UNIVERSITY OF STOCKHOLM

INSTITUTE OF PHYSICS

REPORT

RELATIVISTIC KINEMATICS FOR STORAGE RINGS

by

P.J. Carlson and O. Leistam

USIP-REPORT 71-11 December 1971

CONTENTS

1. INTRODUCTION

The knowledge of relativistic kinematics is necessary for the study of elementary particle reactions. This subject has been treated by several authors [1]. In this note we treat the special features of the relativistic transformations appropriate to the CERN intersecting storage rings (ISR), where two proton beams are brought to collide at an angle of 15°. In section 2 the notations are described. Section 3 gives the appropriate transformation equations. Section 4 gives some results for the case of elastic scattering.

2. NOTATIONS

P	Energy - momentum four vector $P^2 = \vec{p}^{\,2} - E^2 = - m^2$
$\vec{p}$	momentum
p	absolute value of momentum
E	total energy
$\vec{\beta} c$	velocity of the CM-system in the laboratory
γ	$(1-\beta^2)^{-\frac{1}{2}}$
$*$	superscript, indicates CM-system. No superscript indicates laboratory system
$F(X,Y,Z) =$	$X^2 + Y^2 + Z^2 - 2XZ - 2YZ - 2\,XY$
λ	azimuthal angle in the CM-system
s	$-(P_a + P_b)^2$ = total energy in the CM-system squared $s = m_a^2 + m_b^2 + 2\,E_a E_b - 2\,\vec{p}_a \vec{p}_b$
t_{ac}	$-(P_a - P_c)^2$ = four momentum transfer squared $t = m_a^2 + m_c^2 - 2\,E_a E_c + 2\,\vec{p}_a \vec{p}_c$

3. TRANSFORMATION EQUATIONS

At the ISR two particles a and b are brought to collide at an angle α. As an example we take a reaction

$$a + b \rightarrow c + d$$

This is illustrated in Fig. 1. At the ISR $\alpha = 15^{o}$.

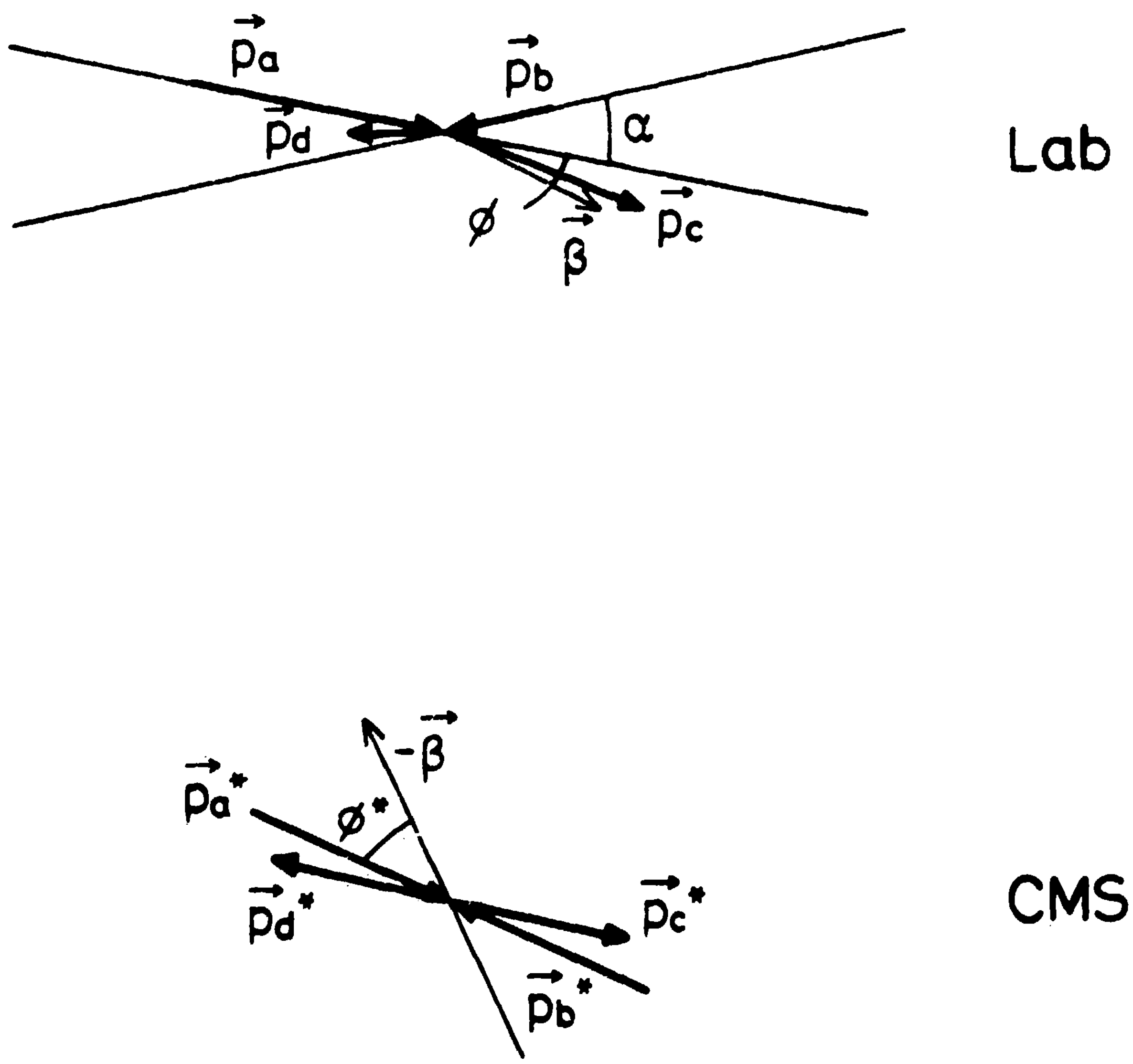

Fig. 1

3.1 Velocity of the CM frame

The velocity of the CM frame is given by

$$\vec{\beta} = \frac{\vec{p}_a + \vec{p}_b}{E_a + E_b} \tag{1}$$

$$\beta^2 = \frac{p_a^2 + p_b^2 - 2p_a p_b \cos\alpha}{(E_a + E_b)^2} \tag{2}$$

The angles ϕ and ϕ^* are given by

$$\tan\phi = \frac{\sin\alpha}{\frac{p_a}{p_b} - \cos\alpha} \tag{3}$$

$$\tan\phi^* = \gamma^{-1}\left(1 - \frac{\beta E_a}{p_a \cos\phi}\right)^{-1} \tan\phi \tag{4}$$

Numerical example

Table 1. p_a = 25 GeV/c

p_b GeV/c	25	20	15	10	5
ϕ degrees	82.50	42.34	20.27	9.58	3.67
ϕ^* degrees	90.00	49.42	26.82	15.38	8.28
β	0.130	0.171	0.280	0.444	0.671

3.2 The momentum in the CM system

The transformation of p_a and p_b to the CM system is easily deduced from the definition of s, and takes this form:

$$p^{*2}_{a,b} = \frac{1}{4\,s}\,F(s, m^2_a, m^2_b) \qquad (5)$$

where F is defined in section 2.

3.3 Transformation formulas

In this section we give a set of transformation formulas for an arbitrary momentum vector.

3.3.1 Transformation from CM to laboratory

In the CM system the particle is emitted with a momentum p^*_c at an angle θ^*_c with respect to $\vec{p}^{\,*}_a$. We introduce the CM azimuthal angle λ^* according to Fig. 2. Particle c emitted in the plane defined by $\vec{p}^{\,*}_a$, $\vec{\beta}$ corresponds to $\lambda = \pi/2$ and $\lambda = 3\,\pi/2$.

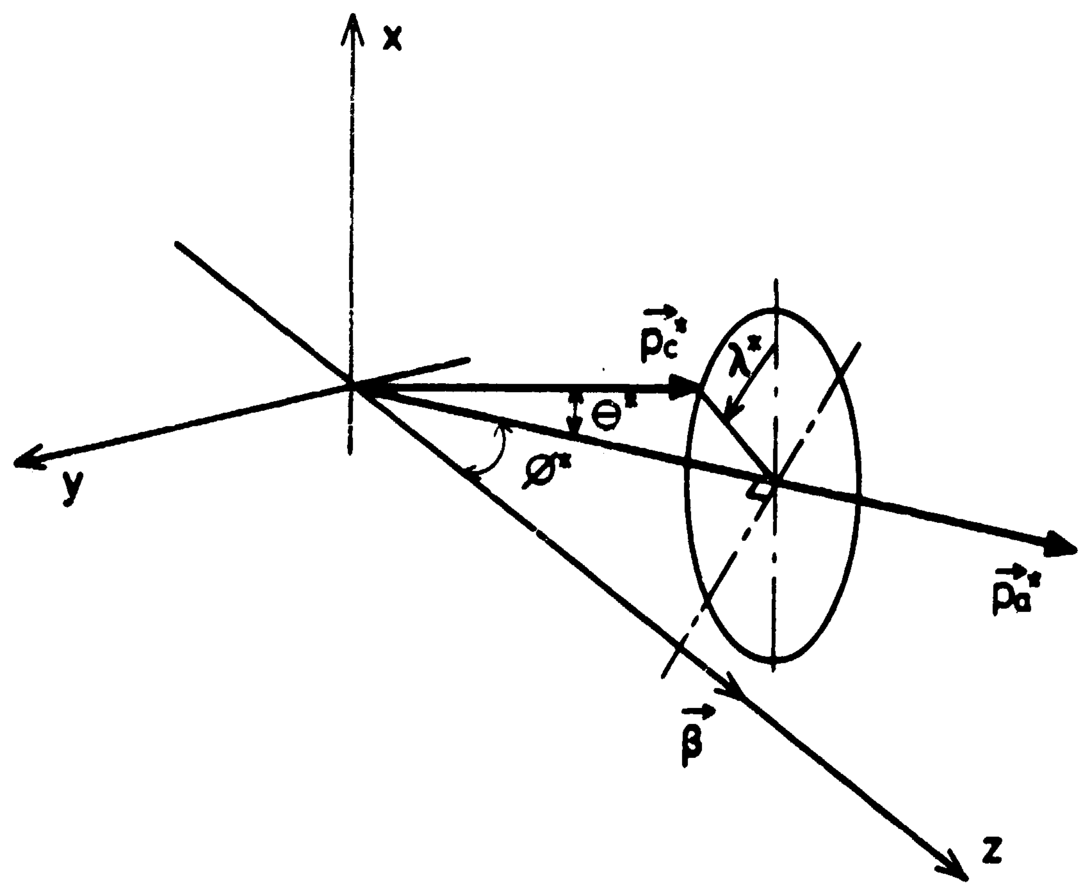

Fig. 2

To perform the transformation we introduce a coordinate system with the z-axis along the direction of $\vec{\beta}$ and with $\vec{p}_a^{\,*}$ in the yz-plane (Fig. 2).

We get ($p^* = p_c^*$)

$$(6)\quad \begin{cases} p_z^* = p^* \cos\theta^* \cos\phi^* + p^* \sin\theta^* \sin\lambda^* \sin\phi^* \\ p_y^* = p^* \cos\theta^* \sin\phi^* - p^* \sin\theta^* \sin\lambda^* \cos\phi^* \\ p_x^* = p^* \sin\theta^* \cos\lambda^* \end{cases}$$

Transformation to the laboratory, rembering the p_z^* is parallel to $\vec{\beta}$ and that p_y^* and p_x^* both are perpendicular to $\vec{\beta}$, gives

$$(7)\quad \begin{cases} p_z = \gamma p^* (\cos\theta^* \cos\phi^* + \sin\theta^* \sin\lambda^* \sin\phi^*) + \gamma\beta E^* \\ p_y = p^* \cos\theta^* \sin\phi^* - p^* \sin\theta^* \sin\lambda^* \cos\phi^* \\ p_x = p^* \sin\theta^* \cos\lambda^* \end{cases}$$

Notice that all three components of p depend upon the CM azimuthal angle λ^*. From equation (7) one gets the laboratory momentum p:

$$(8)\qquad p^2 = \gamma^2 [E^* + \beta p^* (\sin\theta^* \sin\lambda^* \sin\phi^* + \cos\theta^* \cos\phi^*)]^2 - m^2$$

Special case $p_a = p_b$, $\phi^* = \pi/2$

In this simple case we get the following expression for p

$$(9)\qquad p^2 = \gamma^2 [E^* + \beta p^* \sin\theta^* \sin\lambda^*]^2 - m^2$$

p has a maximum for $\lambda^* = \pi/2$ and a minimum for $\lambda^* = 3\pi/2$. As a numerial example, we give in table 2 a few values for the case of elastic scattering

Table 2. $p_a = p_b$ = 25 GeV/c, proton-proton elastic scattering

$-t$ (GeV/c)^2	1	5	10	50	100	1000
θ^* degrees	2.31	5.17	7.31	16.40	23.28	79.27
p_{max} GeV/c	25.13	25.29	25.41	25.92	26.29	28.20
p_{min} GeV/c	24.87	24.71	24.58	24.08	23.71	21.79

Notice that for production of a particle at $\phi^* = 90^\circ$ and $\lambda^* = \pi/2$ we get the usual formula

(10) $$p = \gamma(p^* + \beta E^*)$$

Therefore, although β is small (β = 0.13), for particles with low CM-momenta p^*, the term βE^* gives a large contribution to p.

The laboratory angle θ is derived from equations (7):

(11) $$\cos\theta = \frac{1}{p}\left[\gamma p^* \cos\phi\,(\cos\theta^* \cos\phi^* + \sin\theta^* \sin\lambda^* \sin\phi^*) + \gamma\beta E^* \cos\phi + p^* \sin\phi(\cos\theta^* \sin\phi^* - \sin\theta^* \sin\lambda^* \cos\phi^*)\right]$$

where p^*, p, ϕ^* and ϕ are given by equations 5, 7, 4 and 3 respectively,

Special case $p_a = p_b$

(12) $$\cos\theta = \frac{1}{p}\left[p^* \gamma \sin\theta^* \sin\lambda^* \cos\phi + \gamma\beta E^* \cos\phi + p^* \cos\theta^* \sin\phi\right]$$

where $\phi = 82.5^\circ$ and p^* and p are given by equations 5 and 7 respectively.

3.3.2 Transformation from laboratory to CM

With the notations of Fig. 2, except that all variables are in the laboratory system, we get the equation corresponding to (7) for the CM-momenta.

$$(13)\quad \begin{cases} p_z^* = \gamma p\,(\cos\theta\cos\phi + \sin\theta\sin\lambda\sin\phi) - \gamma\beta E \\ p_y^* = p\cos\theta\sin\phi - p\sin\theta\sin\lambda\cos\phi \\ p_x^* = p\sin\theta\cos\lambda \end{cases}$$

The CM-momentum p^* is given by an expression similar to (8):

$$(14)\quad p^{*2} = \gamma^2\,[\,E - \beta p\,(\sin\theta\sin\lambda\sin\phi + \cos\theta\cos\phi)\,]^2 - m^2$$

The angle θ^* is given by an expression similar to (9)

$$(15)\quad \cos\theta^* = \frac{1}{p^*}\,[\,\gamma p\cos\phi^*(\cos\theta\cos\phi + \sin\theta\sin\lambda\sin\phi) - \gamma\beta E\cos\phi^* - p\sin\phi^*(\cos\theta\sin\phi - \sin\theta\sin\lambda\cos\phi)\,]$$

4. ELASTIC SCATTERING WITH $p_a = p_b$

We now consider the case of elastic scattering (all masses are equal) and with $p_a = p_b$. In Fig. 3 is illustrated how, for given CM scattering angle θ^*, the momentum vectors lie on a circular cone with a component $p^* \sin \theta^*$ parallel to $\vec{\beta}$.

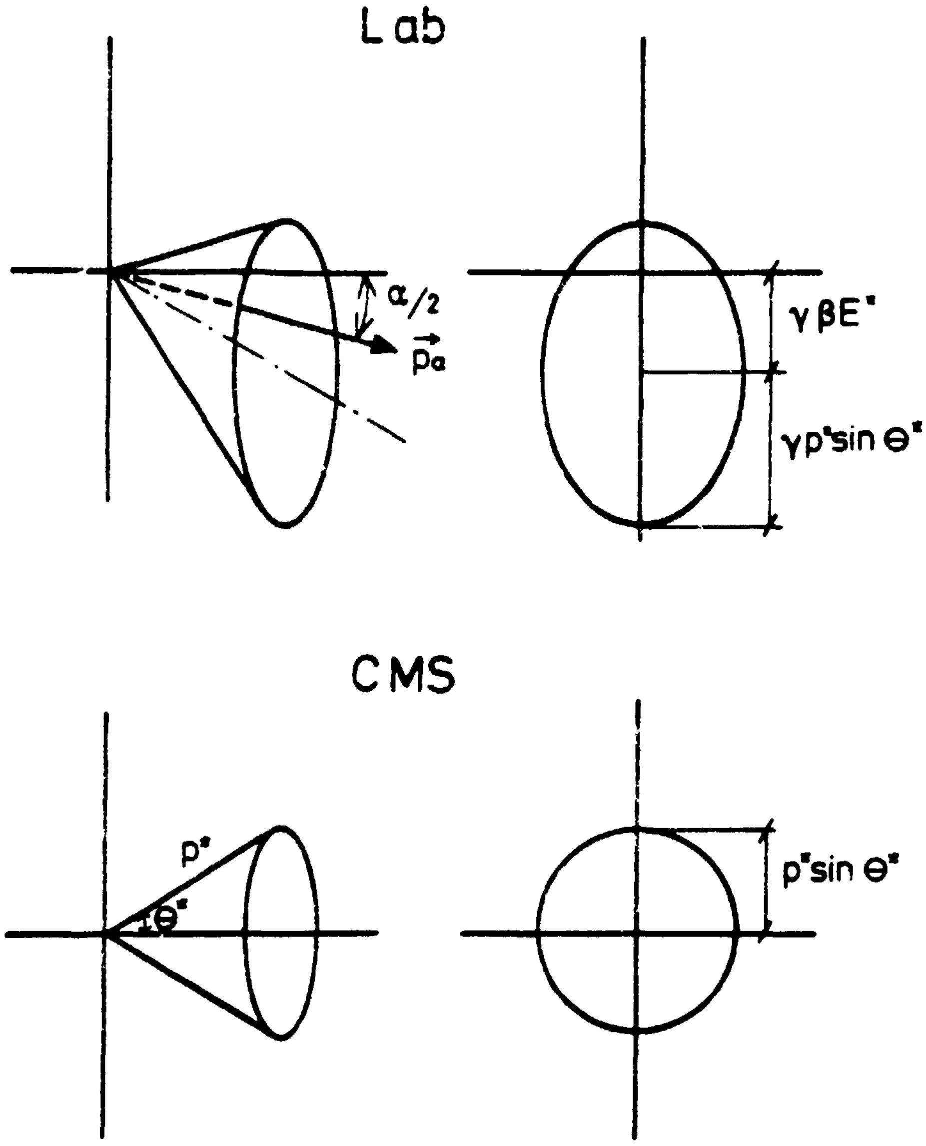

Fig. 3

In the laboratory this becomes a tilted cone and the locus of the momentum vectors is an ellips with half axis $\gamma p^* \sin \theta$ respectively $p^* \sin \theta^*$.

The angle between the axis of the tilted laboratory cone and the direction of $\vec{p}_a$ - also the direction of the vacuum tube - is given by

$$\delta = \frac{\alpha}{2} + \arctan \frac{\gamma \beta E^*}{p^* \cos \theta^*} \tag{16}$$

This is illustrated in Fig. 4.

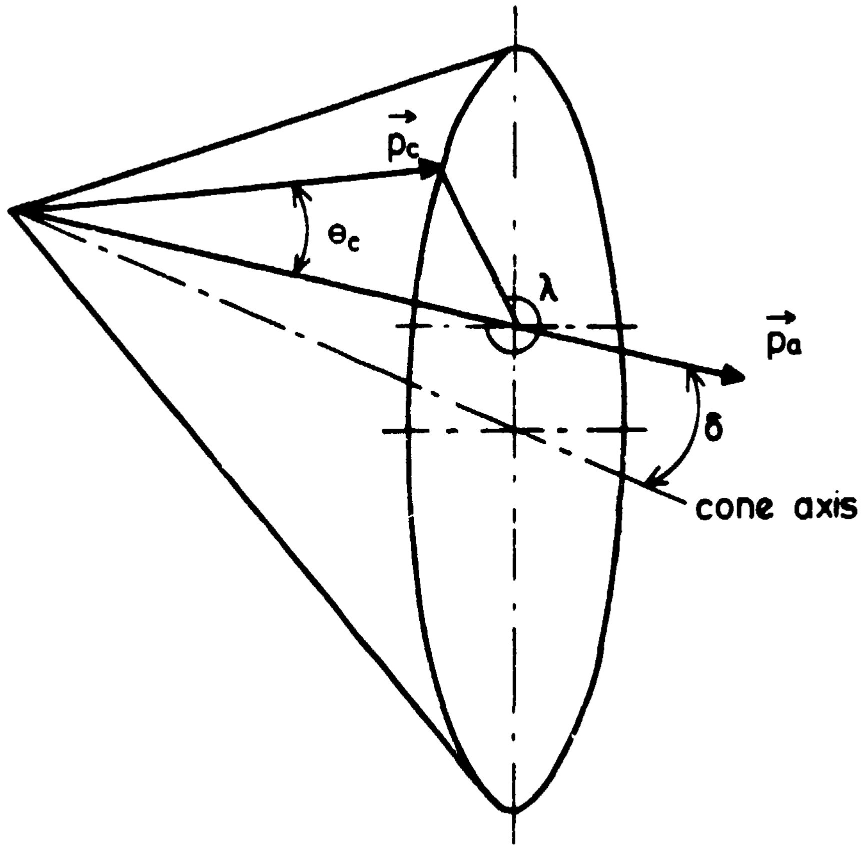

Fig 4

The expression for the laboratory scattering angle θ, formula (11) shows that θ has a minimum for $\lambda = \pi/2$ and a maximum for $\lambda = 3\,\pi/2$. Table 3 gives θ and δ for some different t-values.

Table 3. $p_a = p_b = 25$ GeV/c, proton proton elastic scattering

All angles are given in degrees.

$-t$ $(GeV/c)^2$	θ^*	θ_{min}	θ_{max}	δ
0.5	1.63	1.60	1.62	0.003
1.0	2.31	2.27	2.30	0.006
2.0	3.27	3.21	3.25	0.012
3.0	4.00	3.94	3.99	0.018
4.0	4.62	4.55	4.61	0.024
5.0	5.17	5.08	5.16	0.030
10.0	7.31	7.18	7.31	0.061
50.0	16.40	15.96	16.57	0.314

The difference between θ_{max} and θ_{min} amounts to 1 mrad for $t = 3(GeV/c)^2$.

REFERENCES

1. See for example

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