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On the Poincaré Non-invariance of a Recent Alternative to the Dirac Equation.

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ABSTRACT

The two component factorization of the Klein-Gordon operator, proposed by Biedenharn, Han and van Dam, as an alternate to the Dirac equation for spin one-half massive leptons is investigated for its Poincaré invariance. It is found that the BHvD equation is not Poincaré invariant--and thus must be rejected. The approach is to search for the existence of ten independent infinitesimal generators H, $\underset{\sim}{P}$, $\underset{\sim}{J}$ and $\underset{\sim}{K}$ that satisfy the Lie bracket relations, characteristic of the Poincaré group. We find that the algebra does not close except for mass zero leptons, the case of the Weyl equation.

I. Introduction

In a recent set of papers,[1] Biedenharn et. al. have proposed an alternative to the Dirac equation[2] for spin one-half, finite mass particles, the so called Stigma equation. The Stigma equation is a first order, two component equation and involves factorizing space-time-dependent matrices, even in the case of no interaction. In I (footnote 6) one finds an insufficiently cautious statement, by the authors about the Lorentz invariance of their work. It is the object of this paper to examine systematically the Lorentz invariance of their equation. We achieve this by explicitly constructing the infinitesimal generators of the Poincare´ group and we find that the Poincare´ algebra closes only for the case of zero mass.[3] Hence, the Stigma equation is not Poincare´ invariant except when the leptonic mass is zero. The equation, therefore, does not offer a viable alternative to the Dirac equation. Now, the invariance of the theory under the operations of the inhomogeneous Lorentz group is merely a reflection of the principle of special relativity, which states that the laws of physics should be invariant under transformations of inertial reference frames. This symmetry is guaranteed by postulating the existence of the ten independent infinitesimal generators H, $\underset{\sim}{P}$, $\underset{\sim}{J}$, and $\underset{\sim}{K}$ satisfying the Lie bracket relations[4] characteristic of the inhomogeneous Lorentz group. These relations read:

$$[P_i , P_j] = 0 \tag{1}$$

$$[P_i , H] = 0 \tag{2}$$

$$[J_i , P_j] = i \, \varepsilon_{ijk} P_k \tag{3}$$

$$[J_i , H] = 0 \tag{4}$$

$$[J_i , J_j] = i \, \varepsilon_{ijk} J_k \tag{5}$$

$$[P_i , K_j] = -i \, \delta_{ij} H \tag{6}$$

$$[H \quad , K_i] \quad = -iP_i \tag{7}$$

$$[J_i \quad , K_j] \quad = i\, \varepsilon_{ijk}\, K_k \tag{8}$$

$$[K_i \quad , K_j] \quad = -i\, \varepsilon_{ijk}\, J_k \tag{9}$$

In Section II, we give the explicit form of the Poincaré group generators for the Dirac equation, in the standard frame and also in Bose-Gamba-Sudarshan-Cini-Touschek[5] frame.

In Section III, we search for the realization of the Poincaré group generators for the Stigma equation and check the commutation relations of the corresponding Lie algebra. In the final section, we discuss the consequences of our finding. Some useful commutators are listed in the Appendix.

II The Dirac Equation.

For the Dirac equation the realization of the Poincaré group (up to unitary equivalence) generators are:

$$\begin{aligned} \underset{\sim}{P} &= \underset{\sim}{p}, \\ \underset{\sim}{H}_D &= \underset{\sim}{\alpha} \cdot \underset{\sim}{p} + \beta m \\ \underset{\sim}{J} &= \underset{\sim}{r} \times \underset{\sim}{p} + \tfrac{1}{2}\underset{\sim}{\sigma} \\ \underset{\sim}{K}_D &= \tfrac{1}{2}[\underset{\sim}{r}\, (\beta m + \underset{\sim}{\alpha} \cdot \underset{\sim}{p}) + (\beta m + \underset{\sim}{\alpha} \cdot \underset{\sim}{p})\, \underset{\sim}{r}] - t\underset{\sim}{p} \end{aligned} \tag{10}$$

The forms of $\underset{\sim}{P}$, H_D, and $\underset{\sim}{J}$ are elementary. The form of $\underset{\sim}{K}_D$ follows immediately by observing that

$$\left\{ K_j = M_{jo} \ , \text{ where } M_{\mu\nu} = i(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}). \right\}$$

and making use of the expression for the time derivative in terms of the Hamiltonian. Furthermore, $\bar{\psi}\, \gamma_\mu\, \psi$ transforms as a vector with this choice of generators.

One verifies by direct calculation that the identifications (10) do satisfy the Poincaré group requirements (1) - (9).[4]

It is also known that the Dirac equation admits other representations in which the Hamiltonian is free of either "odd" or "even" operators respectively. The former case arises as a result of performing the Foldy-Wouthuysen-Tani[6] transformation on the Dirac Hamiltonian. The latter case, which proves to be of interest here, is obtained by performing the Bose-Gamba-Sudarshan-Cini-Touschek[5] transformation. In this case $\psi_E = U\psi_D$, $O_E = UO_D U^+$ with $U = \exp[-\frac{1}{2}\beta\,\underset{\sim}{\alpha}\cdot\hat{p}\,\tan^{-1}(m/p)]$. The subscript D refers to the Dirac representation and the subscript E to the so called extreme relativistic representation. The realizations of the Poincaré generators in the 'E' representation are seen to take the form:

$$
\begin{aligned}
\underset{\sim}{P} &= \underset{\sim}{p} \\
H_E &= \underset{\sim}{\alpha}\cdot\hat{p}\,(p^2+m^2)^{\frac{1}{2}} \\
\underset{\sim}{J} &= \underset{\sim}{r}\times\underset{\sim}{p}+\tfrac{1}{2}\underset{\sim}{\sigma} \\
\underset{\sim}{K}_E &= \tfrac{1}{2}\left[\underset{\sim}{\alpha}\cdot\hat{p}\,(p^2+m^2)^{\frac{1}{2}}\,\underset{\sim}{r}+\underset{\sim}{r}\,\underset{\sim}{\alpha}\cdot\hat{p}(p^2+m^2)^{\frac{1}{2}}\right]+\frac{im}{4p}\beta[\underset{\sim}{\alpha},\underset{\sim}{\alpha}\cdot\hat{p}]-t\underset{\sim}{p},
\end{aligned}
\qquad (11)
$$

where $p^2 = \underset{\sim}{p}\cdot\underset{\sim}{p}$. The form of $\underset{\sim}{K}_E$ has been obtained by explicitly performing the unitary transformation U on $\underset{\sim}{K}_D$. It may be pointed out here that the second term in the expression for $\underset{\sim}{K}_E$ arises due to the fact that $UrU^+ \neq \underset{\sim}{r}$, though $UpU^+ = \underset{\sim}{p}$.

III The Biedenharn-Han-Van Dam Equation.[1]

We now turn to the main purpose of this paper, namely, the proposed alternative to the Dirac equation.

In I, Biedenharn and his associates have advocated a different factorization of the Klein-Gordon operator $E^2 = p^2 + m^2$. These authors define a set of three anticommuting operators with unit square:

$$\eta_1 = \vec{\sigma}\cdot\hat{r}$$

$$\eta_2 = i\eta_1\eta_3$$

and

$$\eta_3 = (-1)^{j+\frac{1}{2}} \frac{\vec{\sigma}\cdot\vec{L}+1}{|\vec{\sigma}\cdot\vec{L}+1|}$$

and introduce the Hamiltonian:

$$H_I = \underset{\sim}{\sigma} \cdot \underset{\sim}{p} + \eta_3 m, \tag{12}$$

where $\underset{\sim}{\sigma}$ are the usual the Pauli spin matrices and η_3 introduced above is a space-time dependent operator with $\eta_3{}^2 = 1$ and satisfies the commutation relations:

$$[\eta_3, \sigma_i]_- = o \ , \ \{\eta_3, p_i\}_+ = o \ \text{ and } \ \{\eta_3, r_i\}_+ = o. \tag{13}$$

At this point, one observes a basic difference between the Dirac equation and Eq. (12): in the latter case, $\underset{\sim}{p}$, the generator of spatial displacements does __not__ commute with H_I, thus violating (2). Hence $\underset{\sim}{p}$ cannot be identified with $\underset{\sim}{P}$. Furthermore, as pointed out by Streater[7], $e^{i\underset{\sim}{d}\cdot\underset{\sim}{p}}$, the generator of finite translations also does not commute with η_3, contrary to the assertion of I, since under parity $\underset{\sim}{p} \to -\underset{\sim}{p}$ so $\underset{\sim}{d}\cdot\underset{\sim}{p} \to -\underset{\sim}{d}\cdot\underset{\sim}{p}$($\underset{\sim}{d}$ does not alter as it is __not__ the position operator but real numbers). This means that already at this stage Poincaré invariance is lost! This is indeed the case and we explicitly demonstrate in the following that Poincaré invariance can be restored only in the case of m = 0.

In analogy with the Dirac case, one can consider the two other representations of equation (12). One, in which the Hamiltonian is free of $\underset{\sim}{\sigma} \cdot \underset{\sim}{p}$, that is:

$$H_{II} = \eta_3 (p^2 + m^2)^{\frac{1}{2}} \tag{14}$$

This presents the same difficulty as (12), that is, $\underset{\sim}{p}$ does not again commute with the Hamiltonian H_{II}. Instead one considers the other representation in which the Hamiltonian is free of η_3. For this representation, the operator of unitary transformation is

$$S = \exp[-\tfrac{1}{2}\ \eta_3\ \underset{\sim}{\sigma} \cdot \hat{p}\ \tan^{-1}(\tfrac{m}{p})] \tag{15}$$

and the transformed Hamiltonian turns out to be

$$H_{III} = \underset{\sim}{\sigma} \cdot \hat{p}\ (p^2 + m^2)^{\frac{1}{2}} \tag{16}$$

One notes that $\underset{\sim}{p}$ <u>does</u> commute with H_{III} and hence is a suitable candidate for the generator of translations in space. The realization of this operator in the untransformed representation of I has the unnatural form:

$$\underset{\sim}{P} = \left[\frac{p}{(p^2 + m^2)^{\frac{1}{2}}}\right] \underset{\sim}{p} + \eta_3 \underset{\sim}{\sigma} \cdot \hat{p} \left[-\frac{m}{(p^2 + m^2)^{\frac{1}{2}}}\right] \underset{\sim}{p}\ . \tag{17}$$

It is easily verified that $\underset{\sim}{P}$ in Eq. (17) satisfies (1) and (2), but it is rather unwieldy to check the rest of the algebra in equations (3) - (9). It is more convenient to work in the extreme relativistic representation of Eq. (16), <u>thereby also bypassing any questions about the physical interpretation</u>

of η_3, that may arise. The only role which η_3 plays in this demonstration of non-invariance is through its algebraic property Eq. (13). It is to be emphasized, however, that if so desired, one could carry out the analysis in the original representation of Eq. (12).

In the extreme relativistic representation $\underset{\sim}{P} = \underset{\sim}{p}$ and $\underset{\sim}{J} = \underset{\sim}{r} \times \underset{\sim}{p} + \frac{1}{2}\underset{\sim}{\sigma}$, and it is easily verified that relations (3)-(5) are satisfied. This is not surprising since $[\underset{\sim}{J}, S] = 0$ and Eq. (12) is rotationally invariant.

Now the crucial step in establishing Poincaré invariance lies in constructing the generator $\underset{\sim}{K}$ of pure Lorentz transformations and showing that it obeys the bracket relations (6) - (9) of the Poincaré group. This we now set out to do by first securing a link between the transformed representation (III) Eq. (16) and the extreme relativistic representation Eq. (11). Define:

$$\psi_- = \frac{1}{2}(1-\gamma_5)\psi_E .$$

Since $\underset{\sim}{\alpha} = -\gamma_5 \underset{\sim}{\sigma}$, it follows that

$$H_{III}\,\psi_- = \underset{\sim}{\sigma}\cdot\hat{p}(p^2 + m^2)^{\frac{1}{2}}\,\psi_- . \tag{18}$$

Thus we may identify the wave function in the transformed representation (III) with the chiral projection ψ_- of the Dirac wave function in the "extreme relativistic" representation. The corresponding relation between the operators in these representations is simply

$$O_{III} = \frac{1}{2}(1-\gamma_5)\; O_E\,\frac{1}{2}(1-\gamma_5). \tag{19}$$

From equation (11) for $\underset{\sim}{K}_E$, we thus obtain[8]

$$K_{III} = \frac{1}{2} \left\{ \underset{\sim}{\sigma} \cdot \hat{p}(p^2 + m^2)^{1/2} \underset{\sim}{r} + \underset{\sim}{r} \underset{\sim}{\sigma} \cdot \hat{p}(p^2 + m^2)^{1/2} \right\} - t\underset{\sim}{p}, \qquad (20)$$

since $(1-\gamma_5)\ \beta(1-\gamma_5) = o$, $[\gamma_5, \underset{\sim}{\alpha}]_- = o$ and

$$\frac{1}{2}(1-\gamma_5)\psi_- = \psi_- .$$

By straightforward algebra, we find that K_{III} satisfies the Poincaré relations (6) - (8). As a typical example, let us consider the commutator (7):

$$[H, K_i]_- .$$
$$= \frac{1}{2} [\underset{\sim}{\sigma} \cdot \hat{p}\, E, \underset{\sim}{\sigma} \cdot \hat{p}\, E\ r_i + r_i\, \underset{\sim}{\sigma} \cdot \hat{p}\, E]_-$$
$$= \frac{1}{2}\, \underset{\sim}{\sigma} \cdot \hat{p}\, E\, [\underset{\sim}{\sigma} \cdot \hat{p}\, E, r_i]_- + \frac{1}{2}[\underset{\sim}{\sigma} \cdot \hat{p}\, E, r_i]_-\, \underset{\sim}{\sigma} \cdot \hat{p}\, E$$
$$= \frac{1}{2}\, \underset{\sim}{\sigma} \cdot \hat{p}\, E \left\{ [\underset{\sim}{\sigma} \cdot \hat{p}, r_i]_- E + \underset{\sim}{\sigma} \cdot \hat{p}\, [E, r_i]_- \right\}$$
$$+ \frac{1}{2} \left\{ [\underset{\sim}{\sigma} \cdot \hat{p}, r_i]_-\, \underset{\sim}{\sigma} \cdot \hat{p}\, E^2 + \underset{\sim}{\sigma} \cdot \hat{p}\, [E, r_i]_-\, \underset{\sim}{\sigma} \cdot \hat{p}\, E \right\}$$
$$= -ip_i + i\frac{E^2}{p^2}\, p_i - \frac{i}{2} \left\{ \underset{\sim}{\sigma} \cdot \hat{p}, \underset{\sim}{\sigma}_i \right\}_+ \frac{E^2}{p}$$
$$= -ip_i \text{ , Q. E. D.}$$

Finally, for the crucial commutator $[\underline{K_i}, \underline{K_j}]_-$. After a tedious algebra, one discovers

$$[\underline{K_i}, \underline{K_j}] = -i\, \varepsilon_{ijk} \left\{ \frac{p^2 + m^2}{p^2}\ \frac{\sigma_k}{2} + (\underline{r} \times \underline{p})_k \right\} \qquad (21)$$

clearly, the algebra does not close for the case of finite mass. We are, therefore, forced to the conclusion that the stigma equation of I is Poincaré -invariant only for the case of zero mass leptons. In this limit, Eq. (12) collapses into the Weyl equation, in both the original and the transformed representations. The Weyl equation, of course, is known to be Poincaré invariant and it can be derived from the Dirac equation for zero mass.[9]

IV Discussion.

Some remarks are in order.

(a) The failure of Poincaré invariance of the stigma equation is clearly due to the presence of the space-time dependent operator η_3 in the Hamiltonian. It is worth recalling here the original powerful argument of Dirac[10] that in the case of no external field, all points in space-time must be equivalent and hence the operator in the wave equation must not depend on the r_i's. This simple and fundamental argument of Dirac has been subtly violated in I.

(b) The elegance and power of the unitary transformation employed[5] is worth emphasizing. We have been able to bypass completely all questions about the physical interpretation given to (or to be given to) the space-time dependent operator η_3. The latter can be "transformed away" via the unitary transformation and our analysis depends only on the algebraic property of commutation relations satisfied by η_3.

(c) We must perforce conclude that Poincaré invariance which provides a precise mathematical expression to the principle of special relativity offers an additional proof of the uniqueness of the description of a finite mass, spin $\frac{1}{2}$ particle by means of the Dirac equation.

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8. Additional terms, in the expression for $\underset{\sim}{K}_{III}$, of the type $(\underset{\sim}{\sigma} \times \underset{\sim}{p})$ or $\eta_3(\underset{\sim}{\sigma} \times \underset{\sim}{p})$ do not allow the commutators (6) and (7) to be simultaneously satisfied and hence cannot occur, in contradistinction to the Dirac case.

9. R. E. Marshak, Riazuddin and C. P. Ryan, "Theory of Weak Interactions in Particle Physics." Wiley - Interscience (1969), page 70.

10. P. A. M. Dirac, "Quantum Mechanics" - Oxford (1958), page 255.

Appendix

In this, we tabulate some commutators that were employed in the text.

$$[\eta_3,\sigma_i]_- = o; \ \{\eta_3, p_i\}_+ = \{\eta_3, r_i\}_+ = o$$

$$[\underset{\sim}{\sigma}.\underset{\sim}{p},\sigma_i]_- = 2_i(\underset{\sim}{\sigma} \times \underset{\sim}{p})_i$$

$$[r_i, E]_- = i\,\frac{p_i}{E}$$

$$[r_i, p] = i\frac{p_i}{p}$$

$$[r_i. \frac{1}{E}]_- = -\frac{ip_i}{E^3}$$

$$[r_i,\frac{1}{\sqrt{E}}]_- = \frac{-ip_i}{2\,E^{5/2}}$$

$$[r_i,\frac{1}{\sqrt{2E(E+p)}}]_- = \frac{-ip_i\sqrt{E+p}}{2\sqrt{2}\;E^{5/2}\,p}$$

$$[r_i, \sqrt{\frac{E+p}{2E}}\,]_- = \frac{ip_i\;m\sqrt{E-p}}{2\sqrt{2}\;p\,E^{5/2}}$$

$$[r_i, \underset{\sim}{\sigma}.\hat{p}]_- = \frac{i}{p^2}\left\{p\,\sigma_i - (\underset{\sim}{\sigma}.\hat{p})\,p_i\right\}$$

$$[r_i, \underset{\sim}{\alpha}.\hat{p}]_- = \frac{i}{p^2}\left\{p\,\alpha_i - (\underset{\sim}{\alpha}.\hat{p})\,p_i\right\}$$