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NUMERICAL SOLUTION OF ABEL'S INTEGRAL EQUATION WITH SPLINE FUNCTIONS

(Numerisk lösning av Abels integralekvation med ri-funktioner)

Bo Einarsson

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Antal blad 22

(English summary is given on page 3)

Sammanfattning

Denna rapport bygger på första delen av ett föredrag med titeln "Användning av ri-funktioner för numerisk lösning av integralekvationer", som hölls av författaren vid NordDATA-70 i Köpenhamn inom grupp L (EDB:s användning inom forskning) onsdagen den 26 augusti 1970.

I december 1970 utgavs FOA 2 rapport C 2446-11 med titeln "Om numerisk beräkning av fourierintegraler", som behandlar den andra delen av föredraget.

Syfte: Att studera olika metoder att lösa Abels integralekvation med hjälp av ri-funktioner.

Metod: Integralekvationen löstes dels med en indirekt metod, dels med en indirekt minsta kvadratmetod.

Resultat: En indirekt minsta kvadratmetod med ett relativt lågt antal delintervall visade sig fungera bäst.

En FORTRAN-rutin för denna metod återfinns i FOA 2 försöksstation programbibliotek och har nr G 132.

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Contents

NUMERICAL SOLUTION OF ABEL'S INTEGRAL EQUATION WITH SPLINE FUNCTIONS

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Abstract: The integral equation of Abel can be solved with the help of spline functions in several ways. In this report it is assumed that the function which is not integrated in the integral equation is given at a finite number of points, not necessarily equidistant. The following three methods are discussed.

It is very natural to approximate the known function by its spline approximation. In the special case studied here, this was not realistic, since this function was not differentiable at one of the boundary points.

Another possibility is to approximate the unknown function with a spline approximation with unknown parameters both for the function values and the moments. From the integral equation we then obtain a system of linear equations, which is solved together with the continuity conditions for the spline approximation.

In the third case a least squares method was added to the indirect method above in order to take care of rounding and measurements error. This method appears to be the best one.

This paper was presented at the Scandinavian computer congress NordDATA-70 in Copenhagen, August 26-28, 1970.

Introduction

The mathematical problem was initiated by the following physical problem. A cylinder-symmetric object is radiated with X-rays from a source at a large distance. A detector is placed behind the object and gives the intensity as a function of the coordinate z, cf. figure 1.

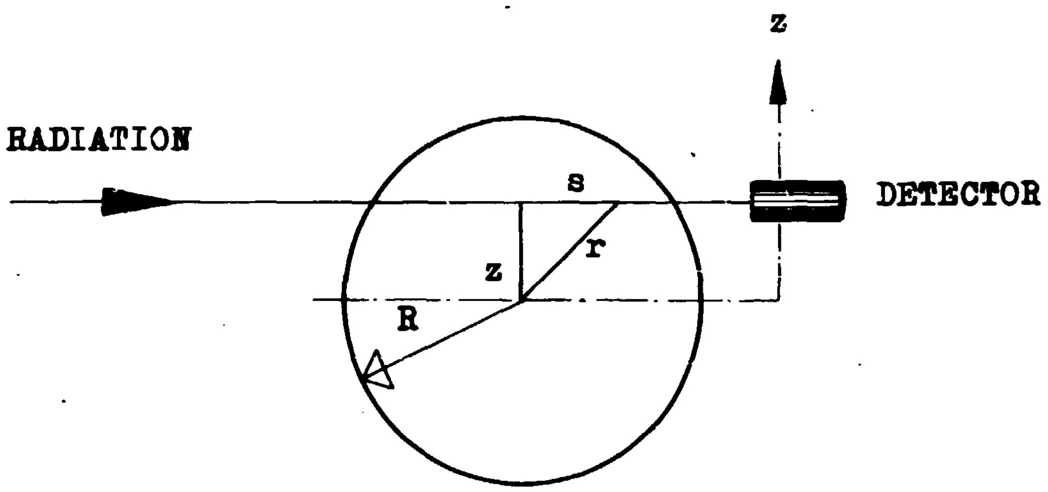

Figure 1

The radiation intensity I after passage of the object depends on the absorption by

$$(1) \qquad I = I_0 \cdot f\left(\int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \rho(r)\, ds \right),$$

where I_0 is the intensity before the passage, $\rho(r)$ is the density at the distance r from the center and s is the distance the radiation passes in the object. We put

$$(2) \qquad \mu(z) = f^{-1}\left(\frac{I}{I_0}\right)$$

and obtain the integral equation

$$(3) \qquad 2\int_z^R \rho(r)\,\frac{r\,dr}{\sqrt{r^2-z^2}} = \mu(z).$$

If we assume that the function $f(\mu)$ has been determined by a measurement process, the problem is reduced to obtaining $\rho(r)$ for known $\mu(z)$.

In order to solve the integral equation we introduce the coordinates

$$(4) \qquad \xi = \sqrt{1-\left(\frac{r}{R}\right)^2}$$

$$(5) \qquad \eta = \sqrt{1-\left(\frac{z}{R}\right)^2}$$

and the functions $F(\xi)$ and $G(\eta)$ by

$$(6) \qquad F(\xi) = R\cdot\rho(r) = R\cdot\rho(R\sqrt{1-\xi^2})$$

$$(7) \qquad G(\eta) = \mu(z) = \mu(R\sqrt{1-\eta^2}).$$

The integral equation (3) is thus transformed to

$$(8) \qquad G(\eta) = \int_0^\eta F(\xi)\,\frac{2\,\xi\,d\xi}{\sqrt{\eta^2-\xi^2}} \quad .$$

We now assume that

$$(9) \qquad G(\eta) = \alpha\eta + g_1(\eta^2),$$

where $g_1(t)$ has a continuous derivative on $[0,1]$, or that

$$(10) \qquad \mu(z) = \alpha \sqrt{1 - \left(\frac{z}{R}\right)^2} + g_2(z),$$

where $g_2(z)$ has a derivative for $0 \leq z \leq R$ and with $g_2'(0) = 0$. Also this derivative is assumed continuous.

We now use the following theorem, which is a slight modification of the theorem of Bôcher (1914).

Theorem. The integral equation of Abel

$$(11) \qquad g(y) = \int_0^y \frac{f(x)}{\sqrt{y-x}}\, dx$$

has the continuous solution

$$(12) \qquad f(x) = \frac{1}{\pi} \int_0^x \frac{g'(y)}{\sqrt{x-y}}\, dy$$

if $g(0) = 0$ and $g(y) = \alpha\sqrt{y} + g_0(y)$, where $g_0(y)$ has a continuous derivative in the considered interval.

The solution of (8) is then given by

$$(13) \qquad F(\xi) = \frac{1}{\pi} \int_0^\xi \frac{G'(\eta)}{\sqrt{\xi^2-\eta^2}}\, d\eta$$

which gives

$$(14) \qquad \rho(r) = -\frac{1}{\pi}\int_r^R \frac{\mu'(z)dz}{\sqrt{z^2-r^2}} \quad .$$

From the symmetry it follows that $\mu'(0) = 0$, and it only remains to discuss the behaviour for $z = R$. If we assume the density $\rho(r)$ to be continuously differentiable we find that the regularity conditions on $\mu(z)$ are satisfied.

Numerical solution with the direct method

For the numerical solution with the first method the interval $[0,R]$ is divided into two parts $[0,R']$ and $[R',R]$ in order to take care of the singularity at $z = R$, where $\mu(z)$ has an infinite derivative.

In the interval $[R',R]$ the density is assumed to be a constant ρ_0. This assumption is valid in the physical example. We obtain for $R' < z < R$

$$(15) \qquad \mu(z) = 2\int_z^R \frac{\rho_0 r\, dr}{\sqrt{r^2-z^2}} = 2\rho_0 \sqrt{R^2-z^2}$$

and

$$(16) \qquad \mu'(z) = -\frac{2\rho_0 z}{\sqrt{R^2-z^2}} \quad .$$

For $0 \leq r < R'$ we get

$$(17) \qquad \rho(r) = \rho_0\left(\frac{1}{2} - \frac{1}{\pi}\arcsin\frac{2R'^2-R^2-r^2}{R^2-r^2}\right) - \frac{1}{\pi}\int_r^{R'} \frac{\mu'(z)}{\sqrt{z^2-r^2}}\, dz .$$

We now introduce the spline approximation of $\mu(z)$ on the interval $[0,R']$. After differentiation, multiplication with the weight function and integration, the solution $\rho(r)$ is obtained.

This method has been documented with complete FORTRAN routines in Einarsson (1968). The disadvantage of the method is the difficulty of obtaining the point R' from the observed values of $\mu(z)$ whereas the results depend strongly on the choice of this value. It is therefore natural to look for a method where the use of two expressions is unnecessary. We obtain such a method by approximating the unknown instead of the known function with its spline fit.

Numerical solution with the indirect method

In the second method we assume that the density $\rho(r)$ is twice continuously differentiable and determine the interpolating spline $S(r)$ to the set (r_i, ρ_i). For the determination of $S(r)$ we need two boundary conditions. For $r = 0$ we use the symmetry to obtain $S'(0) = 0$ and for $r = R$ we have two possibilities, $S'(R) = 0$ for a distinct boundary and $S''(R) = 0$ for a smooth one.

The spline fit is determined by, cf. Ahlberg et al. (1967),

$$(18) \qquad S(x) = M_{i-1}\frac{(x_i-x)^3}{6h_i} + M_i\frac{(x-x_{i-1})^3}{6h_i} + \\ + \left(y_{i-1} - \frac{M_{i-1}h_i^2}{6}\right)\frac{x_i-x}{h_i} + \left(y_i - \frac{M_ih_i^2}{6}\right)\frac{x-x_{i-1}}{h_i}$$

for $x_{i-1} \leq x \leq x_i$ and with $h_i = x_i - x_{i-1}$.

We will also use

$$\mu(z) = 2 \int_z^R \frac{x\, S(x)}{\sqrt{x^2 - z^2}}\, dx \tag{19}$$

to determine y_i and M_i from the values $\mu(z_j)$.

At the numerical solution we consider the case R = 1 and divide the interval $[0,1]$ into N subintervals with partition points $\{x_i\}_0^N$. Two unknown parameters, namely y_i and M_i, belong to each of these points. These are to be interpreted as the parameters in (18) for the spline fit of $\rho(r)$. This equation is substituted into (3), for $z = z_0, z_1, \ldots, z_N$ with the given values of $\mu(z_j)$, and gives (19).

With this method it is natural to choose $z_j = x_j$. We then find that for j = N equation (3) becomes trivial, since $z_N = 1$ and $\mu(1) = 0$. We therefore obtain only N equations from equation (3). We obtain N + 1 additional equations from the continuity conditions of the spline approximation together with its boundary equations. The remaining equation is obtained rather arbitrarily from the condition that $S'''(x)$ is continuous in the last interior point, i.e.

$$S'''(x_{N-1}) = \frac{M_N - M_{N-1}}{h_N} = \frac{M_{N-1} - M_{N-2}}{h_{N-1}} \tag{20}$$

We then have 2(N+1) equations and 2(N+1) unknowns. The formulas for obtaining the linear equation system are rather complicated but the system is easily solved by standard methods. In the test program the SSP routine DGELG, Gauss elimination with complete pivoting, is used, see IBM (1969).

This indirect method showed very good convergence, the error at the computation of $\rho(r)$ being of order h^4, where h is the largest step length h_i. There was, however, also a problem in connection with this metod. For small values of h the round-off error (and the measurement error) in $\mu(z)$ gave oscillations in $\rho(r)$, and $\rho''(r)$ sometimes had alternating signs in the nodes. To take care of this problem a third method, which uses a fit in the least squares sense, has been studied.

Numerical solution with the least squares indirect method

The main difference between this and the previous method is that we no longer take $z_j = x_j$, but instead use a much denser distribution of z_j. The number of points z_j is indicated by L and that of points $\left\{x_i\right\}_0^N$ is still N + 1. We now have 2(N + 1) unknown parameters, the first N + 1 equations are obtained from the continuity of the spline approximation and the following L > N + 1 equations are the discrete counterparts of (19). We thus have an overdetermined system of equations and solve this with the method of Björck and Golub (1967), where the first $m_1 = N + 1$ equations are satisfied exactly and the following $m - m_1 = L$ equations are satisfied in the least squares sense. The number of unknowns is $n = 2(N + 1)$. The original algorithm by Björck and Golub is in Algol, but here I have used a FORTRAN version by Roy H. Wampler, National Bureau of Standards. This routine is written in single precision with the inner loop of the iteration (accumulation of inner products) in double precision. The test computations showed that the single precision of IBM with only six decimal digits is insufficient. The routine was therefore changed to double precision (16 decimal digits) and the accumulation of inner products was improved by adding the products in increasing order, starting with the smallest product. This course gives a higher precision than the random order,but the use of multiple precision accumulation would be advantageous.

The problem with oscillations in $\rho(r)$ was much less pronounced with this method.

Description of the numerical tests

In order to test the two algorithms we put

$$(21) \qquad \rho(r) = a + br + cr^2 + dr^3 + er^4.$$

The two boundary conditions $\rho'(0) = \rho'(1) = 0$ give

$$(22) \qquad b = 0$$

and

$$(23) \qquad 2c + 3d + 4e = 0.$$

Substituting (21) in (3) gives

$$(24) \qquad \mu(z) = \left\{2a + \frac{2}{3}c + \frac{d}{2} + \frac{2}{5}e + z^2\left(\frac{4}{3}c + \frac{3}{4}d + \frac{8}{15}e\right) + \frac{16}{15}z^4 e\right\}\sqrt{1 - z^2} + \frac{3}{4}dz^4 \ln\frac{1+\sqrt{1-z^2}}{z}.$$

We find immediately that this expression satisfies $\mu(1) = 0$ and $\mu'(0) = 0$. In the numerical tests the interval $[0,1]$ was divided into N equal subintervals of length $h = 1/N$, but the routines were written for the general case.

All test computations were performed on an IBM 360/75 in double precision (16 significant figures). The values of $\mu(z)$ from (24) were used as input in (19) and the obtained values $\{y_i\}$ and $\{M_i\}$ of the spline approximation of $\rho(r)$ was the output of the computations.

The first test that was performed was a = 1 and b=c=d=e=0, giving $\rho(r) = 1$ and $\mu(z) = 2\sqrt{1 - z^2}$. In this simple case the computations gave exact results, as was to be expected. The next test was c = 1 and a=b=d=e=0, giving $\rho(r) = r^2$ and $\mu(z) = \frac{2}{3}(1 + 2z^2)\sqrt{1 - z^2}$. In this case we obtained reasonable convergence for $\rho(r)$, but rather bad results for $\rho''(r)$, since equation (23), which is concerned with the condition $\rho'(1) = 0$, was not satisfied. Both these tests were performed for both the indirect method and the least squares method.

We now study the indirect method in the case with no round-off error in the given values of $\mu(z)$.

In the tables below we give the errors for the function $\rho(r)$ and its second derivative with four different stepsizes $h = 1/N$ when a = 0.5, b = 0, c = -0.5, d = -1 and e = 1.

r	$\rho(r)$	Error in computed $\rho(r)\cdot 10^6$			
		N = 5	N = 10	N = 15	N = 20
0.0	0.500000	-55	5	1	0
0.2	0.473600	175	4	1	0
0.4	0.381600	-229	5	0	0
0.6	0.233600	402	10	1	0
0.8	0.077600	-830	25	-3	1
1.0	0.000000	1552	99	20	6

Table 1 a

r	$\rho''(r)$	Error in computed $\rho''(r)\cdot 10^6$			
		N = 5	N = 10	N = 15	N = 20
0.0	-1.000000	-33575	-21357	-9110	-5138
0.2	-1.720000	-138407	-21560	-8958	-5004
0.4	-1.480000	12253	-24362	-8524	-5039
0.6	-0.280000	-235486	-32843	-10767	-5347
0.8	1.880000	170342	-58295	822	-8148
1.0	5.000000	-383829	-96422	-42907	-24149

Table 1 b

These tables indicate that the error of the computed $\rho(r)$ is of order h^4 in accordance with the convergence of the spline approximation, see Hall (1968), the error of $\rho''(r)$ is of the order h^2.

It is very natural to try a single Richardson extrapolation of the values in table 1 in order to obtain smaller errors. Such an extrapolation did not give higher accuracies at all the considered points. This is probably due to an oscillating component in the computed solution. By a simple smoothing process we can reduce the oscillating component,and perform extrapolation of the smoothed values. Some tests have indicated that this is a possible way of increasing the accuracy.

It is also important to study the behaviour of the computed solution when the primary data (in this case $\mu(z)$) have round-off or measurement errors. This is simulated by adding a random function (with values between ϵ and $-\epsilon$). The random function is such that repeated calculations give the same value relative to ϵ.

From the tables below we find that the solution is very sensitive to this disturbance. The tables are for $a = 1$, $b = 0$, $c = 2$, $d = -2$ and $e = 0.5$ and $h = 1/10$.

r	ρ(r)	Error in computed ρ(r)·10^6 ϵ = 0.0001	ϵ = 0.001	ϵ = 0.01
0.0	1.000000	-264	-2668	-26701
0.2	1.064800	-383	-3849	-38505
0.4	1.204800	-99	-1014	-10169
0.6	1.352800	-102	-1069	-10732
0.8	1.460800	-142	-1528	-15395
1.0	1.500000	204	1594	15494

Table 2 a

r	ρ"(r)	Error in computed ρ"(r) ϵ = 0.0001	ϵ = 0.001	ϵ = 0.01
0.0	4.00	0.13	1.37	13.78
0.1	2.86	-0.18	-1.71	-17.06
0.2	1.84	0.25	2.62	26.27
0.3	0.94	-0.29	-2.85	-28.40
0.4	0.16	0.19	2.01	20.17
0.5	-0.50	-0.15	-1.43	-14.24
0.6	-1.04	0.09	1.01	10.27
0.7	-1.46	-0.08	-0.77	-7.69
0.8	-1.76	0.03	0.57	6.01
0.9	-1.94	0.02	0.06	0.37
1.0	-2.00	0.10	-0.58	-5.38

Table 2 b

These tables show that the error is proportional to the magnitude of the random disturbance and that oscillations in the second derivative (or spline moments) are obtained.

We now turn to the indirect least squares method and first consider the same test example as in the tables 1 . We thus have a = 0.5, b = 0, c = -0.5, d = -1 and e = 1. We use the step length h = 1/N and L = 30 (the number of measurement points).

r	ρ(r)	Error in computed $\rho(r)\cdot 10^6$			
		N = 5	N = 10	N = 15	N = 20
0.0	0.500000	-809	6	1	0
0.2	0.473600	483	5	1	0
0.4	0.381600	-543	6	1	0
0.6	0.233600	589	11	1	0
0.8	0.077600	-303	19	0	0
1.0	0.000000	521	19	0	0

Table 3 a

r	ρ"(r)	Error in computed $\rho''(r)\cdot 10^6$			
		N = 5	N = 10	N = 15	N = 20
0.0	-1.000000	103582	-23106	-9017	-4173
0.2	-1.720000	-253282	-22938	-8885	-4720
0.4	-1.480000	81714	-25297	-8849	-4854
0.6	-0.280000	-229852	-32011	-9069	-4885
0.8	1.880000	54081	-35870	-8883	-4803
1.0	5.000000	-208904	-33420	-5087	-4321

Table 3 b

A comparison between tables 1 and 3 shows that all entries are of the same order, but that the least squares method gives smaller errors, except with the coarsest partitition. As in the previous method we can improve the solution by Richardson extrapolation after smoothing.

We will now study the indirect least squares method and the influence of random terms. We use the parameters a = 1, b = 0, c = 2, d = -2, and e = 0.5.

Error in computed $\rho(r) \cdot 10^6$

r	$\rho(r)$	N = 10 L = 15	N = 10 L = 60	N = 20 L = 30	N = 20 L = 60
0.0	1.000000	-1	4	0	0
0.2	1.064800	1	3	0	0
0.4	1.204800	1	4	0	0
0.6	1.352800	1	8	0	0
0.8	1.460800	0	17	0	0
1.0	1.500000	1	19	0	1

Table 4 a

Error in computed $\rho''(r) \cdot 10^4$

r	$\rho''(r)$	N = 10 L = 15	N = 10 L = 60	N = 20 L = 30	N = 20 L = 60
0.0	4.0000	-83	-126	-21	-25
0.2	1.8400	-90	-125	-24	-25
0.4	0.1600	-93	-144	-24	-25
0.6	-1.0400	-94	-202	-24	-27
0.8	-1.7600	-86	-266	-24	-38
1.0	-2.0000	-85	-275	-22	-42

Table 4 b

We find that we can use a rather limited number of nodes against a larger number of measurement points z_j.

It is interesting to note that test computations with equation (20) included among the equations to be satisfied exactly, gave a significant decrease in the accuracy.

The slight decrease in the accuracy when L is increased is partly due to increased oscillations.

We now study the behaviour of the solution when a random term is present. The same parameters as above are used and $h = 1/10$ and $L = 30$.

r	$\rho(r)$	Error in computed $\rho(r) \cdot 10^6$ $\epsilon = 0.0001$	$\epsilon = 0.001$	$\epsilon = 0.01$
0.0	1.000000	-121	-1236	-12387
0.1	1.018050	-250	-2507	-25079
0.2	1.064800	111	1089	10865
0.3	1.130050	-11	-107	-1071
0.4	1.204800	11	83	797
0.5	1.281250	-97	-958	-9568
0.6	1.352800	21	160	1546
0.7	1.414050	-24	-169	-1620
0.8	1.460800	-13	-218	-2262
0.9	1.490050	67	682	6835
1.0	1.500000	-51	-594	-6027

Table 5 a

r	$\rho''(r)$	Error in computed $\rho''(r)$ $\epsilon = 0.0001$	$\epsilon = 0.001$	$\epsilon = 0.01$
0.0	4.00	-0.11	-1.03	-10.24
0.1	2.86	0.12	1.28	12.83
0.2	1.84	-0.13	-1.21	-11.96
0.3	0.94	0.05	0.62	6.23
0.4	0.16	-0.06	-0.48	-4.73
0.5	-0.50	0.05	0.52	5.30
0.6	-1.04	-0.05	-0.38	-3.63
0.7	-1.46	0.00	0.06	0.59
0.8	-1.76	0.01	0.25	2.71
0.9	-1.94	-0.06	-0.57	-5.65
1.0	-2.00	0.05	0.65	6.67

Table 5 b

We see that the error is proportional to the magnitude of the random error.

In the following table we give the corresponding results for L = 60.

r	$\rho(r)$	Error in computed $\rho(r)\cdot 10^6$		
		ϵ = 0.0001	ϵ = 0.001	ϵ = 0.01
0.0	1.000000	-105	-1091	-10952
0.1	1.018050	-126	-1258	-12576
0.2	1.064800	57	538	5354
0.3	1.130050	-6	-55	-541
0.4	1.204800	9	51	472
0.5	1.281250	-53	-499	-4961
0.6	1.352800	19	113	1050
0.7	1.414050	-26	-145	-1339
0.8	1.460800	14	-19	-351
0.9	1.490050	15	245	2544
1.0	1.500000	-2	-186	-2026

Table 6 a

r	$\rho''(r)$	Error in computed $\rho''(r)$		
		ϵ = 0.0001	ϵ = 0.001	ϵ = 0.01
0.0	4.00	-0.04	-0.32	-3.04
0.1	2.86	0.04	0.50	5.08
0.2	1.84	-0.07	-0.57	-5.62
0.3	0.94	0.02	0.30	3.04
0.4	0.16	-0.04	-0.26	-2.46
0.5	-0.50	0.03	0.28	2.87
0.6	-1.04	-0.04	-0.24	-2.22
0.7	-1.46	0.01	0.10	0.92
0.8	-1.76	-0.02	0.03	0.50
0.9	-1.94	-0.01	-0.18	-1.83
1.0	-2.00	0.00	0.20	2.27

Table 6 b

We thus find that the error from the random term is reduced when more points are used as measurement points. Note however that the random error occurs only in certain points. The measurement points are given by $z_j = j/L, (j=0,1,2,\ldots,L-1)$ and the random error is included only for $j = 0(\frac{L}{15})\ L-1$. This means that the same random error is used independently of whether L = 15, 30 or 60. The solution of the problem is however different in these three cases, since the values of $\mu(z)$ used at the other points are those corresponding to no random error. If tables 5 and 6 are compared it is therefore necessary to remember that they correspond to different functions $\mu(z)$.

In order to understand the influence of the random error it is necessary to consider the function $\mu(z)$. This function is given in table 7 in the case of no random error and a = 1, b = 0, c = 2, d = -2, and e = 0.5.

z	$\mu(z)$
0.0	2.533333
0.1	2.534500
0.2	2.533659
0.3	2.521058
0.4	2.484374
0.5	2.409660
0.6	2.281192
0.7	2.079631
0.8	1.775602
0.9	1.303108
1.0	0.000000

Table 7

From this table we understand that an error of ±0.001 has a great influence on the solution. The reason we have studied an absolute error, constant in magnitude over z, is that this gives a good simulation of the background radiation and the calibration errors.

Finally, we want to point out that decreasing the step length h in this case with random error increases the oscillations. In the test case, N = 10 and L ≐ 30 gave slighter higher accuracy than N = 20 and L = 60. It is therefore recommended to use a small number of interpolation nodes (small N) and a high number of measurement points (large L).

We finally give the results with three different step sizes for the same parameters as above, the number of measurement points L equal to 60 and $\epsilon = 0.001$.

r	$\rho(r)$	Error in computed $\rho(r)\cdot 10^6$		
		N = 5	N = 10	N = 20
0.0	1.000000	-2158	-1091	-4113
0.2	1.064800	459	538	1617
0.4	1.204800	-484	51	-565
0.6	1.352800	220	113	644
0.8	1.460800	-152	-19	-198
1.0	1.500000	301	-186	307

Table 8 a

r	$\rho''(r)$	Error in computed $\rho''(r)$		
		N = 5	N = 10	N = 20
0.0	4.00	0.28	-0.32	3.74
0.2	1.84	-0.29	-0.57	-4.53
0.4	0.16	0.11	-0.26	1.62
0.6	-1.04	-0.14	-0.24	-2.38
0.8	-1.76	0.03	0.03	0.32
1.0	-2.00	-0.11	0.20	-1.98

Table 8 b

That the use of a least squares method to smooth a curve requires many sample points is a wellknown fact. It is often difficult to determine such parameters as the order of the least squares polynomial or the number and locations for the nodes of the spline fit in advance. The use of interactive systems for solving problems of this type is therefore recommended, see for example two recent articles by Lyle B. Smith (Oct. and Dec. 1970). Another useful article is that of Horsley et al. (1968).

Conclusions

The investigations on the numerical solution of Abel's integral equation reported here show that the spline function is a powerful tool. Three methods are considered: the direct method (approximation of the measured function with a spline fit), the indirect method (approximation of the unknown function with a spline fit), and the indirect least squares method. The test computations and the theoretical considerations have shown that the last method is superior.

The FORTRAN listing for this method is available from the author.

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