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Algorithm for Computing the Total Vertex Irregularity Strength of Some Cubic Graphs

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Abstract

Determining the total vertex irregularity strength of any regular graph is a challenging problem. In 2010, a lower bound for this value has been given by Nurdin et al. A cubic graph is a graph of regular degree three. In this paper, we propose an algorithm to compute the total vertex irregularity strength of cubic graphs. By this algorithm we have a unified method to derive the total vertex irregularity strength of some cubic graphs, namely three cubic platonic graphs and k-prism graphs.

1 Introduction

In this paper, we only deal with simple and connected graphs. We use the standard definitions and notations based on [1]. Let $G = (V, E)$ be a graph. Any mapping $\alpha : V \cup E \to \{1, 2, \cdots, t\}$ is called a t-labeling on G. The *weight* $wt(u)$ of a vertex u, under α, is defined as $wt(u) = \alpha(u) + \sum_{uw \in E} \alpha(uw)$. The labeling α of G is called a *total vertex irregular labeling* if all the (vertex)

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weights are distinct, namely $wt(u) \neq wt(w)$ for any $u \neq w$. The *total vertex irregularity strength* of the graph G is the least integer s such that G admits a total vertex irregular s-labeling. It is denoted by tvs(G). Bača et al. [2] initiated the study of such a labeling. In fact, this notion is a generalization of the vertex irregularity strength of a graph introduced in [3].

In [2], it is shown that for a star $K_{1,n}$ and a complete graph K_n we have that $\text{tvs}(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$ and $\text{tvs}(K_n) = 2$ for $n \geq 2$. Moreover, if a graph G has n vertices with minimum and maximum degrees δ and Δ, respectively then the lower and upper bounds of the tvs(G) is as follows:

$$\left\lceil \frac{n+\delta}{\Delta+1} \right\rceil \leq tvs(G) \leq n + \Delta - 2\delta + 1. \tag{1.1}$$

Therefore, if G is a regular graph of degree r, then we have:

$$\left\lceil \frac{n+r}{r+1} \right\rceil \leq tvs(G) \leq n - r + 1. \tag{1.2}$$

In the following, we present a better lower bound derived by Nurdin et al. [4] regarding the total vertex irregularity strength for any graph G of order n with the minimum and maximum degrees δ and Δ respectively. Let n_j be the set of all vertices of degree j in G. Then, we have that:

$$tvs(G) \geq \max \left\{ \left\lceil \frac{\delta + n_\delta}{\delta+1} \right\rceil, \left\lceil \frac{\delta + n_\delta + n_{\delta+1}}{\delta+2} \right\rceil, \cdots, \left\lceil \frac{\delta + \sum_{j=\delta}^{\Delta} n_j}{\Delta+1} \right\rceil \right\}. \tag{1.3}$$

Furthermore, Nurdin et al. [4] conjectured that the sign "=" in (1.3) holds for any graph G. In this paper, we focus on finding the total vertex irregularity strength (tvs) of a cubic graph G. In this case, if the conjecture is true then we have that $\text{tvs}(G) = \lceil \frac{n+3}{4} \rceil$ for any cubic graph G. Ahmad et al. [5] has found the values of the total vertex irregularity strength for some cubic graphs, such as certain convex polytopes containing a pair of t-sided faces, $2t$ 5-sided faces and tm 6-sided faces embedded in the plane, Goldberg snarks, the necklace graphs, certain plane graph, and the crossed prisms. Rajasingh et al. [6] also derived the tvs(G) for certain cubic graphs G. These results strengthen the above conjecture.

2 Main results

In this paper, we consider several classes of cubic graphs such as some platonic graphs, e.g., tetrahedral, hypercube Q_3 and dodecahedral and n-prism graphs. An n-prism graph Y_n is the graph $P_2 \times C_n$ with the vertex set:

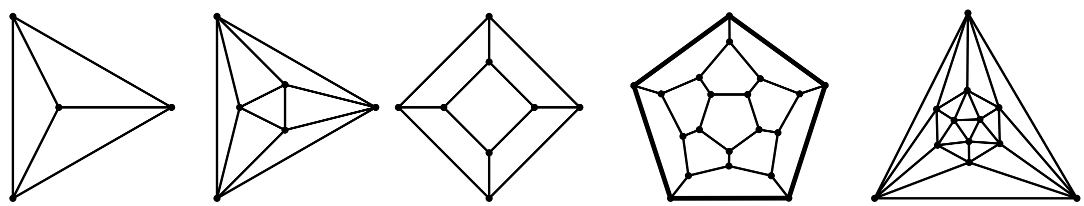

Figure 1: Platonic graphs: (a) tetrahedral, (b) octahedral, (c) cube, (d) dodecahedral, (e) icosahedral

$$V(Y_n) = \{a_i, b_i | 1 \leq i \leq n\}, \tag{2.4}$$

and the edge set

$$E(Y_n) = \{a_i b_i | 1 \leq i \leq n\} \cup \{a_i a_{i+1}, b_i b_{i+1} | 1 \leq i \leq n-1\} \cup \{a_n a_1, b_n b_1\}. \tag{2.5}$$

We are going to propose a new algorithm to compute the total vertex irregularity strength of such graphs. The main contribution of this paper is to propose **one** algorithm for computing the total vertex irregularity strength for **different classes** of some cubic graphs. We do not present a method of labeling for calculating the total vertex irregularity strength in an explicit formula (as it was done in many previous papers). In Figure 1, we present five platonic graphs.

In the following, we propose an algorithm for computing the total vertex irregularity strength of any cubic graph. By given a cubic graph H with n vertices, this Algorithm 1 results a total vertex irregular labeling f of H. Such a labeling has the property that $f(xy) \leq \left\lfloor \frac{n+3}{4} \right\rfloor < \left\lceil \frac{n+3}{4} \right\rceil = \text{tvs}(H)$, for all $xy \in E(H)$. However, the vertex-labels $f(x)$ for some vertices x may exceed the value of $\text{tvs}(H)$. In order to get a total vertex irregular $\text{tvs}(H)$-labeling from the labeling f, we employ Algorithm 2.

Algorithm 1. Total vertex irregularity labeling f on a cubic graph H
Input: A cubic graph $H = (V, E)$ with $|V| = n$.
Output: A total vertex irregular labeling of graph $H = (V, E)$

1. Choose x_1 as random.
2. Order all the vertices of H starting from x_1, and then its neighbors, denoted by x_2, x_3 and x_4 in any order, and then assign by x_j $(j \geq 5)$ the

first unlabeled vertex which is adjacent to x_k, with k is the smallest index as possible. If there are two or more such unlabeled vertices adjacent to x_k then select (if any) the one which is also adjacent to other x_t, with t is the smallest integer greater than k, otherwise select any vertex. Do this until the last vertex, and as a result we have ordered vertices: $x_1, x_2, x_3, \cdots, x_n$ in graph H.

3. Distribute the weights to all the vertices such that $wt(x_k) = 3 + k$, for $k = 1, 2, \cdots, n$.
4. Calculate the edge labels f by using a formula $f(xy) = \min\left\{\left\lfloor \frac{wt(x)}{4} \right\rfloor, \left\lfloor \frac{wt(y)}{4} \right\rfloor\right\}$.
5. Calculate the vertex labels f by using a formula $f(x) = wt(x) - \sum_{xy \in E} f(xy)$.

End of Algorithm 1.

Algorithm 2. Reduction Algorithm
Input: A graph $H = (V, E)$ with $|V| = n$, and a labeling f produced by Algorithm 1.
Output: A total vertex irregular tvs(H)-labeling of $H = (V, E)$.

1. Assign $A \leftarrow \{x \in V(H) | f(x) > \lceil \frac{n+3}{4} \rceil\}$
2. Assign total labeling $r(x) \leftarrow f(x)$, for each $x \in V(H) \cup E(H)$
3. Repeat until $A = \varnothing$
   - Let $x_i \in A$ with the smallest index
   - Find the edge $e = x_i y$ which $r(e) < \lceil \frac{n+3}{4} \rceil$ and $r(y) > 1$, then: $r(x_i) \leftarrow r(x_i) - 1, r(y) \leftarrow r(y) - 1$ and $r(e) \leftarrow r(e) + 1$
   - If $r(x_i) \leq \lceil \frac{n+3}{4} \rceil$ then $A \leftarrow A - \{x_i\}$

End of Algorithm 2.

Step 4 of Algorithm 1 produces a total irregular vertex labeling f with all the labels of the edges $f(xy) \leq tvs(H) - 1$,
since $f(xy) = \min\left\{\left\lfloor \frac{wt(x)}{4} \right\rfloor, \left\lfloor \frac{wt(y)}{4} \right\rfloor\right\} \leq \lfloor \frac{n+3}{4} \rfloor < \lceil \frac{n+3}{4} \rceil = tvs(H)$. The rightmost inequality holds since n is always even for a cubic graph. From Step 5 of Algorithm 1, we may have $f(x) > \lceil \frac{n+3}{4} \rceil$ for some vertices x.

However, with Algorithm 2, we can produce a total vertex irregular tvs(H)-labeling on graph H. Note that the total vertex irregularity strength tvs(H) of any cubic graph H on n vertices is $\lceil \frac{n+3}{4} \rceil$.

By applying Algorithm 1 to the three cubic platonic graphs we obtain their total vertex irregular labeling as in Figure 2. The largest label of such labelings lies on their vertices, the value differs at least 3 from the value of tvs(H).

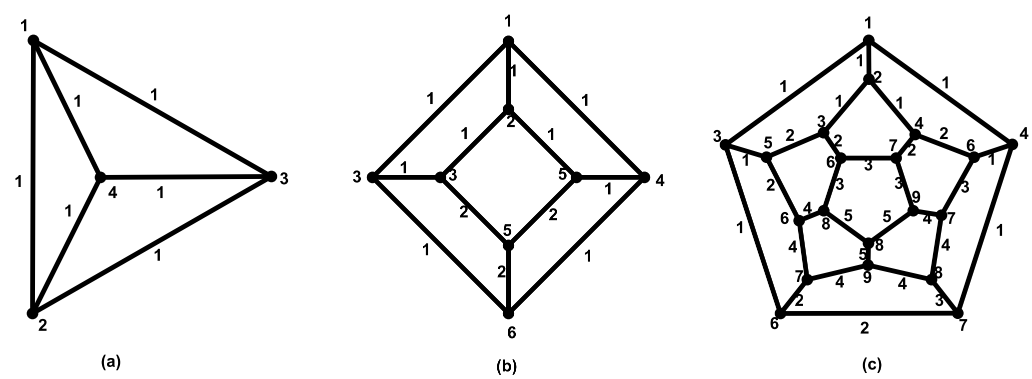

Figure 2: Result from Algorithm 1: (a) tetrahedral, (b) cube, (c) dodecahedral.

By applying Algorithm 2 (Reduction) with the input of the function f, we obtain a total vertex irregular labeling with the largest label equal to the value of tvs(H). The resulted labelings can be seen in Figure 3.

Theorem 2.1. *For $n \geq 3$, let H be the n-prism graph Y_n. Then, tvs(H) = $\lceil \frac{2n+3}{4} \rceil$.*

Proof. By Eq.(1.3), $\text{tvs}(Y_n) \geq \lceil \frac{2n+3}{4} \rceil$. By using Algorithms 1 and 2 we are going to show that $\text{tvs}(Y_n) = \lceil \frac{2n+3}{4} \rceil$. Let the vertex and edge sets of the graph Y_n be given in Eqs (2.4) and (2.5). Now, set $x_1 = a_1$ and $x_2 = b_1$. By Step 2 of Algorithm 1 to obtain ordered vertices (after x_1 and x_2): $x_3, x_4, \cdots, x_{2n}$ so that $wt(x_i) = i + 3$, for $i = 1, 2, 3, \cdots, 2n$. By the use of this ordering, we obtain that $|wt(a_i) - wt(a_j)| \leq 4$ for $a_i \sim a_j$, $|wt(b_i) - wt(b_j)| \leq 4$ for $b_i \sim b_j$, and $|wt(a_i) - wt(b_i)| \leq 2$ for any i. By Steps 3, 4 and 5 of Algorithm 1 applying to the n-prism, we obtain $f(xy) = \min\{\lfloor \frac{wt(x)}{4} \rfloor, \lfloor \frac{wt(y)}{4} \rfloor\} \leq \lfloor \frac{2n+3}{4} \rfloor \leq \lceil \frac{2n+3}{4} \rceil - 1$ for each $xy \in E(Y_n)$, and $f(x) = wt(x) - \sum_{x \sim z} f(xz)$ for each $x \in V(Y_n)$. For any $x \in V(H)$, we will show

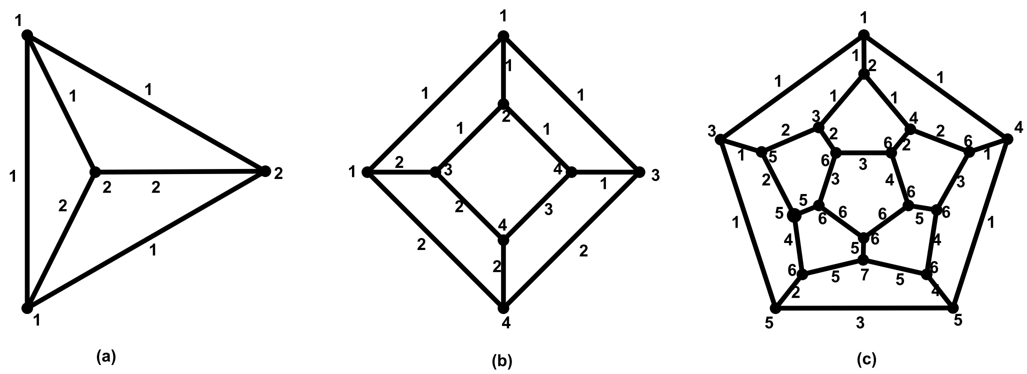

Figure 3: Result from Algorithm 2: (a) tetrahedral, (b) cube, (c) dodecahedral.

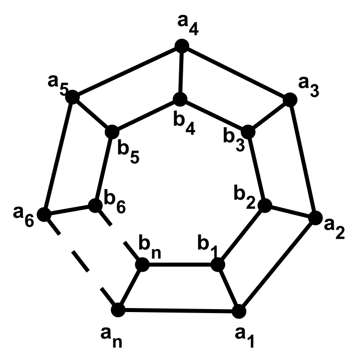

Figure 4: Prism graph Y_n

that $f(x) - \lceil \frac{2n+3}{4} \rceil \leq 4$. Let u be any vertex of Y_n and y, w, z be the three neighbors of u. If $u = x_1$ then we have that $wt(u) < wt(y) < wt(w) < wt(z)$, and by Algorithm 1 we have $f(u) = 1$ and $f(uy) = f(uw) = f(uz) = 1$. If $u \neq x_1$ then one of the following three cases holds.

- Case 1. $wt(y) < wt(u) < wt(w) < wt(z)$.

  In this case, we must have that $u = a_i$ for some i, and so $wt(u) \equiv 2(mod\ 4)$ or $wt(u) \equiv 3(mod\ 4)$.

  Let $wt(u) = t$, then

$$f(uy) = \lfloor \tfrac{t}{4} \rfloor - 1 \text{ and } f(uz) = f(uw) = \lfloor \tfrac{t}{4} \rfloor.$$

Therefore, $f(u) = wt(u) - f(uy) - f(uw) - f(uz) = t - 3\lfloor \frac{t}{4} \rfloor + 1$. Thus, $f(u) = \lfloor \frac{t}{4} \rfloor + 3$ for $t \equiv 2 \ (mod\ 4)$ or $f(u) = \lfloor \frac{t}{4} \rfloor + 4$ for $t \equiv 3 \ (mod\ 4)$. In this case, we have that $|f(u) - \lceil \frac{2n+3}{4} \rceil| \leq 2$ for any u. Therefore by Algorithm 2, the value of $f(u)$ can be reduced to so that $f(u) \leq \lceil \frac{2n+3}{4} \rceil$.

- Case 2. $wt(y) < wt(w) < wt(u) < wt(z)$.

  In this case, we have that $u = b_i$ for some i. Let $wt(u) = t$,

  $$f(uy) = f(uw) = \lfloor \tfrac{t}{4} \rfloor - 1 \text{ and } f(uz) = \lfloor \tfrac{t}{4} \rfloor.$$

  Therefore, $f(u) = wt(u) - f(uy) - f(uw) - f(uz) = t - 3\lfloor \frac{t}{4} \rfloor + 2$. Thus, $f(u) \leq \lceil \frac{2n+3}{4} \rceil + 2$. By Algorithm 2, the value of $f(u)$ can be reduced so that $f(u) \leq \lceil \frac{2n+3}{4} \rceil$.

- Case 3. $wt(y) < wt(w) < wt(z) < wt(u)$.

  In this case, vertex u must receive the largest weight. Thus, $wt(u) = 2n + 3$ and

  $$f(uy) = f(uw) = \lfloor \tfrac{2n+3}{4} \rfloor - 1 \text{ and } f(uz) = \lfloor \tfrac{2n+3}{4} \rfloor.$$

  Therefore, $f(u) = wt(u) - f(uy) - f(uw) - f(uz) = 2n - 3\lfloor \frac{t}{4} \rfloor + 2$. Thus, $f(u) \leq \lceil \frac{2n+3}{4} \rceil + 3$. By Algorithm 2, the value of $f(u)$ can be reduced to so that $f(u) \leq \lceil \frac{2n+3}{4} \rceil$.

In any case, by Algorithms 1 and 2 we obtain a total vertex irregular labeling on Y_n with the largest label $\lceil \frac{2n+3}{4} \rceil$. □

3 Conclusion

In this paper, we propose an algorithm to determine the total vertex irregularity strength of some cubic graphs. In particular we apply it to some platonic graphs and n-prism graph. All the total vertex irregularity strengths of such graphs satisfy the conjecture stated in [4].

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