INSTITUTE OF PLASMA PHYSICS
CZECHOSLOVAK ACADEMY OF SCIENCES

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PROPAGATION AND REFLECTION OF AN ELECTROMAGNETIC WAVE IN A HOT INHOMOGENEOUS PLASMA

V. A. Petržílka, J. Preinhaelter

IPPCZ - 153

August 1971

RESEARCH REPORT

NADEMLÝNSKÁ 600, PRAGUE 9
CZECHOSLOVAKIA

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ABSTRACT

Using the WKB method a local dispersion equation of a right hand circularly polarized wave and an expression for its electric field are derived, the wave having the frequency smaller than the electron cyclotron frequency and propagating in a hot inhomogeneous plasma. The density gradient of a plasma is constant and parallel to an external magnetostatic field. Further the reflection coefficient of this wave is derived.

1. INTRODUCTION

Recently a considerable interest has been devoted to the propagation of waves in an inhomogeneous plasma, especially in connection with its instabilities. In [1] and [2] a method has been developed to investigate the propagation of electromagnetic waves in a inhomogeneous hot magnetoactive plasma in the WKB approximation. H. S. Hsieh [3] studied the propagation of the circularly polarized electromagnetic wave in such a plasma when the electrostatic field parallel to the magnetostatic field is present. As we show, some of his conclusions are not correct.

In this paper the propagation of the right-hand circularly polarized electromagnetic wave in a hot magnetoactive plasma is investigated. The plasma inhomogeneity is simulated by the homogeneous gravitational field parallel to the external magnetostatic field. The rigth-hand wave propagates in two frequency bands, namely for $\omega < \omega_c$ ($\omega_c > 0$ is the electron cyclotron frequency) and for $\omega > \omega_c/2 + (\omega_c^2/4 + \omega_0^2)^{1/2}$ (ω_0 is the electron plasma frequency). The plasma temperature affects the propagation most strongly within the first frequency band and particularly in the vicinity of ω_c , where the phase velocity of the wave is much smaller than the velocity of light c. Within this frequency region the influence of the ion movement on the propagation of the wave is negligible. In the hot not too dense plasma, collisions can be neglected and the wave propagation can be described by the linearized Vlasov equation for electrons together with the Maxwell equations. Assuming the wavelength to be much smaller than the characteristic length of the inhomogeneity, we solve this set of equations using the WKB method. We obtain the dispersion equation for the wave, the expression for the electric field and in the second order of the WKB approximation the reflection coefficient of the wave propagating from vacuum into plasma.

2. BASIC EQUATIONS AND THEIR WKB SOLUTION

We shall first derive an integro-differential equation for the electric field of the right-hand polarized wave propagating in an inhomogeneous plasma. The z-axis will be chosen in the direction of the external homogeneous magnetostatic field $\vec{H} \equiv (0, 0, H)$. A gravitational force which models the inhomogeneity is of the form $\vec{G} \equiv (0, 0, mg)$. In what follows we confine ourselves to the investigation of a stable plasma so that the stationary distribution function of electrons f_0 can be assumed to be Maxwell - Boltzmann,

$$f_0(v_\perp, v_z, z) = n(z)\, e^{-\frac{v_\perp^2 + v_z^2}{v_T^2}} \cdot \pi^{-3/2} v_T^{-3}$$

where $n(z)$ is the local density given by the barometric formula,

$$n(z) = n_0\, e^{\frac{z}{L}}, \qquad L = \frac{v_T^2}{2g}$$

v_z and $v_\perp$ are components of electron velocity parallel and perpendicular to the magnetostatic field respectively, v_T is the mean thermal velocity of electrons, $n_0 = n(0)$. As long as the amplitude of the electric field of the wave remains sufficiently small, the distribution function for electrons can be expressed as the sum of the stationary distribution function f_0 and the perturbation f. The perturbation f is governed by the linearised Vlasov equation

$$(1)\quad \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial t} - [\vec{v}_\perp, \omega_c]\frac{\partial f}{\partial \vec{v}_\perp} + g\frac{\partial f}{\partial v_x} = \frac{|e|}{m}\vec{E}_\perp \frac{\partial f_0}{\partial \vec{v}_\perp}$$

Here e and m are the charge and mass of the electron respectively, $\vec{\omega}_c = (0, 0, \omega_c)$, ω_c is the electron cyclotron frequency $(\omega_c > 0)$. $\vec{E}_\perp(x, t)$ is the electric field of the wave. We can assume a time dependence of all quantities to be of the form $\exp[-i\omega t]$. Then, from the Maxwell equations the following equation for the field $\vec{E}_{\perp}\omega$ is obtained

$$(2)\quad \frac{\partial^2 \vec{E}_{\perp\omega}}{\partial x^2} + \frac{\omega^2}{c^2}\vec{E}_{\perp\omega} = i\frac{4\pi e\omega}{c^2}\int_{-\infty}^{\infty} \vec{v}_\perp f_\omega d^3v$$

The electric field of the propagating wave can be divided into a right-hand and a left-hand polarized component

$$E_- = \frac{1}{2}(E_x - iE_y), \qquad E_+ = \frac{1}{2}(E_x + iE_y)$$

Eq. (1) is solved by using the method of characteristics and after introducing f_ω into eq. (2) an integro-differential equation for the wave E_- is obtained

$$(3)\quad \frac{d^2 E_-}{dx^2} + \frac{\omega^2}{c^2}E_- =$$

$$= -i\frac{\omega_p^2(x)\omega}{c^2}\int_{-\infty} dv_x f_0(v_x)\int_0 d\tau\, e^{i(\omega-\omega_c)\tau} E_-(x(\tau))$$

where $\omega_0^2(x) = 4\pi n(x) e^2/m$. The function $x(\tau)$ is the characteristics of eq. (1), $x(\tau) = x - v_x\tau + \tau^2 g/2$. For the waves whose wavelength is small comparing with the characteristic length of the inhomogeneity, eq. (3) can be solved by using the WKB method developed for this type of integrodifferential equations by M.N. Rosenbluth and others [1], [2]. The electric field can then be written in the form

$$(4) \qquad E_- = C(x)\, e^{i\int_0^x k(x')\,dx'}$$

where the amplitude C and the wave vector k vary slowly along the wavelength so that

$$\frac{1}{k^2}\frac{dk}{dx} \sim \frac{1}{kC}\frac{dC}{dx} \sim \frac{1}{kL} << 1$$

By introducing expression (4) into eq. (3) the dispersion equation for the local wave vector $k(x)$ is obtained in the lowest order of the WKB approximation

$$(5) \quad D \equiv k^2(x) - k_v^2 - \frac{i\sqrt{\pi}\,\omega_0^2(x)\,\omega}{k(x)\,v_T c^2}\, W\!\left(\frac{\omega-\omega_c}{k(x)\,v_T}\right) = 0$$

where

$$k_v^2 = \frac{\omega^2}{c^2}, \qquad W(x) = e^{-x^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}dt\right).$$

Eq. (5) is valid under the assumption $Im((\omega-\omega_c)/k v_\perp) > 0$. We obtain the following expression for the amplitude of the wave $C(x)$ in the next order of the WKB approximation

$$C = A\left(\frac{\partial D}{\partial k}\Big|_{k=k(x)}\right)^{-1/2}$$

where A is an arbitrary constant. The solution of the dispersion equation can easily be found in the region of small damping $\left(|(\omega-\omega_c)/k v_T| \gg 1, \quad \omega < \omega_c\right)$, where the wave vector has a small imaginary part, i.e. $k = k_0 + i k_1$ and $|k_0| \gg k_1$. For the real and imaginary part of the wave vector we have from eq. (5)

$$(6) \quad k_0 = \left(k_v^2 + \frac{\omega_0^2(x)\,\omega}{(\omega_c-\omega)\,c^2}\right)^{1/2}, \qquad k_1 = \frac{\omega_0^2(x)\,\omega\sqrt{\pi}}{2k_0^2(x)\,v_T c^2}\, e^{-\left(\frac{\omega-\omega_c}{k_0(x)\,v_T}\right)^2}$$

A complete set of the solutions of eq. (3) in the first order of the WKB approximation can be written in the form

$$(7) \quad E_- = \left(\frac{\partial D}{\partial k}\right)^{-1/2}\left(A_1 e^{i\int_0^x k(x')dx'} + A_2 e^{-i\int_0^x k(x')dx'}\right)$$

The constants A_1, A_2 are independent and can be determined from boundary conditions. Apparently, we can see from the preceding expression that both the waves propagating in the direction of an inhomogeneity and the wave propagating in the opposite direction are damped. For the electric field of the wave propagating into the plasma from vacuum (i.e. from region $x \to \infty$) having the unit amplitude and the phase equal to $i(k_v x-\omega t)$, we get from (5), (6) and (7)

$$E_- = \frac{\bar{A}_1}{\sqrt{k_0(x)}} \exp\left\{2iLk_0(x) + iLk_v \lg\frac{k_0(x)-k_v}{k_0(x)+k_v} - \right.$$
$$\left. -\frac{\sqrt{\pi}}{2} L k_0(x) \frac{k_0(x) v_T}{\omega_c-\omega} e^{-\left(\frac{\omega_c-\omega}{k_0(x) v_T}\right)^2}\right\}$$

where

$$\bar{A}_1 = \sqrt{k_v} \exp\left\{-iLk_v\left(2+\lg\frac{k_0^2(0)-k^2 v}{4k^2 v}\right) + \right.$$
$$\left. + \frac{\sqrt{\pi}}{2} L k_v \frac{k_v v_T}{\omega_c-\omega} e^{-\left(\frac{\omega-\omega_c}{k_v v_T}\right)^2}\right\} .$$

The preceding formula for the electric field can be used only for those x, for which the condition $(\omega_c-\omega)/k_0(x) v_T \gg 1$ is fulfilled.

3. REFLECTION COEFFICIENT

The reflection coefficient for a plasma the density of which slowly varies in space is different from zero only in the second order of the WKB approximation [4]. In paper [2] the kinetic effects were shown, for electrostatic waves, to lead to essentially greater wave reflection than follows from the simple magnetohydrodynamical theory. In the following, the reflection coefficient is derived for the right-hand polarised wave propagating into the plasma from vacuum. The solution of eq. (3) is thus supposed to have the form

$$E_- = C\left(\frac{\partial D}{\partial k}\right)^{-1/2}\left(e^{i\int^x k\,dx'} + \psi(x) e^{-i\int^x k\,dx'}\right)$$

where $\psi(x)$ is at least a first order small quantity in the WKB expansion. For the function $\psi(k)$ we then have

$$(8)\quad \psi(x) = \frac{i}{L}\int_{k(x)}^{k(\infty)} dy\, \exp\left\{2iL\int_{k(0)}^{y}\frac{t^2 P(t)\,dt}{(t^2-k_v^2)W\left(\frac{\omega-\omega_c}{k v_T}\right)}\right\}\sum_{n=0}^{4}\frac{B_n(y)}{P^n(y)}$$

where

$$P(k) = 2W\left(\frac{\omega-\omega_c}{k v_T}\right) - (k^2-k_v^2)\frac{\partial}{\partial k}\left(\frac{1}{k}W\left(\frac{\omega-\omega_c}{k v_T}\right)\right).$$

An expression for $B_n(k)$ is complicated but for what follows we need only the result that for $\mathrm{Im}\, k \geq 0$ the value of $\lim_{k\to 0} B_n(k)$ is finite and

$$B_4(0) = 5\, k_v^2\, v_T^4\, i\pi(\omega-\omega_c)^4.$$

In fact, the terms with $n = 0, 1, 2, 3$ in integral (8) can be shown to be small comparing with the term with $n = 4$. To compute the integral in (8) the method of saddle points is used. An equation for saddle points of integral (8) is of the form

$$(9)\quad \frac{2iLk^2P(k)}{(k^2-k_v^2)W\left(\frac{\omega-\omega_c}{k v_T}\right)} - \frac{4}{P(k)}\frac{dP(k)}{dk} = 0$$

The functions $W((\omega-\omega_c)/k v_T)$ and $P(k)$ are analytic and unequal to zero in the upper half of the k-plane ($\mathrm{Im}\, k \geq 0$) with the exception of the point $k = 0$ [5]. For $\mathrm{Re}\, k \geq 0$, $\mathrm{Im}\, k \geq 0$, $k \neq 0$ eq. (9) has a unique root which lies in the region where $|k v_T/(\omega-\omega_c)| \ll 1$. Solving eq. (9) within this region of the k-plane the asymptotic expression of the functions P and W can be used,

$$P \sim 2W \sim \frac{2ikv_T}{\sqrt{\pi}(\omega-\omega_c)} .$$

For the saddle point k_s we then get

$$k_s = k_v (L k_v)^{-1/3} e^{i\frac{\pi}{6}} .$$

The integration path C_1 in integral (8) can be shifted in the complex plane in order that it passes through the saddle point and runs along the curve of steepest descent. Integrals on the sections of the deformed integration path (the curve C_2 in Fig. 1) connecting the original curve C_1 with line of steepest descent can be neglected for $x \to -\infty$, i.e. $k(x) \to k_v$. Then, for $\psi(-\infty)$ we obtain

$$(10)\quad \psi(-\infty) = \frac{5\sqrt{2\pi}}{32\sqrt{3}} \exp\left\{2iL \int_{k(0)}^{k_s} \frac{t^2 P(t)\, dt}{(t^2-k_v^2)W}\right\}$$

The complex reflection coefficient $R(x)$ is given by

$$R(x) = \psi(x) \exp\left\{-2i \int_0^x k(x')\, dx'\right\} .$$

Using (10) we obtain the following expression for the absolute value of the reflection coefficient $|R(-\infty)|$ of the wave propagating from vacuum into the plasma

$$|R(-\infty)| = \frac{5\sqrt{2\pi}\, e^{4/3}}{32\sqrt{3}} e^{-2\pi L k_v} \tag{11}$$

4. CONCLUSION

Using the WKB method we have derived an expression for the electric field of the right-hand circularly polarized wave in a hot collisionless plasma with the density increasing exponentially in space. The wave is damped collisionlessly propagating both in and opposite to the direction of the density gradient. The amplification of this wave predicted in [3] was obtained from an uncorrect dispersion equation. The author included some terms proportional to $(L k_v)^{-1}$ that are of the first order of the WKB the approximation. Recently, M.P. Bachynski and B.W. Gibbs experimentally verified the Hsieh's theory and they observed no space amplification of the waves as a consequence of an inhomogeneity.

When a right-hand polarized wave propagates into the plasma from vacuum the movement of particles plays no important role. The reflection coefficient (11) is the same as that derived from the magnetohydrodynamics with $v_T = 0$ [7] and consequently the wave reflection slight. A weak influence of the kinetic effects on the reflection coefficient of the right-hand polarized wave can be explained by an interaction of this wave with the resonant particles by means of the anomalous Doppler effect. The electrostatic waves, investigated in [2], interact with the resonant particles by means of the Čerenkov effect; as a consequence, in computing the integral analogous to (8) new saddle points appear in the upper half of the k-plane. In the case of the right-hand polarized wave these saddle points, connected with kinetic effects, shift to the lower half-plane and in calculation of the reflection coefficient they do not play any role.

Stimulating discussions with Dr. J. Václavík are gratefully acknowledged.

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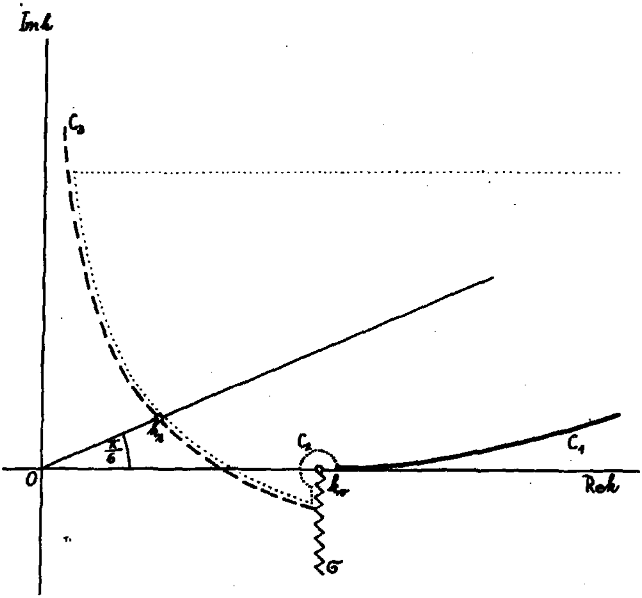

Im k
Re k
O
C_0
C_1
C_2
k_0
k_1
$\frac{\pi}{6}$
σ

Fig. 1.: C_1 is the original integration path, C_2 is the distorted integration path, C_3 is the curve of steepest descent, k_s is a saddle point, k_v is the wave number in vacuum, σ is a cut of the complex plane.

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