Dependence of salt intrusion length on fresh water discharge and estuary shape

The following formulas are taken from the book "Salinity and Tides in Alluvial Estuaries" by H.H.G. Savenije (2005):

$$L = a\ln\left(\frac{1}{\beta}+1\right) = a\ln\left(\frac{1+\beta}{\beta}\right) \tag{5.48}$$

$$\beta = \frac{KaQ_f}{D_0 A_0} \tag{5.47}$$

where L is the intrusion length, a is the convergence length of an exponentially varying cross-section A, while β is the salt intrusion parameter that defines the balance between advection of the fresh water discharge Q_f and the mixing of saline and fresh water characterised by the dispersion D. The subscript 0 relates to the downstream boundary condition at the mouth of the estuary. Savenije (2005) provides a predictive equation for D_0 :

$$\frac{D_0}{\upsilon_0 E_0} = 1400\frac{h_0}{a}\mathrm{N}_R^{0.5} = 1400\frac{h_0}{a}\frac{c_0}{\upsilon_0}\sqrt{\frac{\Delta\rho}{\rho}\frac{Q_f T}{E_0 A_0}} \tag{5.70}$$

where E is the tidal excursion, h is the depth, c_0 is the wave celerity of a progressive wave and υ is the tidal velocity amplitude. Further elaboration yields:

$$D_0 = 1400 h_0 E_0 \frac{c_0}{a}\sqrt{\frac{\Delta\rho}{\rho}\frac{Q_f T}{E_0 A_0}}$$

$$\beta = \frac{KaQ_f}{D_0 A_0} = \frac{K}{1400}\frac{a}{h_0}\frac{a}{c_0 T}\sqrt{\frac{\rho}{\Delta\rho}\frac{Q_f T}{E_0 A_0}} = \frac{K}{1400}\frac{a}{h_0}\frac{a}{\lambda_0}\sqrt{\frac{\rho}{\Delta\rho}N}$$

Hence the general solution for the dependence of the intrusion length on fresh water discharge:

$$\frac{\partial L}{\partial Q_f} = a\frac{\beta}{1+\beta}\left(\frac{-1}{\beta^2}\right)\frac{\partial\beta}{\partial Q_f} = a\frac{\beta}{1+\beta}\left(\frac{-1}{\beta^2}\right)\left(\frac{1}{2}\frac{\beta}{Q_f}\right) = -\frac{1}{2}\frac{a}{Q_f}\frac{1}{1+\beta}$$

There are two interesting asymptotic solutions:

1. prismatic

$$\beta \gg 1;\ \frac{L}{a} = \ln\left(\frac{1}{\beta}+1\right) \approx \frac{1}{\beta}$$

$$\frac{\partial L}{\partial Q_f} = -\frac{1}{2}\frac{a}{Q_f}\frac{1}{1+\beta} \approx -\frac{1}{2}\frac{a}{Q_f}\frac{1}{\beta} = -\frac{1}{2}\frac{a}{Q_f}\frac{L}{a} = -\frac{1}{2}\frac{L}{Q_f}$$

$$L \propto Q_f^{-0.5}$$

2. strongly convergent

$$\beta \ll 1;\ \frac{1}{1+\beta} \approx 1$$

$$\frac{\partial L}{\partial Q_f} = -\frac{1}{2}\frac{a}{Q_f}\frac{1}{1+\beta} \approx -\frac{1}{2}\frac{a}{Q_f}$$, which depends on *a*, the so-called "buffering".

$$\frac{L}{a} \approx -\frac{1}{2}\ln\left(\frac{Q_f}{Q_0}\right) = \ln\left(\left(\frac{Q_f}{Q_0}\right)^{-0.5}\right)$$

were Q_0 is the discharge where L=0. This equation can be approached by a power function:

$$\frac{L}{a} \propto Q_f^{-0.17}$$

So depending on the convergence, the exponent of the fresh water discharge in this relationship varies between -0.5 for prismatic channels to -0.17 for strongly tapering channels, or in the symbols of G&O'D: $0.17<\gamma<0.5$, or $\gamma_{max}=0.5$. Hence the tapering strongly affects the dependency of the salt intrusion length on the fresh water discharge, which G&O'D call "buffering". Moreover, this relationship strongly depends on the convergence length.

General expression for L:

$$L = a\ln\left(\frac{1}{\beta}+1\right) = a\ln\left(\frac{1400}{K}\frac{h_0}{a}\frac{\lambda_0}{a}\sqrt{\frac{\Delta\rho}{\rho}}N^{-0.5}+1\right)$$

$$\frac{L}{a} = \ln\left(\frac{1400}{K}\frac{h_0}{a}\frac{\lambda_0}{a}\sqrt{\frac{\Delta\rho}{\rho}\frac{A_0E_0}{Q_fT}}+1\right)$$