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ON THE FIERZ-PAULI EQUATION FOR PARTICLES WITH SPIN 3/2

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The following discussion is motivated by recent work on higher spin equations, in particular by that of Velo and Zwanziger[1]. The subject of their investigation are relativistic wave equations which can be brought into the general form

$$(\beta^\mu \partial_\mu + m + B(x))\,\Psi(x) = 0 \qquad (1)$$

The β^μ's $(\mu = 0,1,2,3)$ are nxn-matrices and $B(x)$ a matrix-valued function of $x \in R^4$. The operator $L_B = \beta^\mu \partial_\mu + m + B(x)$ is constructed so that for $B = 0$ the family of solutions Ψ of (1) carries a Hilbert space structure with an irreducible representation of the inhomogeneous Lorentz group -or rather its universal covering group i SL(2,C)- characterized by a positive mass m and a spin s. It has been known[2] since the earliest studies of relativistic wave equations, that this property together with a stability condition[3] makes the Cauchy problem with data on $x^0 = 0$ for L_B particularly nasty from a mathematical point of view. One aspect of the trouble with L_B comes from the possible singularity of the matrix β^0

$$\det \beta^0 = 0 . \qquad (2)$$

This gives rise to slogans as "the plane $x^0 = 0$ is characteristic" or "L_B is singular".

The difficulty with the Cauchy problem originating in (2) has been circumvented by Velo and Zwanziger for several wave equations with the help of the following trick : Instead of the original equation (1), they consider a new set of equations of the form

$$(\gamma^\mu(x)\partial_\mu + m + C(x))\,\Psi(x) = 0 \qquad (3)$$

with properties

$\gamma^\mu(x)$ are matrix valued functions of $x \in R^4$. (4a)

$\det \gamma^0(x) \neq 0$, $\forall\ x \in R^4$. (4b)

Each solution of (1) is a solution of (3) . (4c)

Each solution (3) , if a solution of (1) for $x^0 = 0$, is a solution of (1) . (4d)

This method avoids the difficulty with the original equations mentioned earlier (2) because of (4b) . However there remain some unsatisfactory aspects even on this formal level :

The construction of the γ^{μ}'s seem quite arbitrary. (5a)

The consistency relation (4d) is in general hard to prove[4]. (5b)

Yet another difficulty with this method arises if we leave the formal level and look at the Cauchy problem for the partial differential operators L_B and $M_C = \gamma^{\mu}(x)\partial_{\mu} + m + C(x)$ in a function space. We are going to show that M_C does not belong in general to eigher one of the classes of partial differential operators -they will be specified latter- for which strong results are known about existence and uniqueness of the Cauchy problem. To be more specific we consider the Fierz-Pauli equation for $s = 3/2$ with minimal electromagnetic interaction. For convenience we use the formalism of Rarita and Schwinger[5]. The wave equation is given by [6] :

$$(L(D)\psi)_{\mu} = (\not{D} + m)\psi_{\mu} - (\gamma_{\mu}D^{\nu} + D_{\mu}\gamma^{\nu})\psi_{\nu} - \gamma_{\mu}\not{D}\gamma^{\nu}\psi_{\nu} + m\gamma_{\mu}\gamma^{\nu}\psi_{\nu} = 0 \quad (6)$$

Velo and Zwanziger proposed two possible M's for this particular case:

$$(M_1\psi)_{\mu} = (\not{D} + m)\psi_{\mu} - g\,(D_{\mu} + \tfrac{1}{2}m\gamma_{\mu})\gamma_5\,\gamma\cdot\tilde{F}\cdot\psi$$

$$\begin{aligned}(M_2\psi)_{\mu} &= (\not{D} + m)\psi_{\mu} - g\,(D_{\mu} + \tfrac{1}{2}m\gamma_{\mu})\gamma_5\,\gamma\cdot\tilde{F}\,\psi - \\ &\quad - g\tilde{F}_{\mu}\cdot\gamma\,\gamma_5(D^{\nu} + \tfrac{1}{2}m\gamma^{\nu})\psi_{\nu} - g^2\tilde{F}_{\mu}\cdot\gamma\,\gamma_5(\not{D} - 2m)\gamma_5\,\gamma\cdot\tilde{F}\cdot\psi\end{aligned} \quad (7)$$

The partial differential operators of the first class have strictly hyperbolic determinants[7]. In our case the determinants can both be calculated :

$$\det M_1 = \left\{(p^2 - m^2)^2 + \frac{4p^2}{9m^4}(p\tilde{F})^2\right\}(p^2 - m^2)^6 \quad (8)$$

$$\det M_2 = \left\{(p^2 - m^2)^2 + \frac{4p^2}{9m^4}(p\hat{F})^2\right\}^2(p^2 - m^2)^4 \quad (9)$$

Both determinants contain several times the factor $(p^2 - m^2)$ and are therefore not strictly hyperbolic. The operators of the second class are of the general form $T^{\mu}(x)\partial_{\mu} + S(x)$ where the $T^{\mu}(x)$'s are matrix valued functions hermitian with respect to a positive definite scalar product[8].

The positive definitness is essential for derivation and application of an energy inequality which is at the heart of existence and uniqueness proof for the Cauchy problem within this class. Of the two operators M_1 and M_2 the second comes near to belonging this class because it is hermitian (for real external field) with respect to the scalar product

$$\langle \phi, \psi \rangle = \bar{\phi}_\mu \gamma^0 g^{\mu\nu} \psi_\nu \tag{10}$$

in the 16-dimensional vector space $C^4 \otimes C^4$. However this scalar product is not positive definite[9]. This indicates that the Cauchy problem for M_1 and M_2 might not always have a solution -respectively a unique solution- in a space of functions with a finite number of derivatives, as is typically the case for strictly hyperbolic operators and symmetric systems[10]. However there is a unique solution to the problem in a more complicated space of quasi-analytic functions[11].

Now we show how the knowledge of a fundamental solution E for M_1^+

$$(M_1^+)^\mu{}_\nu = (\not{D} + m) g^\mu{}_\nu - g \tilde{F}^\mu \cdot \gamma \gamma_5 (D_\nu + \tfrac{i}{2} m \gamma_\nu) \tag{11}$$

allows the construction of a fundamental solution E' for L(D) by a purely algebraic procedure. Such a solution is most important for a quantized theory with external fields A (x) [12] as well as for the discussion of the classical Cauchy problem[13]. A fundamental solution of M_2 could be used the same way[18),19]. M_1^+ is related to $M_1(A) = T(A)^\mu \partial_\mu + S(A)$ as follows :

$$M_1^+ = T(\bar{A})^{\mu *} \partial_\mu + S(\bar{A})^*$$

where the star denotes hermitian conjugation with respect to (10). The method to be deployed is a variant of the one used previously[14),15] in the analysis of the Proca equation.

A part of the construction is based on a result valid for arbitrary relativistic wave equations of type (1) . We denote by $d(\partial)$ the corresponding Klein-Gordon divisor[16)]

$$d(\partial) = m - \beta^\mu\partial_\mu + \frac{1}{m}(g^{\mu_1\mu_2} - \beta^{\mu_1}\beta^{\mu_2})\partial_{\mu_1}\partial_{\mu_2} - \cdots \tag{12}$$

$$+ (-1)^n \frac{1}{m^{n-1}}(g^{\mu_1\mu_2} - \beta^{\mu_1}\beta^{\mu_2})\beta^{\mu_3}\cdots\beta^{\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n} \tag{13}$$

$$L_B\, d(\partial) = (\Box + m^2)1 \;, \quad B = 0 \,.$$

Looking for a generalization of (13) for the case of minimal coupling one gets the following partial result :

<u>LEMMA</u>

The order of $L_{i\not{A}}\, d(\partial_\mu + iA_\mu)$ does not exceed $n-1$ if $n > 2$, respectively 2 if $n = 2$.

Proof : $L_{i\not{A}}\, d(D)$, $D_\mu = \partial_\mu + iA_\mu$ can be written in the form

$$\sum_{|\mu|=n+1} \alpha^\mu_{n+1} D_\mu + \sum_{|\mu|=n} \alpha^\mu_n D_\mu + \sum_{|\mu|<n} \alpha^\mu_{n-1} D_\mu \;, \quad D_\mu = D_{\mu_1}\cdots D_{\mu_k} \,, \tag{14}$$

where we used the notation

$\mu = (\mu_1, \ldots, \mu_k)$, $|\mu| = k$ and the α 's denote tensor valued functions of $x \in R^4$. The totally symmetric part of α^μ_{n+1} -in the μ_i 's- is zero due to (13) . Since the commutator $[D_{\mu_i}, D_{\mu_k}]$ is only of zeroth order in ∂ the first term in (14) contains only contributions of order $n-1$. This is enough to demonstrate the lemma in the case $n = 2$. If $n > 2$, we know again from (13) that the symmetric part of α^μ_n vanishes. Hence the second term contains again only terms of order $n-2$ in ∂ . ∎

From the lemma one draws readily the following conclusion. If $n = 2s$ -this occurs in many cases[17)]- the order of $L_{i\not{A}}\, d(D)$ is independent of the external field provided $s \leq 3/2$.

Turning back to the Fierz-Pauli equation for $s = 3/2$ we recall the explicit form of the Klein Gordon divisor,

$$d(\partial) = d'_r(\partial)(m - \not\partial) = (m - \not\partial)\, d'_\ell(\partial)$$

$$d'_r(\partial)^\mu{}_\nu = g^\mu{}_\nu + \frac{1}{3}\gamma^\mu\gamma_\nu - \frac{1}{3m}(\gamma^\mu\partial_\nu - \partial^\mu\gamma_\nu) + \frac{2}{3m^2}\partial^\mu\partial_\nu \tag{15}$$

$$d'_\ell(\partial)^\mu{}_\nu = g^\mu{}_\nu + \frac{1}{3}\gamma^\mu\gamma_\nu + \frac{1}{3m}(\gamma^\mu\partial_\nu - \partial^\mu\gamma_\nu) + \frac{2}{3m^2}\partial^\mu\partial_\nu$$

A straightforward computation yields the following relations between

$L(D)$ defined by (6) , the Klein Gordon divisor and the Velo Zwanziger operator M_1[18)],

$$L(D)\, d'_r(D) = M_1^+ \qquad (16)$$

$$d'_e(D)\, L(D) = M_1 .$$

Due to (16) $E' = d'_r(D)\, E$ is a fundamental solution of $L(D)$ if E is one for M_1^+[19)].

$$L(D)(d'_r(D)E) = M_1^+ E = \delta \quad . \qquad (17)$$

The simple relations between $L(D)$, $d(D)$ and M_1 , as they are expressed by (16), are remarkable for two reasons :

i) They yield a canonical way of constructing M_1 .

ii) The construction of a fundamental solution for $L(D)$ can be achieved without even touching upon the problem of inconsistencies mentioned at the beginning (5) .

-- REFERENCES AND FOOTNOTES -

1) G. VELO and D. ZWANZIGER

Phys. Rev. <u>186</u>, 1337, (1969).

Phys. Rev. <u>188</u>, 2218, (1969).

2) M. FIERZ and W. PAULI

Proc. Roy. Soc. Lond. <u>173 A</u>, 211, (1939).

3) A relativistic wave equation is called stable if the substitution $\partial_\mu \longrightarrow D_\mu = \partial_\mu + i A_\mu(x)$ with $A_\mu(x)$ an external field, leads to a family of solutions $\{ \Psi_A(x) \}$ depending smoothly on A.
For a precise definition see :
A.S. Wightman : Proceedings of Fifth Coral Cables Conference (1968).

4) For s = 1 and s = 3/2 see :

B. SCHROER, A. SWIECA and R. SEILER

Phys. Rev. <u>D2</u>, 2927 (1970).

For s = 2 see :

G. VELO

"Anomalous Behaviour of a Massive Spin Two Charged Particle in an External Electromagnetic Field"
Preprint. CERN TH 1473, March 1972.

5) The connection between the two formalisms are given by the invertible c-number transformation (A real, $A \neq -\frac{1}{2}$)

$$\psi^\mu = \sigma^\mu_{\alpha\dot\beta}\begin{pmatrix}\chi^{\alpha\dot\beta\gamma}\\ \varphi^{\kappa\dot\beta\dot\delta}\end{pmatrix}, \quad (\varepsilon_{\alpha\beta}) = (\varepsilon_{\dot\alpha\dot\beta}) = \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \quad \varphi^{\gamma}_{\dot\alpha\dot\beta} = \varphi^{\dot\rho\dot\sigma\gamma}\varepsilon_{\dot\rho\dot\alpha}\,\varepsilon_{\dot\sigma\dot\beta}$$

$$a^\gamma_{\dot\alpha\dot\beta} = -\frac{1}{2}\left(\varphi^\gamma_{\dot\alpha\dot\beta}+\varphi^\gamma_{\dot\beta\dot\alpha}\right) \qquad c_{\dot\beta} = \left(A+\frac{1}{2}\right)\chi^{\rho}_{\rho\dot\beta}$$

$$b^{\kappa\beta}_{\dot\gamma} = \frac{1}{2}\left(\chi^{\alpha\beta}_{\dot\gamma}+\chi^{\beta\alpha}_{\dot\gamma}\right) \qquad d^\beta = \left(A+\frac{1}{2}\right)\varphi^{\beta}_{\dot\sigma}{}^{\dot\sigma}$$

The Fradkin parameter A corresponding to (6) equals minus one.

6) Our notations are : $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$; $g^{00} = -g^{ii} = 1$, $i = 1,2,3.$

$$D_\mu = \partial_\mu + iA_\mu(x)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\tilde{F}_{01} = F^{23}, \ldots ; \quad \tilde{F}^\mu . \gamma = \tilde{F}^{\mu\rho} \gamma_\rho$$

$$g = 2i/3m^2, \quad P = i\partial, \quad P^2 = P^\mu P_\mu$$

7) J. LERAY and Y. OHYA

"Systèmes linéaires hyperboliques non-stricts".
Colloque C.B.M. Louvain (1964) ; Reprint in Battelle Rencontres on Hyperbolic Equations and Waves, Seattle 1968 .

8) FRITZ JOHN

"Partial Differential Equations"
Proceedings of the Summer Seminar, Boulder, Colorado 1957 ; Interscience Publishers, New York (1964) .

9) It seems that this difficulty has not been noticed until now.

10) Ref. 7 , Introduction.

11) This is going to be explained in a futur publication.

12) A.S. WIGHTMAN

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Preprint, Princeton University (1972) .

13) LARS HORMANDER

"Linear Partial Differential Operators"
Springer (1963) .

14) [illegible]

15) J. BELLISSARD

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16) E. SPEER

"Generalized Feynmann Amplitudes"
Annals of Mathematics Studies, # 62 , Princeton Press (1969)
Appendix.

17) A.S. GLASS

"On the Harish-Chandra Condition for First-Order Relativistically-Invariant Free Field Equations". Preprint Joseph Henry Laboratories of Physics Princeton University (1972) .

18) There is an analoguous relation linking L(D) and ,

$$L(D)\, S = M_2 ,$$

$$S^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \left(\frac{2}{3m^2} D^{\mu} - \frac{1}{3m}\gamma^{\mu}\right)\left(g \not{D} \gamma_5 \gamma \cdot \tilde{F}_{\nu} - \frac{1}{2} m g \gamma_5 \gamma \cdot \tilde{F}_{\nu} - m\gamma_{\nu} + D_{\nu}\right) + \frac{1}{m} D^{\mu} \left(\gamma_{\nu} - g \gamma_5 \gamma \cdot \tilde{F}_{\nu}\right).$$

19) M_1^{+} and E can be replaced by M_2 and a fundamental solution of M_2 , see[18] .