THE INDEPENDENCE NUMBER PROJECT: α-BOUNDS

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1. Introduction

A lower bound for the independence number of a graph is a graph invariant l such that, for every graph G, $l(G) \leq \alpha(G)$. Similarly, an upper bound for the independence number is a graph invariant u such that, for every graph G, $\alpha(G) \leq u(G)$. Many efficiently computable upper and lower bounds, called **α-bounds** here, have been published and these are surveyed in the following section. They can be used to predict the value of α. Suppose $l_1, l_2, \ldots, l_k$ are efficiently computable lower bounds for the independence number of a graph; then $l = \max\{l_1, l_2, \ldots, l_k\}$ is also an efficiently computable lower bound for the independence number. Similarly, if $u_1, u_2, \ldots, u_m$ are efficiently computable upper bounds for the independence number, then $u = \min\{u_1, u_2, \ldots, u_m\}$ is also an efficiently computable upper bound for the independence number. For some graphs G, $l(G) = u(G)$ and, in such cases, it follows that the independence number $\alpha(G) = l(G) = u(G)$ can be directly computed from its bounds. For instance, consider the graph consisting of the cycle C_4 with a diagonal. It is known that, for every graph $\alpha_c \leq \alpha$ and, for every graph, $\alpha \leq \alpha_f$, where α_c is the critical independence number and α_f is the fractional independence number. These bounds are both efficiently computable and, for this graph, equal 2. Thus the theory implies that $\alpha = 2$.

New efficiently computable independence number bounds are also of practical interest: they can lead to faster independence number computations. New bounds can lead to new exact predictions of the independence number of a graph, without any need for computer search of subsets of vertices or calculating independence numbers of subgraphs of the given graph. If it is known that α must lie in the interval $[l, u]$ then only subsets of sizes in this range must be considered. In some instances theoretical upper and lower bounds for α can be used to predict the independence number with no further search (in this case the theory predicts that α lies in an

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interval where $l = u$). It is known that the independence of a graph is equal to the larger of the independence numbers of two proper subgraphs [43]. So computation of the independence number depends on the computation of the independence number of two proper subgraphs G_1 and G_2. If the theoretical bounds predict that either of these has independence number no more than l then no further computation for that graph is needed and, if the theoretical bounds imply that one of these graphs has independence number u then the original graph must have independence number u.

2. α-Upper Bounds

The following bounds are all efficiently computable.

(1) $\alpha \leq n - \frac{e}{\Delta}$. This bound is credited to Kwok in [47], but may belong to "folklore." Here Δ is the *maximum degree* of the graph and e is the number of edges.
(2) $\alpha \leq p_0 + \min\{p_-, p_+\}$ [6]. Here p_-, p_0, p_+ denote the number of eigenvalues of the adjacency matrix of a graph G smaller than, equal to, and greater than zero respectively. This is the Cvetković bound and is very good for many graphs. For the Petersen graph there are 4 non-positive eigenvalues. Thus, $\alpha \leq 4$. In fact, $\alpha = 4$ for this graph.
(3) $\alpha \leq \lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e} \rfloor$ [17].
(4) $\alpha \leq \alpha_f$. Here α_f is the *fractional independence number* of a graph. It is the sum of the largest weights in $[0, 1]$ that can be assigned to the vertices of a graph such that the sum of the weights on the vertices of any edge does not exceed 1. It can be formulated as the optimal value of a linear program and can, thus, be computed efficiently. The number is not explicitly defined in [37], but all of these ideas can be found there. Larson has found a characterization of graphs where these invariants are equal: a graph is König-Egerváry if, and only if, $\alpha = \alpha_f$ [28]. Thus it is possible in polynomial-time to check if these invariants are equal.
(5) $\alpha \leq a$ [38]. Here a is the *annihilation number* of a graph. If the degree sequence of a graph is listed in non-decreasing order $d_1 \leq d_2 \leq \ldots \leq d_n$ then a is the largest index such that the sum of the first a degrees is no more than the sum of the remaining degrees. Larson and Pepper have found a polynomial-time characterization of graphs where $\alpha = a$ [30].
(6) $\alpha \leq n - \mu$ [33]. Here μ is the *matching number* of the graph, the largest number of independent edges. This is an early result of König. For bipartite graphs the König-Egerváry Theorem states that $\alpha = n - \mu$.
(7) $\alpha \leq n - \lceil \frac{n-1}{\Delta} \rceil = \lfloor \frac{(\Delta-1)n+1}{\Delta} \rfloor$ [2].
(8) $\alpha \leq n - \delta$. Here δ is the *minimum degree* of the graph. This bound probably belongs to "folklore".

(9) $\alpha \leq \Theta$ [31, 32, 26]. This is the Lovász Theta Function bound, perhaps the most important upper bound in both theory and practice.

(10) $\alpha \leq \min\{\sum_{i=k+1}^{n} \frac{-\lambda_{min}(A)}{\lambda_i(A)-\lambda_{min}(A)} \times [(e+y)^T u_i]^2 : y \in Y\}$ [35]. Here e is a vector of all 1's, u_i is an orthonormal eigenvector associated with eigenvalue λ_i, and $Y = \{y : y \geq 0 \text{ and } (e+y)^T u_i = 0, \forall i = 1 \ldots k\}$.

(11) $\alpha \leq n - \frac{C}{2} - \frac{1}{2}$ [29]. Here C is the number of *cut vertices* of the graph.

(12) $\alpha \leq n + \Delta - \lceil 2\sqrt{n-1} \rceil$ [1]. This is a conjecture of the AGX graph theory conjecture-making software.

(13) $\alpha \leq \frac{\Delta}{2} \lceil \frac{n}{2} \rceil$ (for graphs with at least 3 vertices) [1]. This is a conjecture of the AGX graph theory conjecture-making software.

(14) $\alpha \leq n - \frac{M}{2} - \frac{1}{2}$ [29], where M is the median degree.

(15) $\alpha \leq n - \frac{1}{2} rank$ [3], where *rank* is the rank of any adjacency matrix of the graph.

3. α-Lower Bounds

The following bounds are all efficiently computable.

(1) $\alpha \geq \frac{n}{1+\bar{d}}$ [44, 15]. Here $\bar{d}$ is the *average degree* of the graph. This may be the oldest non-trivial bound and is a consequence of the celebrated theorem of Turán.

(2) $\alpha \geq \sum_{v \in V} \frac{1}{1+d(v)}$ [4, 45]. This is the Caro-Wei bound. Here $d(v)$ is the *degree* of vertex v. Griggs improved this bound for triangle-free graphs not including odd paths or cycles [14].

(3) $\alpha \geq \frac{n}{1+\lambda_1}$ [48]. Here λ_1 is the largest eigenvalue of the graph.

(4) $\alpha \geq \left\lceil n - \frac{2e}{(1+\lfloor 2e/n \rfloor)} \right\rceil + \left\lceil \frac{n - \left\lceil n - \frac{2e}{1+\lfloor 2e/n \rfloor} \right\rceil \cdot (1+\lfloor 2e/n \rfloor)}{(2+\lfloor 2e/n \rfloor)} \right\rceil = \left\lceil \frac{2n - \frac{2e}{\lceil 2e/n \rceil}}{\lceil 2e/n \rceil + 1} \right\rceil$ [17].

(5) $\alpha \geq \bar{D}$ [5]. Here $\bar{D}$ is the average distance between distinct vertices of the graph. This bound was conjectured by Graffiti [10].

(6) $\alpha \geq R$ [13]. Here R is the *residue* of the graph, namely the number of zeros remaining after termination of repeated application of the Havel-Hakimi process. This bound was conjectured by Graffiti [10]. Several proofs of this bound have now been published. [15] illuminates the connection between the residue of a graph and applications of the greedy heuristic Maxine.

(7) $\alpha \geq r$ [8, 12]. r is the *radius* of a graph. This bound was conjectured by Graffiti [10]. [11] gives a useful characterization of radius-critical graphs which then led to a characterize those graphs where $\alpha = r$ bt DeLaVina, Larson, Pepper and Waller [7].

(8) $\alpha \geq \alpha_c$ [27]. Here α_c is the *critical independence number* of a graph.

(9) $\alpha \geq \frac{1}{2}[(2e+n+1) - \sqrt{(2e+n+1)^2 - 4n^2}]$ [19].

(10) $\alpha \geq n - 2\mu$. Again μ is the matching number of the graph.
(11) $\alpha \geq \sum_{v \in V} \frac{1}{1+d(v)}(1 + max\{0, \frac{d(v)}{d(v)+1} - \sum_{v \in V} \frac{1}{1+d(v)}\})$ [40]. This is an improvement of the Caro-Wei bound.
(12) $\alpha \geq \frac{n^2}{n(\Delta+1)+(\Delta+1-\lambda_{l1})\max\{(U_1^+)^2,(U_1^-)^2\}}$ [34]. Here $U_1^+ = \min_{(u_j)_i > 0} \frac{1}{(u_j)_i}$ and $U_1^- = \min_{(u_j)_i < 0} \frac{1}{|(u_j)_i|}$. For $1 \leq j \leq n$, where u_j is the normalized eigenvector corresponding to λ_j, $(u_j)_i$ is the i^{th}-entry of u_j.
(13) $\alpha \geq \frac{(CW(G))^2}{CW(G) - \sum_{ij \in E(G)} (d_i - d_j)^2 q_i^2 q_j^2}$ [18]. Here $CW(G)$ denotes the value of the Caro-Wei Bound for a graph G, and $q_i = \frac{1}{1+d_i}$ where d_i is the degree of vertex v_i .
(14) $\alpha \geq \frac{S^2}{S^2+\lambda_1}$ [49]. Here λ_1 be the maximum eigenvalue of the adjacency matrix and S is given by the sum of the entries of the normalized eigenvector corresponding to λ_1.
(15) $\alpha \geq 1 + \frac{C}{2}$ [23]. Here C is the number of *cut vertices* of the graph. This bound was conjectured by Graffiti [10].
(16) $\alpha \geq \max\{e(v) - eh(v)\}$ [10]. Here $e(v)$ is the number of vertices at even distance from vertex v, and $eh(v)$ is the number of *even horizontal* edges with respect to v, that is, the number of edges e where both end-points of e are at even distance from v. This bound was conjectured by Graffiti. This is often a very good bound for regular graphs, a class of graphs where other bounds often fail to predict the independence number. Michelle Grigsby has many interesting results related to this bound, including a polynomial-time characterization of those König-Egerváry graphs where equality holds [16]. The characterization seems to be true for general graphs, but the conjecture is open.
(17) $\alpha \geq \lceil 2\sqrt{n} \rceil - \Delta$ [1]. This is a conjecture of the AGX graph theory conjecture-making software.
(18) $\alpha \geq \frac{n-1}{\Delta}$ (for graphs with at least 5 vertices) [1]. This is a conjecture of the AGX graph theory conjecture-making software.
(19) $\alpha \geq CW + \frac{CW-1}{\Delta(\Delta+1)}$ [36], where CW is the Caro-Wei bound.
(20) $\alpha \geq \frac{2n}{(\bar{d}+1+\frac{2}{n})+\sqrt{(\bar{d}+1+\frac{2}{n})^2-8)}}$ [20], where $\bar{d}$ is the average degree.
(21) $\alpha \geq r + \frac{p}{2} - 1$ [30], where r is the radius, and p is the number of pendants. This is an improvement on the radius lower bound in $p \geq 3$.
(22) $\alpha \geq \frac{n}{\max\{\min\{d_i+1,i\}:i=1,\ldots,i=n\}}$ [46], where the d_i's are the degrees of the vertices. This is the Welsh-Powell bound.
(23) $\alpha \geq \frac{n}{1+\max_{G' \subset G} \delta(G')}$ [42], where the maximum is taken over all subgraphs G' of the given graph G. This is the Szekeres-Wilf bound.

4. Special α-bounds

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The following efficiently computable bounds for the independence number only apply to graphs with special properties. In each case, testing whether a graph has the specified property is itself efficient.

(1) If a graph is triangle-free and $\Delta \geq 4$ then $\alpha \geq \frac{4n}{3}$ [25].
(2) If a graph is regular, then $\alpha \leq \frac{-n\lambda_{min}}{\lambda_{max}-\lambda_{min(A)}}$ [35], where λ_{min} and λ_{max} are the smallest and largest eigenvalues of the graph. The bound is known as the Hoffman-Lovász bound.
(3) If a graph is triangle-free then $\alpha \geq CW + \frac{CW+e-n}{\Delta(\Delta+1)}$ [36], where CW is the Caro-Wei bound.
(4) If a graph is triangle-free and $\Delta \leq 3$ then $\alpha \geq \frac{5n}{14}$ [9, 25, 22, 21].
(5) If the girth of a graph is at least 6 then $\alpha \geq \frac{(2\Delta-1)n}{\Delta^2+2\Delta-1}$ [24].
(6) If a graph is a connected triangle-free graph and neither an odd cycle nor an odd path then $\alpha \leq CW + \frac{n}{\Delta(\Delta+1)}$ [14].
(7) If the girth of a graph is $2k+3$ ($k \geq 2$) then $\alpha \geq 2^{-(\frac{k-1}{k})}[\sum_{v\in V(G)} d(v)^{\frac{1}{k-1}}]^{\frac{k-1}{k}}$ [41].
(8) If the graph is $K_{1,r+1}$-free then $\alpha \leq \frac{1}{2}(2n+2r-1-\sqrt{8e+(2r-1)^2})$ [39].

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