CO-279

VISUALIZATION OF MAP PROJECTION DISTORTIONS BY USING FINITE CIRCLES: AN ALTERNATIVE APPROACH TO TISSOT'S INDICATRIX

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ABSTRACT

The most common way of visualizing distortions caused by map projections is the use of Tissot's indicatrix, the projection of an infinitesimal circle around a point on the datum surface. The appearance of such a circle on the projection plane is an ellipse unless the projection is conformal. Indicatrices distributed on a regular grid or graticule provides an efficient tool for understanding distortion characteristics of a projection. Other methods for this purpose are isocol representations for area and/or angle distortions, tables showing distortions numerically in certain latitudes or longitudes, and using known shapes, i.e. the projection of a circle or square on the datum surface. For most of the azimutal, conic and cylindrical projections the indicatrices can easily be created. For more complicated projections such as pseudocylindrical, polyconic, pseudoconic projections the positioning of the indicatrix is not easy, because the projected graticule is not orthogonal in such projections. In order to interpret the indicatrix it must be exaggerated, so its scale differs from the principal map scale. Known shapes approach can be used to simulate the indicatrix. In other words the representation of finite circles that are small enough can be used instead of the indicatrix. Doing so, no extra calculations are necessary. A data set including finite circles with an appropriate interval in longitude and latitude can be added to map data. Doing so, a finite circle representation that look like an indicatrix representation can be created with the map data automatically.

BACKGROUND AND OBJECTIVES

The basic problem of map projections is the representation of a curved surface, the earth, in a plane, the map. The figure of the earth is usually represented by a surface of revolution, either the ellipsoid or the sphere (Richardus & Adler 1972, p.1). Both the surfaces are non-developable, i.e. they can not be opened to the plane without distortions. This is actually a transformation process from a curved surface to the plane, in which areas, shapes and angles are distorted. One of these properties can be preserved by using an appropriate map projection. To construct such projections, distortion characteristics must be known extensively. On the other hand, it is also necessary to know distortion characteristics of map projections. So the geometry of the map can be understood correctly. It is especially important when using small-scale maps, because projection distortions are more significant at small scales.

Tissot's theory provides a useful mechanism to understand projection distortions. The theory states that a circle on the datum surface, either the ellipsoid or the sphere, with a center P and a radius, ds, can be assumed to be a plane figure within its infinitely small area, which will remain infinitely small on the plane. In general, the circle will be portrayed as an ellipse, which is called the ellipse of distortion or Tissot's indicatrix. When using particular map projections, in which datum and image planes are parallel, the circle will be portrayed as a circle, though at a different scale. Such projections are called conformal.

The axes of the indicatrix show the maximum and minimum linear distortions on the projection surface or on the plane of the map. It provides a useful medium for analyzing or visualizing projection distortions. Since the indicatrix is infinitesimally small, it is exaggerated when depicting on the map, i.e. the scale of the indicatrix differs from the scale of the map. In practice, an indicatrix representation is made on the intersections of a graticule, a network of meridians and parallels on the map plane.

In case of azimuthal, cylindrical and conic projections the graticule is orthogonal. Here the semi-axes of the indicatrix are coincident with the directions of the meridians and the parallels. It means that the directions of the maximum and minimum linear distortions are coincident with the directions of the meridians and parallels. An indicatrix representation for these projections is rather straightforward, because the direction of the indicatrix is obvious.

In mathematical projections, such as pseudocylindirical projections, the graticule is not orthogonal. It means that the directions of maximum and minimum linear distortions are not coincident with the graticule. In this case the direction of the indicatrix must be determined. Since the forward projection equations are more complicated and partial derivatives are necessary, creating an indicatrix representation is not an easy task.

Known shapes can also be used for distortion visualization. A human head drawn in Mollweide projection and transferred to Mercator and cylindrical equal-area projection in the famous text book of Arthur

Robinson is a famous representation of distortion visualization (Robinson et al 1995, p.69). Similarly a circle can be created on the sphere and represented on the map plane for the same purpose. Such shapes show the distortion pattern globally.

Known shapes approach can also be used in a way that the indicatrix representation do. In other words circles, which are small enough, created on the intersections of a graticule can simulate the indicatrix representation. Here, these circles are a part of the map content and visualizes the distortion pattern in a simple and efficient way. Partial derivatives and extra calculations are not necessary. This approach is also very useful when using table-based projections.

TISSOT'S INDICATRIX

Tissot's indicatrix or ellipse of distortion is a concept presented by French mathematician Nicolas Auguste Tissot in 1859 and 1871 to measure and to visualize distortions caused by map projection. It is the diagram that results from projecting a circle of infinitesimal radius from a curved geometric model onto the projection plane. Tissot proved that this diagram is normally an ellipse whose axes indicate the two directions along which its scale is maximal and minimal. These directions are called principal directions (URL1).

In general, forward projection is expressed with two functions:

$$x = f_1(\varphi, \lambda)$$
$$y = f_2(\varphi, \lambda) \quad (1)$$

Here x and y represent Cartesian coordinates on the map surface, φ and λ geographical coordinates. The local scales or linear distortions along meridians (h) and parallels (k) can be obtained as follows:

$$h = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2}, \quad k = \frac{1}{\cos\varphi}\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2} \quad (2)$$

If the graticule is orthogonal on the map plane, meridians and parallels are the principal directions. So h and k are coincident with the semi-axes or the radii of the indicatrix (a,b) (Fig.1).

Any point (P) on the infinitesimal circle on the sphere are transformed to a point on the indicatrix or the ellipse (P'). Assuming the radius of the circle 1, and radii of the ellipse a and b, the distortion of the azimuth around the point O can be obtained as follows (Fig.1).

$$\tan\alpha = \frac{x}{y}, \quad \tan\alpha' = \frac{x'}{y'}$$

$$y' = by, \quad x' = ax \quad (3)$$

$$\frac{\tan\alpha'}{\tan\alpha} = \frac{a}{b} \Rightarrow \tan\alpha' = \frac{a\sin\alpha}{b\cos\alpha} \quad (4)$$

Direction distortion in the azimuth;

$$\sin(\alpha' - \alpha) = \frac{a-b}{a+b}\sin(\alpha' + \alpha) \quad (5)$$

Maximal direction distortion occurs when $(\alpha' + \alpha) = \frac{\pi}{2}$

$$\sin(\alpha' - \alpha)_{\max} = \sin\omega = \frac{a-b}{a+b} \quad (6)$$

Angular distortion:

$$w = 2\omega \quad (7)$$

The area of the indicatrix delivers the local area distortion.

$$p = ab \quad (8)$$

The local scale or linear distortion in the direction of can be obtained by the equation of the ellipse.

$$r^2 = a^2 \sin^2 \alpha' + b^2 \cos^2 \alpha' \quad (9)$$

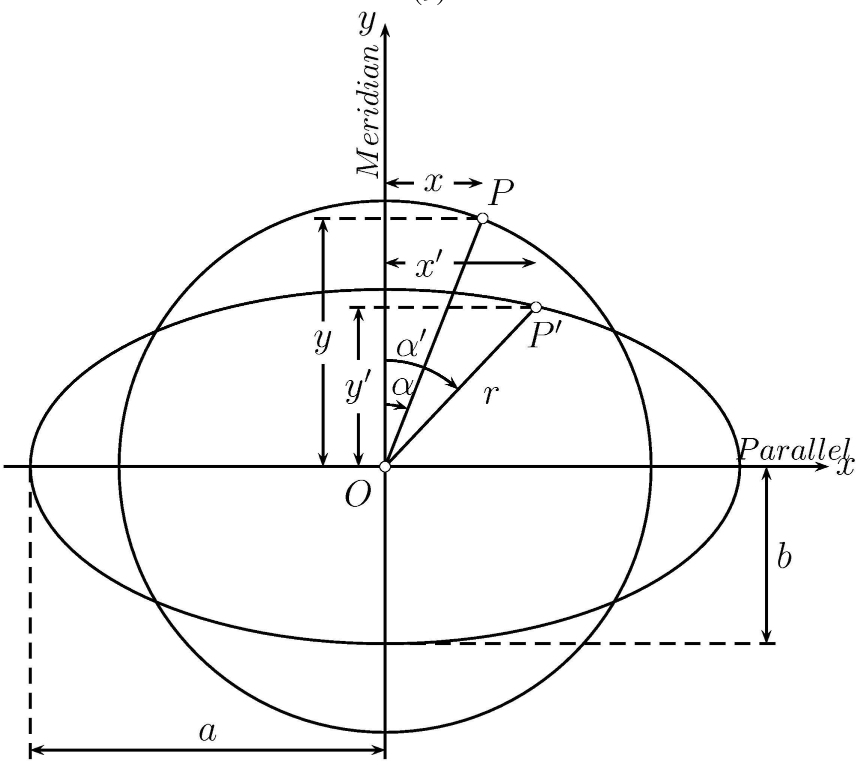

Figure 1: Tissot's indicatrix

If the graticule is not orthogonal, the radii of the indicatrix or minimal and maximal local scales around a point can be obtained as follows (Bildirici et al 2006):

$$(a+b)^2 = \left(\frac{\partial y}{\partial \varphi} + \frac{1}{\cos\varphi}\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial x}{\partial \varphi} - \frac{1}{\cos\varphi}\frac{\partial y}{\partial \lambda}\right)^2$$

$$(a-b)^2 = \left(\frac{\partial y}{\partial \varphi} - \frac{1}{\cos\varphi}\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial x}{\partial \varphi} + \frac{1}{\cos\varphi}\frac{\partial y}{\partial \lambda}\right)^2 \quad (10)$$

The direction of the indicatrix can be determined by using the direction angles of the projected meridians and parallels (β_φ , β_λ, see Fig.2). We assume the projected graticule is not orthogonal, which is the general case.

$$\tan\beta_\varphi = \frac{\frac{\partial y}{\partial \lambda}}{\frac{\partial x}{\partial \lambda}}, \quad \tan\beta_\lambda = \frac{\frac{\partial y}{\partial \varphi}}{\frac{\partial x}{\partial \varphi}} \quad (11)$$

The angle between horizontal axis (y) and the parallel (Richardus and Adler 1972, p.131):

$$\tan \nu' = \pm \sqrt{\frac{1-\frac{k^2}{b^2}}{\frac{k^2}{a^2}-1}} \qquad (12)$$

The direction of the major radius of the indicatrix (*a*):

$$\gamma = \beta_\varphi + \nu' \qquad (13)$$

After the radii (*a,b*) and the direction of the indicatrix are known, it should be scaled, i.e. the radii are multiplied with an appropriate scale factor. So the indicatrix can be drawn.

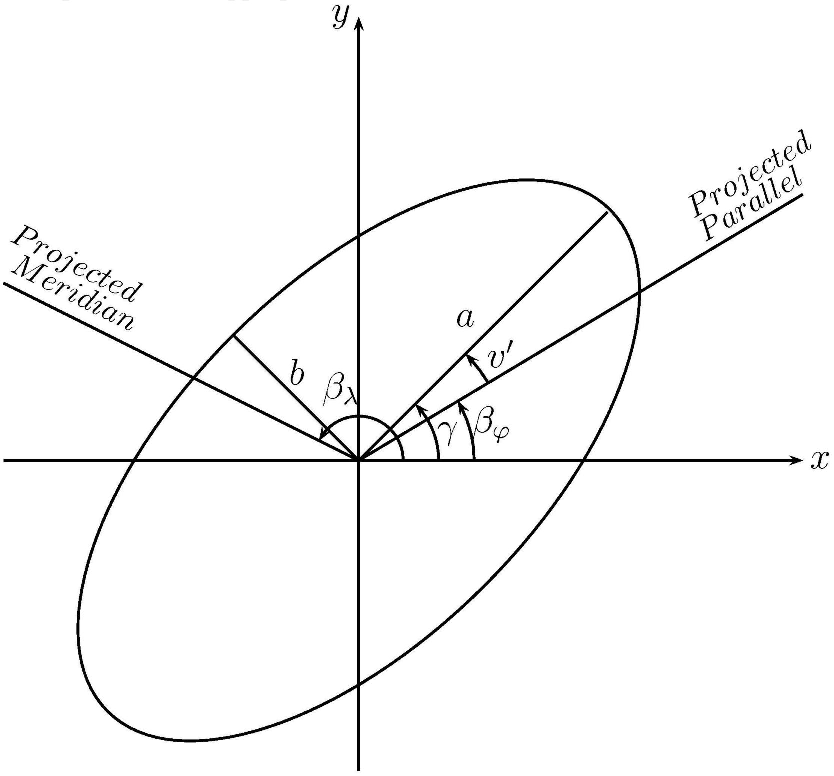

Figure 2: Direction of the indicatrix

FINITE CIRCLES APPROACH

Instead of indicatrix representation, finite circles can be created on the datum surface, sphere or ellipsoid, in accordance with the graticule. Such circles can simulate the indicatrix representation. The circles can be created and added to the map data. The resulting map shows ellipses just like indicatrices.

In order to create circles on the sphere, the direct solution is used.

$$\sin \varphi_P = \sin \varphi_O \cos\delta + \cos\varphi_O \sin\delta \cos\alpha_O$$

$$\tan(\lambda_P - \lambda_O) = \frac{\sin \alpha_O}{\frac{\cos\varphi_O}{\tan\delta} - \sin \varphi_O \cos\alpha_O} \qquad (14)$$

Here δ denotes the great circle distance from point *O* to point *P*, α_O is the azimuth at point *O*. If point *O* stays unchanged, point *P* draws a circle with a constant distance δ (the radius). In practice a polygon is created by incrementing the azimuth (Fig.3). For instance a polygon can be created in 5° increments in azimuth.

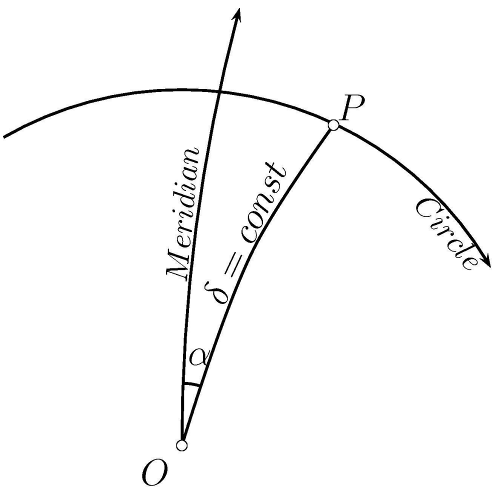

Figure 3: Direct solution on the sphere

For an efficient use of this approach an appropriate radius and angular increment for the change in azimuth are to be determined. The radius can be a value around 1 cm or the less on the map. A polygon that represents a circle can be created in a linear increment of 0.2 m, minimum space between two points on the map. In this case the increment in the azimuth is 1.4°. Since 0.2 mm is the minimum distance on the map, an increment of 5° in azimuth is sufficient for simulating a circle with a polygon. It is also obvious that the more the radius is the less the increment in azimuth should be.

Since no partial derivatives and other quantities that depend on the forward projection formulas are required, the application of the finite circles approach is straightforward. After addition of circles to the map data, the indicatrix-like representation is created together with the actual map data. Circle data with different properties (scale, graticule interval, radius etc) can be created as separate layers in GIS systems. The coordinate system of such data must be longitude and latitude (the geographical coordinate system). Such layers can be added to any map in a GIS system, and distortion distribution can be visualized. Such a representation makes possible to understand how the data are distorted.

A COMPARISON

In order to examine whether the finite circles simulate the indicatrix representation, both the approaches should be compared numerically. For this purpose a simple projection, the equal-area cylindrical projection, is chosen, where the graticule is orthogonal. Here the principal directions are coincident with the meridians and parallels. The forward projection equations:

$$x = R\lambda$$

$$y = R\sin\varphi$$

The local scales:

$$b = h = R\cos\varphi$$

$$a = k = \frac{1}{R\cos\varphi}$$

Choosing an arbitrary point O, we can create an indicatrix. The geographical and plane coordinates and the radii of the indicatrix (R=6371 km);

$$\varphi_O = 30°N \;\; \lambda_O = 30°E \quad x_O = 3185.50km \;\; y_O = 3335.85\text{km}$$

$$a = k = 1.154701 \qquad b = h = 0.866025$$

The local scales around the point O and directions on the sphere and on the map can be calculated with *Eq.*4 and 9. Table 1 shows these magnitudes in 15° interval in the azimuth (see left side of the table).

To simulate the indicatrix a finite circle around the same point (O) can be created with a radius of 100 km, which is small enough for a small scale map. With the same interval in the azimuth the points are calculated on the sphere and projected on the map with the forward projection equations. The azimuths on the map are calculated with plane coordinates. Similarly the distances from the point O to the points on the ellipse (the projected circle) are divided by the radius of the circle, which gives the local scale in this direction, approximately. These values are shown in the table.

Table 1: The direction distortion and local scales around point O (30°N, 30°E)

Indicatrix			Finite Circle					
$\alpha°$	$\alpha_i'°$	r_i	$\varphi_P°$	$\lambda_P°$	$x_{P\ km}$	$y_{P\ km}$	$\alpha_c'°$	r_c
0	0.000000	0.866025	30.899322	30.000000	3271.71	3335.85	0.000000	0.862066
15	19.660060	0.903343	30.868400	30.271164	3268.76	3366.00	19.908430	0.885476
30	37.589089	0.983389	30.777800	30.523359	3260.10	3394.04	37.955849	0.946176
45	53.130102	1.059874	30.633853	30.739049	3246.34	3418.03	53.484926	1.022501
60	66.586776	1.114104	30.446577	30.903415	3228.41	3436.30	66.871365	1.092349
75	78.637085	1.144853	30.228942	31.005401	3207.52	3447.64	78.856738	1.139436
90	90.000000	1.154701	29.995925	31.038419	3185.11	3451.31	90.194709	1.154676
105	101.362915	1.144853	29.763454	31.000696	3162.69	3447.12	101.582673	1.135854
120	113.413224	1.114104	29.547311	30.895265	3141.81	3435.40	113.696595	1.087151
135	126.869898	1.059874	29.362072	30.729638	3123.87	3416.98	127.219964	1.018838
150	142.410911	0.983389	29.220162	30.515208	3110.11	3393.14	142.768687	0.946865
165	160.339940	0.903343	29.131054	30.266457	3101.46	3365.48	160.579773	0.891105
180	180.000000	0.866025	29.100678	30.000000	3098.51	3335.85	180.000000	0.869914
195	199.660060	0.903343	29.131054	29.733543	3101.46	3306.22	199.420227	0.891105
210	217.589089	0.983389	29.220162	29.484792	3110.11	3278.56	217.231313	0.946865
225	233.130102	1.059874	29.362072	29.270362	3123.87	3254.72	232.780036	1.018838
240	246.586776	1.114104	29.547311	29.104735	3141.81	3236.30	246.303405	1.087151
255	258.637085	1.144853	29.763454	28.999304	3162.69	3224.58	258.417327	1.135854
270	270.000000	1.154701	29.995925	28.961581	3185.11	3220.38	269.805291	1.154676
285	281.362915	1.144853	30.228942	28.994599	3207.52	3224.05	281.143262	1.139436
300	293.413224	1.114104	30.446577	29.096585	3228.41	3235.39	293.128635	1.092349
315	306.869898	1.059874	30.633853	29.260951	3246.34	3253.67	306.515074	1.022501
330	322.410911	0.983389	30.777800	29.476641	3260.10	3277.65	322.044151	0.946176
345	340.339940	0.903343	30.868400	29.728836	3268.76	3305.70	340.091570	0.885476

If the azimuths on the plane and local scales are compared, it can be seen that the finite circle simulates the indicatrix. It can be said that the differences, which are given in table 2, can hardly be noticed on a small scale map.

Table 2: Differences in azimuth and local scale

$\alpha°$	$(\alpha_i' - \alpha_c')°$	$(r_i - r_c)$
0	0.000	-0.004
15	0.248	-0.018
30	0.367	-0.037
45	0.355	-0.037
60	0.285	-0.022
75	0.220	-0.005
90	0.195	0.000
105	0.220	-0.009
120	0.283	-0.027
135	0.350	-0.041
150	0.358	-0.037
165	0.240	-0.012
180	0.000	0.004
195	-0.240	-0.012
210	-0.358	-0.037
225	-0.350	-0.041
240	-0.283	-0.027
255	-0.220	-0.009
270	-0.195	0.000
285	-0.220	-0.005
300	-0.285	-0.022
315	-0.355	-0.037
330	-0.367	-0.037
345	-0.248	-0.018

APPLICATION

In this section four projections with different properties are selected and finite circle representations are created. For this purpose world countries data is used, and world maps with finite circle representations are created. Selected projections scale and circle radius are listed in table 3. In figures 4 to 7 world maps in selected projections with finite circle representation are shown. The graticule interval is 15° for all maps. The word countries data are taken from digital chart of the world (URL2). The maps are created with the program "wproj" developed by Bildirici. It is capable of creating small-scale maps in dxf and postscript format. It currently includes 75 projections.

Table 3: Selected projections

Projection	Map Scale	Circle Radius (km)	Azimuth Interval (°)	Map
Mollweide	1:300 Million	300	5	Fig.4
Winkel Tripel	1:300 Million	300	5	Fig.5
Ginzburg V	1:300 Million	300	5	Fig.6
HEALPix (H=4, K=3)	1:300 Million	300	5	Fig.7

Forward projection equations and other information can be found in the following references: For Winkel Tripel projection see Ipbuker & Bildirici (2005), for Ginzburg Projections see Bildirici et. al. (2006), for HEALPix see Calabretta & Roukema (2007).

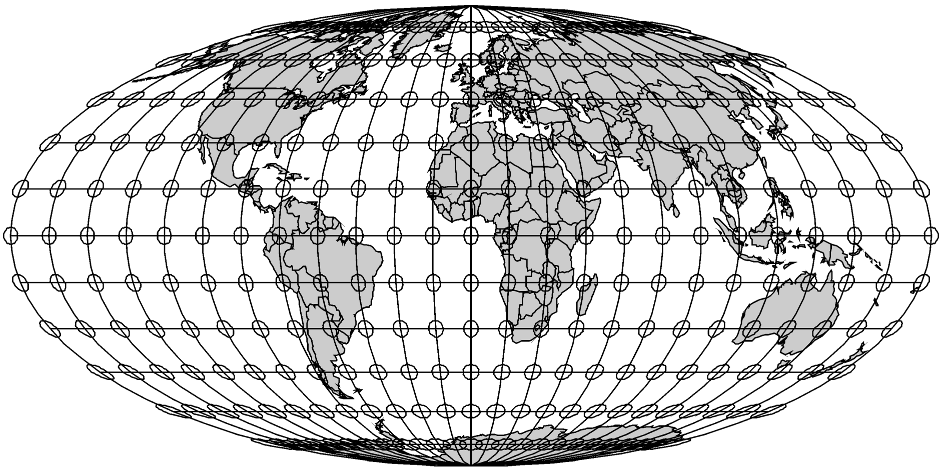

Figure 4: Mollweide Projection

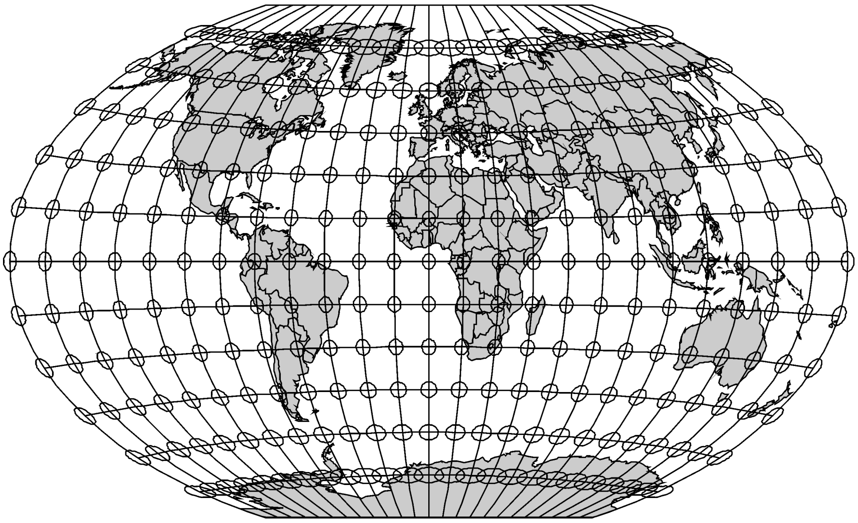

Figure 5: Winkel Tripel Projection

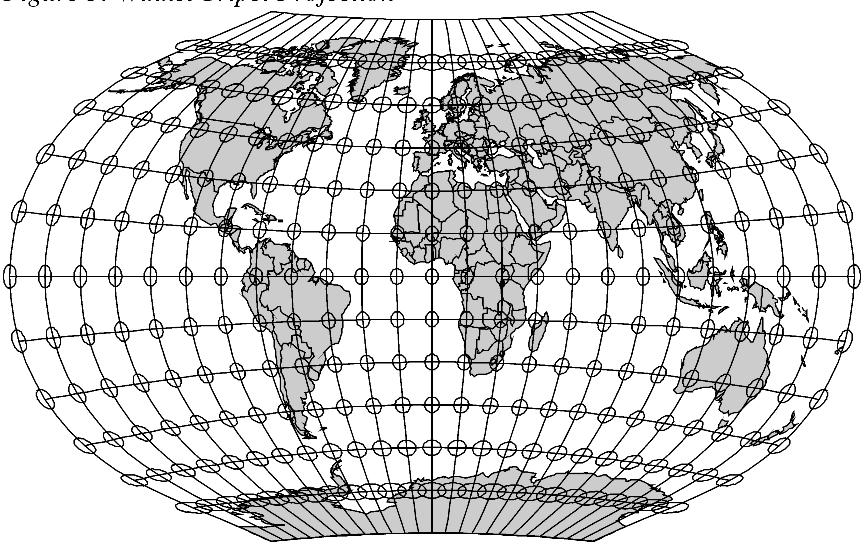

Figure 6: Ginzburg V Projection

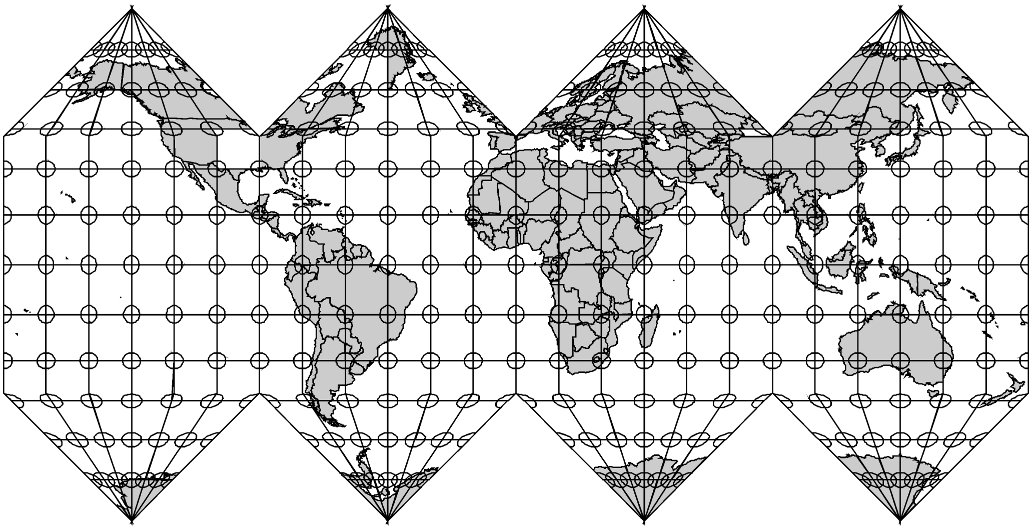

Figure 7: HEALPix Projection (H=4 K=3)

CONCLUSION

Distortions are unavoidable in map projections. It is necessary to know their effects and magnitudes, especially when creating small-scale maps. Tissot's indicatrix provides a useful tool for understanding distortion distribution on map area. In order to create an indicatrix representation, partial derivatives are necessary. In mathematical projections such as pseudocylindirical projections or polyconic projections, partial derivatives can be complicated equations, and direction of the indicatrix (the angle in Eq.12) can have some special cases to be taken into account. The projection of the finite circles that are created on the datum surface can simulate Tissot's indicatrix, if they are small enough. Such circles are prepared for certain scales and purposes, and added to GIS projects as separate layers. So the distortion distribution can be understood easily, for which no extra calculation efforts are necessary. In this paper the applicability of the finite circle method was shown.

REFERENCES

Bildirici, İ.Ö., Ipbüker, C., Yanalak, M.(2006) Function matching for Soviet-era table-based modified polyconic projections , International Journal of Geographical Information Science, 20/7, 769-795 .

Calabretta, M.R., Roukema, B.F., (2007) Mapping on the HEALPix grid, Mon. Not. R. Astron. Soc. 381, 865–872.

Ipbüker, C., Bildirici, İ.Ö.(2005) Computer program for the inverse transformation of the Winkel Projection , ASCE Journal of Surveying Engineering, 131/4, 125-129 .

Richardus, P., Adler, R.K., Map Projections for Geodesists, Cartographers and Geographers (1972): North-Holland Publishing Company, Amsterdam.

Robinson, A.H., Morrison, J.L., Muerchke, P.C., Kimmerling, A.J., Guptill, S.C. (1995): Elements of Cartography, Sixth Edition, John Wiley & Sons Inc, New York.

URL1: http://en.wikipedia.org/wiki/Tissot's_Indicatrix, accessed: 01.02.2011.

URL2: http://en.wikipedia.org/wiki/Digital_Chart_of_the_World, accessed: 01.02.2011.