CPT 143
ORO 3992 - 90
June 7, 1972

# On the Stability of the Classical Electron

By

C. C. Chiang

Center for Particle Theory

University of Texas

Austin, Texas

Abstract: In this paper we show that in the classical theory of the electron, with the introduction of a shadow electromagnetic field, the electron is stable in the point particle limit.

## Introduction

It is well known that the classical theory of electrons suffers from some fundamental difficulties such as the divergences of the self-energy and the self-force. The divergence of the self-force implies the instability of the electron. The usual way to avoid the difficulty of these divergences is based on the idea of renormalization. It has long been considered as proper to assume that the infinite self-energy of isolated electrons is physically meaningless, and can be subtracted away by the renormalization method.[(1)]

The concept of renormalization has been extensively used in quantum field theory to eliminate divergent quantities computed from the theory. The results obtained are in remarkable agreement with experiments. Nevertheless there are reasons not to be satisfied with the conventional treatments of the renormalization procedure. For example, not all field-theoretical interactions are renormalizable. Furthermore, the computed mass differences due to electromagnetic interaction is infinite, while experimentally it is finite. Therefore, a genuine finite theory seems to be favorable, although the renormalization procedure might still be necessary.

The failure of having a finite theory is usually attributed to the fact that within the usual framework of quantized fields it does not seem possible to describe a system with a local interaction. In fact many of the difficulties caused by the use of local interaction are shared both by the classical and the quantum field theories. One of the explanations for the occurrence of the divergences is that classical considerations indicate that for any kind of matter coupled to the metric field in the Einstein way, there are limitations on the energy densities and masses which can be concentrated or built up in a given region. Consequently, that space-time loses its physically meaningful character beyond such limiting densities, and singularities then appear in the solutions.[2] This explanation seems appealing. However, it is not yet quite clear whether the existence of the singularities is essential in the problem of elementary particles. It might be that gravitation does not play an important role as far as the divergence problem is concerned.

Recently it has been emphasized that the introduction of states with negative norm provides a way out of the divergence difficulties in quantum field theories.[3] In order to assure the probability interpretation, the concept of shadow states has also been introduced.[4] In the electrodynamic theory, there is always analogy between the classical and quantum theories. However, it is not yet clear what is the counterpart of the shadow field in the classical field theory. In this note we show that in the theory of classical electrodynamics the introduction of a shadow field makes self-energy of the electron finite and the electron stable.

## Lagrangian and Energy-Stress Tensor Density

Following the idea of shadow field in quantum field theory, we introduce a massive vector field $\tilde{A}_\mu$ as the shadow field accompanying with the ordinary electromagnetic field $A_\mu$. We may write down the Lagrangian for the system of the fields $A_\mu$ and $\tilde{A}_\mu$ interacting with a current as follows:

$$L = -\frac{1}{16\pi} F_{\mu\nu} F^{\nu\mu} + \frac{1}{16\pi} (\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + M^2 \tilde{A}_\alpha \tilde{A}^\alpha) + j_\mu (A^\mu + \tilde{A}^\mu) \tag{1}$$

with

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ \tilde{F}_{\mu\nu} &= \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \end{aligned} \tag{2}$$

M is the mass parameter of the shadow field.

The field equations for $A_\mu$ and $\tilde{A}_\mu$ are obtained by variational methods as usual

$$A_\mu = -4\pi\, j_\mu \tag{3}$$

$$(\Box - M^2)\tilde{A}_\mu = 4\pi\, j_\mu \tag{4}$$

here the Lorentz condition $\partial_\mu A^\mu = 0$ is assumed. The condition $\partial_\mu \tilde{A}^\mu = 0$ already follows from the field equations.

The energy-stress tensor densities for $A_\mu$ and $\tilde{A}_\mu$ are respectively

$$T_\mu{}^\nu = \frac{1}{4\pi}(F_{\mu\alpha}\, F^{\alpha\nu} + \frac{1}{4}\,\eta_\mu{}^\nu\, F_{\alpha\beta}\, F^{\alpha\beta}) \tag{5}$$

$$\tilde{T}_\mu{}^\nu = -\frac{1}{4\pi}\{(\tilde{F}_{\mu\alpha}\, \tilde{F}^{\alpha\nu} - M^2\tilde{A}_\mu\tilde{A}^\nu) + \frac{1}{2}\eta_\mu{}^\nu(\frac{1}{2}\,\tilde{F}_{\alpha\beta}\, \tilde{F}^{\alpha\beta} + M^2\tilde{A}_\alpha\tilde{A}^\alpha)\} \tag{6}$$

The total energy-stress tensor density $\bar{T}_\mu{}^\nu$ is the sum of $T_\mu{}^\nu$ and $\tilde{T}_\mu{}^\nu$

$$\bar{T}_\mu{}^\nu = T_\mu{}^\nu + \tilde{T}_\mu{}^\nu \tag{7}$$

## Electromagnetic Energy-Momentum and Stability of the Electron

The electromagnetic energy-momentum of the electron is defined as

$$P_\mu = \int \bar{T}_\mu{}^\nu\, d\sigma_\nu \tag{8}$$

where $\sigma$ is a space-like plane. If $P_\mu$ defined in (8) is indeed a proper definition for the electromagnetic energy-momentum contributing to the electron's energy and momentum, it is supposed to transform as a 4-vector under Lorentz transformation. It is well known that this is not true when we replace $\bar{T}_\mu{}^\nu$ by $T_\mu{}^\nu$ in (8)[5]. In order that $P_\mu$ defined in (8) is a covariant 4-vector and the electron is stable, the following conditions have to be satisfied,

$$\int \bar{T}_{(o)i}{}^k\, d^3x_{(o)} = 0 \qquad \text{for } i,\, k = 1,\, 2,\, 3. \tag{9}$$

The subscript (o) means that the quantities are computed in the rest frame of the electron.

From the field equations (3) and (4) we have

$$A^{4}_{(o)} = \frac{1}{r} \tag{10}$$

$$\tilde{A}^{4}_{(o)} = -\frac{e^{-Mr}}{r} \tag{11}$$

all other components of the fields vanish. The non-vanishing components of $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ are

$$F_{(o)}^{k4} = \frac{\partial A_{(o)}^{4}}{\partial x_k} = -\frac{x^k}{r^3} \tag{12}$$

$$\tilde{F}_{(o)}^{k4} = \frac{\partial \tilde{A}_{(o)}^{4}}{\partial x_k} = \frac{e^{-Mr}}{r}\left(\frac{1}{r} + M\right)\frac{x_k}{r} \tag{13}$$

From (5), (10), and (12), we have

$$T_{(o)\,k}^{\;\;k} = \frac{1}{4\pi}\,\frac{1}{r^3}\left(\frac{x_k^2}{r^2} - \frac{1}{2}\right)$$

$$T_{(o)\,i}^{\;\;k} = \frac{1}{4\pi}\,\frac{x_i x_k}{r^6} \qquad \text{for } i \neq k. \tag{14}$$

$$T_{(o)\,k}^{\;\;4} = 0$$

$$T_{(o)\,4}^{\;\;4} = \frac{1}{4\pi}\,\frac{1}{2r^4}$$

Similarly, from (6), (11) and (13), we have

$$\tilde{T}_{(o)\,k}^{\;\;k} = -\frac{1}{4\pi}\,\frac{e^{-2Mr}}{r^2}\left\{\left(\frac{1}{r} + M\right)^2\left(\frac{x_k^2}{r^2} - \frac{1}{2}\right) - \frac{M}{2}\right\}$$

$$\tilde{T}_{(o)\,i}{}^{k} = -\frac{1}{4\pi}\frac{e^{-2Mr}}{r^2}\left(\frac{1}{r}+M\right)^2\frac{x_i x_k}{r^2} \qquad \text{for } i \neq k. \tag{15}$$

$$\tilde{T}_{(o)\,4}{}^{4} = -\frac{1}{4\pi}\frac{e^{-2Mr}}{2r^2}\left\{\left(\frac{1}{r}+M\right)^2+M^2\right\}$$

The integration of (14) and (15) over 3-dimensional space yield

$$\int T_{(o)\,k}{}^{k}\, d^3x_{(o)} = \lim_{a\to o}\frac{1}{6a} \tag{16a}$$

$$\int T_{(o)\,i}{}^{k}\, d^3x_{(o)} = 0 \tag{16b}$$

$$\int T_{(o)\,4}{}^{4}\, d^3x_{(o)} = \lim_{a\to o}\frac{1}{2a} \tag{16c}$$

and

$$\int \tilde{T}_{(o)\,k}{}^{k}\, d^3x_{(o)} = -\lim_{a\to o}\frac{1}{6a} \tag{17a}$$

$$\int \tilde{T}_{(o)\,i}{}^{k}\, d^3x_{(o)} = 0 \tag{17b}$$

$$\int \tilde{T}_{(o)\,4}{}^{4}\, d^3x_{(o)} = -\lim_{a\to o}\left(\frac{1}{2a}-\frac{M}{2}\right) \tag{17c}$$

With (7), (16) and (1`˙, it is easy to see that the conditions (9) are indeed satisfied. Note also that the self-energy of the electron in the rest frame of the electron is finite,

$$E_{(o)} = \int \bar{T}_{(o)\,4}{}^{4}\, d^3x_{(o)} = \frac{M}{2}$$

$E_{(o)}$ is interpreted as the electromagnetic mass of the electron. The observed mass is the sum of the electromagnetic mass and the mechanical mass.

## Concluding Remarks

We have shown that with the introduction of a shadow field the electron is stable in the point particle limit. This is due to the fact that the shadow field provides an attractive force to keep the electron together. The idea of introducing an additional "non-electromagnetic" force to compensate for the Maxwell stress, producing stability of the charged particle and making the total self-energy vanish in the rest frame, was first suggested by Poincare a long time ago.[5] Except for the non-electromagnetic character of the force, as it was postulated, the origin of this Poincare's tensor was not clear. In contrast to the non-electromagnetic character of the Poincare's tensor, the interaction between the shadow field and the charged particle is, in terms of the strength of the coupling constant, electromagnetical in character. Physically, the presence of the shadow field is to introduce a small non-local effect in a manifestly local fashion, therefore the electromagnetic character of the interaction is understandable.

Another difficulty encountered in the classical theory of electrons is the existence of the so called "runaway" solution. A way to avoid it is to impose proper boundary conditions on the solution of the equation of motion. The runaway solutions have also been found in a number of simple, exactly soluble quantum field theories. In a separate paper[6] it has been shown by the author that in the quantum electrodynamics with shadow fields, the runaway modes do not occur in the dipole approximation. Whether the runaway solution can also be avoided in the classical theory with the introduction of shadow field is not yet clear.

## References

1. See, e.g. S.N. Gupta, Proc. Phys. Soc. (London) A64, 50(1951).
F. Röhrlich, Am. J. Phys. 28. 639(1960).
2. S. Deser, Rev. Mod. Phys. 29, 417(1957).
3. E.C.G. Sudarshan, Fundamental Problems in Elementary Particle Physics, XIV. Sovay Congress, Interscience Publishers, New York 1968. More references are given here.
4. E.C.G. Sudarshan, Action-at-a-Distance, Fields and Quanta, 2, 175(1972).
Indefinite Metric Shadow States in Quantum Field Theory, CPT, Preprint 81.
5. H. Poincare, R.C. Circ, Mat. Palermo 21, 129(1906).
6. C.C. Chiang, in preparation.

## Acknowledgement

Thanks are due to Professor E.C.G. Sudarshan for useful discussions and careful reading of the manuscript.