT(0)$ space X. Let T be a mapping on C. Assume that T is a Suzuki generalized nonexpansive mapping. Then T has a fixed point.*

Lemma 2.7 ([23, Lemma 2.9]) *Suppose that X is a complete $CAT(0)$ space and $x \in X$. $\{t_n\}$ is a sequence in $[b,c]$ for some $b, c \in (0,1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that, for some $r \ge 0$, we have*

$$\begin{aligned} &\lim_{n\to\infty} \sup d(x_n, x) \le r, \qquad \lim_{n\to\infty} \sup d(y_n, x) \le r \quad \text{and} \\ &\lim_{n\to\infty} \sup d\big(t_n x_n \oplus (1-t_n) y_n, x\big) = r; \end{aligned}$$

then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Table 1 Sequences generated by *K*, Picard-S- and S-iteration processes

	K	Picard-S	S
x_0	0.9	0.9	0.9
x_1	0.9875	0.975	0.95
x_2	0.998561026471100	0.994244105884402	0.976976423537605
x_3	0.999840932805849	0.998727462446794	0.989819699574350
x_4	0.999982839247306	0.999725427956896	0.995606847310338
x_5	0.999998178520930	0.999941712669962	0.998134805438786
x_6	0.999999808905464	0.999987769949668	0.999217276778764
x_7	0.999999980126971	0.999997456252294	0.999674400293643
x_8	0.999999997947369	0.999999474526643	0.999865478820511
x_9	0.999999999789148	0.999999892043912	0.999944726482773
x_{10}	0.999999999978438	0.999999977920327	0.999977390415280
x_{11}	0.999999999997803	0.999999995501064	0.999990786179471
x_{12}	0.999999999999777	0.999999999086208	0.999996257108584
x_{13}	0.999999999999977	0.999999999814902	0.999998483680543
x_{14}	0.999999999999998	0.999999999962595	0.999999387160191
x_{15}	1	0.999999999992457	0.999999752825556
x_{16}	1	0.999999999998482	0.999999900490241
x_{17}	1	0.999999999999695	0.999999960003588
x_{18}	1	0.999999999999939	0.999999983947466
x_{19}	1	0.999999999999988	0.999999993565774
x_{20}	1	0.999999999999997	0.999999997424076

Let $n \geq 0$ and $\{\xi_n\}$ and $\{\zeta_n\}$ be real sequences in $[0,1]$. Hussain et al. [20] introduced a new iteration process namely the K iteration process, thus:

$$\begin{cases} x_0 \in C, \\ z_n = (1-\zeta_n)x_n + \zeta_n T x_n, \\ y_n = T((1-\xi_n)Tx_n + \xi_n T z_n), \\ x_{n+1} = T y_n. \end{cases} \tag{4}$$

They also proved that the K iteration process is faster than the Picard-S- and S-iteration processes with the help of a numerical example. In order to show the efficiency of the K iteration process we use Example 2.4 with $x_0 = 0.9$ and get Table 1. A graphic representation is given in Fig. 1. We can easily see the efficiency of the K iteration process.

3 Convergence results for Suzuki generalized nonexpansive mappings

In this section, we prove some strong and Δ-convergence theorems of a sequence generated by a K iteration process for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ space. The K iteration process in the language of $CAT(0)$ space is given by

$$\begin{aligned} &x_0 \in C, \\ &z_n = (1-\zeta_n)x_n \oplus \zeta_n T x_n, \\ &y_n = T\big((1-\xi_n)Tx_n \oplus \xi_n T z_n\big), \\ &x_{n+1} = T y_n. \end{aligned} \tag{5}$$

Theorem 3.1 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T: C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrarily chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (5) *then $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T)$.*

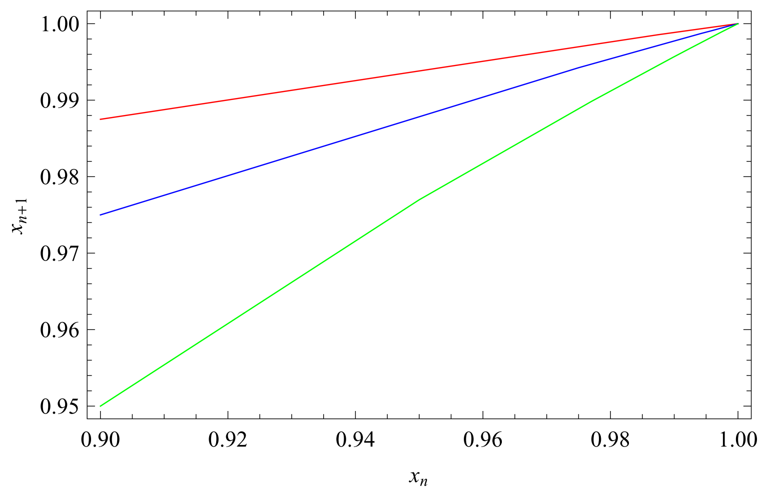

Figure 1 Convergence of iterative sequences generated by *K* (red line), Picard-S (blue line) and *S* (green line) iteration process to the fixed point 1 of the mapping *T* defined in Example 2.4

Proof Let $p \in F(T)$ and $z \in C$. Since T is a Suzuki generalized nonexpansive mapping,

$$\frac{1}{2}d(p, Tp) = 0 \leq d(p,z) \quad \text{implies that} \quad d(Tp, Tz) \leq d(p,z).$$

So by Proposition 2.5(ii), we have

$$\begin{aligned} d(z_n,p) &= d\big(\big((1-\zeta_n)x_n \oplus \zeta_n Tx_n\big),p\big) \\ &\leq (1-\zeta_n)d(x_n,p) + \zeta_n d(Tx_n,p) \\ &\leq (1-\zeta_n)d(x_n,p) + \zeta_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{6}$$

Using (6) we get

$$\begin{aligned} d(y_n,p) &= d\big(\big(T(1-\xi_n)Tx_n \oplus \xi_n Tz_n\big),p\big) \\ &\leq d\big(\big((1-\xi_n)Tx_n \oplus \xi_n Tz_n\big),p\big) \\ &\leq (1-\xi_n)d(Tx_n,p) + \xi_n d(Tz_n,p) \\ &\leq (1-\xi_n)d(x_n,p) + \xi_n d(z_n,p) \\ &\leq (1-\xi_n)d(x_n,p) + \xi_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{7}$$

Similarly by using (7) we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\leq d(y_n,p) \\ &\leq d(x_n,p). \end{aligned} \tag{8}$$

This implies that $\{d(x_n,p)\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim_{n\to\infty} d(x_n,p)$ exists, as required. □

Theorem 3.2 *Let C, X, T and $\{x_n\}$ be as in Theorem* 3.1, *where $\{\xi_n\}$ and $\{\zeta_n\}$ are sequences of real numbers in $[a,b]$ for some a, b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.*

Proof Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Theorem 3.1, $\lim_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n,p) = r. \tag{9}$$

From (6) and (9), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = r. \tag{10}$$

By Proposition 2.5(ii) we have

$$\limsup_{n\to\infty} d(y_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = r. \tag{11}$$

On the other hand by using (6), we have

$$\begin{aligned}
d(x_{n+1},p) &= d(Ty_n,p) \\
&\leq d(y_n,p) \\
&= d\big(\big(T(1-\xi_n)Tx_n \oplus \xi_n Tz_n\big),p\big) \\
&\leq d\big((1-\xi_n)Tx_n \oplus \xi_n Tz_n,p\big) \\
&\leq (1-\xi_n)d(Tx_n,p) + \xi_n d(Tz_n,p) \\
&\leq (1-\xi_n)d(x_n,p) + \xi_n d(z_n,p) \\
&= d(x_n,p) - \xi_n d(x_n,p) + \xi_n d(z_n,p).
\end{aligned}$$

This implies that

$$\frac{d(x_{n+1},p) - d(x_n,p)}{\xi_n} \leq d(z_n,p) - d(x_n,p).$$

So

$$d(x_{n+1},p) - d(x_n,p) \leq \frac{d(x_{n+1},p) - d(x_n,p)}{\xi_n} \leq d(z_n,p) - d(x_n,p)$$

implies that

$$d(x_{n+1},p) \leq d(z_n,p).$$

Therefore

$$r \leq \liminf_{n\to\infty} d(z_n,p). \tag{12}$$

From (10) and (12) we get

$$r = \lim_{n\to\infty} d(z_n, p) = \lim_{n\to\infty} d\big(\big((1-\zeta_n)x_n \oplus \zeta_n Tx_n\big), p\big). \tag{13}$$

From (9), (11), (13) and Lemma 2.7, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Let $p \in A(C, \{x_n\})$. By Proposition 2.5(iii), we have

$$\begin{aligned} r\big(Tp, \{x_n\}\big) &= \limsup_{n\to\infty} d(x_n, Tp) \\ &\leq \limsup_{n\to\infty} \big(3d(Tx_n, x_n) + d(x_n, p)\big) \\ &\leq \limsup_{n\to\infty} d(x_n, p) \\ &= r\big(p, \{x_n\}\big). \end{aligned}$$

This implies that $Tp \in A(C, \{x_n\})$. Since X is uniformly convex, $A(C, \{x_n\})$ is a singleton and hence we have $Tp = p$. Hence $F(T) \neq \emptyset$. □

The proof of the following Δ-convergence theorem is similar to the proof of [24, Theorem 3.3].

Theorem 3.3 *Let C, X, T and $\{x_n\}$ be as in Theorem* 3.2 *with $F(T) \neq \emptyset$. Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Next we prove the strong convergence theorem.

Theorem 3.4 *Let C, X, T and $\{x_n\}$ be as in Theorem* 3.2 *such that C is compact subset of X. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof By Lemma 2.6, we have $F(T) \neq \emptyset$ and so by Theorem 3.1 we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By Proposition 2.5(iii), we have

$$d(x_{n_k}, Tp) \leq 3d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p) \quad \text{for all } n \geq 1.$$

Letting $k \to \infty$, we get $Tp = p$, i.e., $p \in F(T)$. By Theorem 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(T)$ and so the x_n converge strongly to p. □

A strong convergence theorem using condition I introduced by Senter and Dotson [25] is as follows.

Theorem 3.5 *Let C, X, T and $\{x_n\}$ be as in Theorem* 3.2 *with $F(T) \neq \emptyset$. If T satisfies condition* (I), *then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof By Theorem 3.1, we see that $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$ and so $\lim_{n\to\infty} d(x_n,F(T))$ exists. Assume that $\lim_{n\to\infty} d(x_n,p) = r$ for some $r \geq 0$. If $r = 0$ then the result follows. Suppose $r > 0$, from the hypothesis and condition (I),

$$f\bigl(d\bigl(x_n,F(T)\bigr)\bigr) \leq d(Tx_n,x_n). \tag{14}$$

Since $F(T) \neq \emptyset$, by Theorem 3.2, we have $\lim_{n\to\infty} d(Tx_n,x_n) = 0$. So (14) implies that

$$\lim_{n\to\infty} f\bigl(d\bigl(x_n,F(T)\bigr)\bigr) = 0. \tag{15}$$

Since f is a nondecreasing function, from (15) we have $\lim_{n\to\infty} d(x_n,F(T)) = 0$. Thus, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$d(x_{n_k},y_k) < \frac{1}{2^k} \quad \text{for all } k \in \mathbb{N}.$$

So using (9), we get

$$d(x_{n_{k+1}},y_k) \leq d(x_{n_k},y_k) < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} d(y_{k+1},y_k) &\leq d(y_{k+1},x_{k+1}) + d(x_{k+1},y_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(x_n,p)$ exists, we have $x_n \to p \in F(T)$. □

4 Conclusions

The extension of the linear version of fixed point results to nonlinear domains has its own significance. To achieve the objective of replacing a linear domain with a nonlinear one, Takahashi [26] introduced the notion of a convex metric space and studied fixed point results of nonexpansive mappings in this framework. This initiated the study of different convexity structures on a metric space. Here we extend a linear version of convergence results to the fixed point of a mapping satisfying condition C for the newly introduced K iteration process [20] to nonlinear $CAT(0)$ spaces.

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