B. I. P. M.
F-92310 SEVRES

Report BIPM-72/13
NOTICE 1991

A new method for distinguishing between pairs and single pulses

by Jörg W. Müller

1. Introduction

In recent years various groups have focussed their interest on problems related to spurious pulses which are an obvious and permanent nuisance in all measurements of absolute disintegration rates. The current methods available for detecting and measuring afterpulses have been well reviewed quite recently in several papers ([1] to [3]), where also earlier references can be found.

Apart from a few rare cases where pulse-height analysis is possible, all these techniques use in one way or another the time relationship which exists between the corresponding "genuine" and "spurious" pulses. Within this general approach, two main variants can be distinguished. Either the time distribution of the intervals is measured, or one analyzes the relative variance in the number of registered counts which can be deduced from repeated measurements. Let us first have a brief look at some of the merits and drawbacks of these "interval" and "counting" techniques.

In the first place, they all suffer more or less from the fact that the influence of dead times is an essentially unsolved problem for parent-daughter decays or similar two-step processes. However, provided that $\rho\tau$, the product of count rate and dead time, is sufficiently small, simple approximate methods for the corresponding corrections will be adequate.

The interval method, among other virtues, has the advantage of great flexibility, as the time origin can be determined either by a genuine or by an arbitrary pulse, and differential or integral distributions can be measured. This technique represents a direct approach to the problem and is capable of yielding fairly detailed information on the time behaviour of the various mechanisms which may be responsible for the production of spurious pulses. Besides, from an experimental point of view, the measurements are rather straightforward and rapid.

The counting technique, on the other hand, requires a higher degree of sophistication in experimentation as well as in the analysis of the results. This is at least the case in its present form where a variance-to-mean ratio has to be exploited. Apart from the dead-time corrections, which are a more serious problem here, some specific assumption about the time behaviour of the after-pulses is needed (e.g. exponential) to permit unambiguous conclusions. Although fine achievements have been made recently in this field ([4], [5]), much work still remains to be done. For other versions of counting methods with gateing see [1].

Since a characteristic, but often badly known time distribution between a parent (or genuine) pulse and its daughter pulse (or afterpulse) is the only recognizable feature of "pairs" in a train of pulses – the physical causes for the relationship being largely unknown and therefore out of control –, it seems natural that statistical methods have to be applied in any attempt to separate such pairs from single pulses. A distinction can therefore only be expected for large samples, but not for individual events. This is what is actually done in all the techniques mentioned before, but other possibilities for extracting the wanted information might exist.

2. Separation by means of a modulo 2 counter

In order to achieve such a discrimination we are going to suggest a somewhat different approach which is based on a particularly simple variant of the correlation technique. If we restrict ourselves to the case where a primary pulse cannot be followed by more than one secondary pulse (thus neglecting multiple afterpulses), then any measured count in the superimposed process is either a "single" or belongs to a "pair".

Our problem is therefore equivalent to finding a practical way to distinguish between these two classes, e.g. by counting the pairs or the singles alone – if this can be achieved. We think that the special form of the correlation method as used previously ([6], [7]) might offer an interesting and simple solution to this problem.

We recall that in this variant a two-valued function $x(t)$ is associated with the counting process which jumps at each arrival of a pulse from -1 to $+1$ or vice versa, depending on the previous state. This is also done (with the same process) after a delay δ. A simple electronic arrangement then allows us, by measuring an average count rate, to determine the autocorrelation function

$$R(\delta) = E\left\{x(t) \cdot x(t+\delta)\right\} . \tag{1}$$

If $W(k)$ is the probability for measuring exactly k counts within a time interval δ (with random origin), the correlation function may also be written in the form

$$R(\delta) = \sum_{k=0}^{\infty} W(k) \cdot (-1)^k = \text{Prob } (k \text{ even}) - \text{Prob } (k \text{ odd}) . \tag{2}$$

Now, the total number of pulses can always be decomposed into "pairs" and "singles", thus

$$k = 2\, n_p + n_s , \tag{3}$$

where n_p is the number of pairs and n_s of single pulses within the time δ.

Whether k is even or odd depends therefore only on the number of single pulses, hence

$$R(\delta) = \sum_{n_s=0}^{\infty} W(n_s) \cdot (-1)^{n_s} \quad . \tag{4}$$

This relation holds quite generally and is independent of any assumption about the probability distribution.

An experimental measurement of the correlation function $R(\delta)$ is thus not at all affected by the presence of pairs. As a matter of fact, this is an obvious consequence of the construction of the correlator which (in the present form) measures the difference in the probabilities for counting an even or an odd number of events in δ. It is therefore basically a modulo 2 counter.

As in (4) no interval distribution is needed, but only the probability for a given number of (unpaired) events, this relation may also be applied to non-homogeneous processes. With the help of the well-known result ([6], [7]) that for a Poisson process (with count rate ρ) the correlation function is given by

$$R(\delta) = e^{-2\rho|\delta|}, \tag{5}$$

it now follows (see Appendix A) that for a Poisson distribution of the primary events we always have

$$R(\delta) = e^{-2\mu_s} \quad . \tag{6}$$

Here μ_s is the mean number of uncorrelated single pulses in the interval δ.

In order to illustrate more explicitly the effect on the correlation function, let us consider two specific assumptions for the time relationship between main pulse and afterpulse in some more detail.

3. Exponential time distribution

This case has already been treated earlier in connection with the parent-daughter problem. If

ρ = true count rate of the primary events,

τ = average time interval between primary and secondary event,

$e_{1,2}$ = detection probabilities for a primary (secondary) event,

θ = probability for afterpulsing (per genuine pulse) and

b = experimental count rate for background,

then it can be shown [8] that the mean number of uncorrelated single pulses in a time interval δ is given by

$$\mu_s = \rho\left\{\varepsilon_1 + \varepsilon_2 - 2\,\varepsilon_1\,\varepsilon_2\left[1 - \frac{\tau}{|\delta|}\left(1 - e^{-|\delta|/\tau}\right)\right]\right\}|\delta| + b|\delta| . \tag{7}$$

where now $\varepsilon_1 \equiv e_1$, but

$$\varepsilon_2 \equiv \theta \cdot e_2 .$$

In what follows, ε_2 thus always means the "effective efficiency" for afterpulses, with θ included.

By using the abbreviations introduced previously in a similar context [9], namely

$$\begin{aligned} \alpha &\equiv \rho(\varepsilon_1 + \varepsilon_2 - \varepsilon_1\,\varepsilon_2) + b \quad \text{and} \\ \beta &\equiv \rho \cdot \varepsilon_1\,\varepsilon_2 , \end{aligned} \tag{8}$$

the expectation for singles may be written as

$$\mu_s = (\alpha - \beta)\,|\delta| + 2\,\beta\,\tau\left(1 - e^{-|\delta|/\tau}\right) . \tag{9}$$

Since a direct measurement of the total count rate yields

$$\rho_{tot} = \rho\,(\varepsilon_1 + \varepsilon_2) + b = \alpha + \beta , \tag{10}$$

an equivalent form of (9) is also

$$\mu_s = \rho_{tot}\,|\delta| - 2\,\beta\left(|\delta| - \tau + \tau \cdot e^{-|\delta|/\tau}\right) . \tag{9'}$$

From (9) or (9') we obtain readily the limiting cases

$$\mu_s = \begin{cases} \rho_{tot}\,|\delta| = (\alpha + \beta)\,|\delta| & \text{for } |\delta| \ll \tau \\ (\alpha - \beta)\,|\delta| & \text{"}\ \ |\delta| \gg \tau . \end{cases}$$

We may note that both these limits are actually independent of the specific time distribution chosen here (see Appendix B).

The correlation function is now easily obtained inserting (9) or (9') into (6) as

$$\begin{aligned} R(\delta) &= \exp\left\{-2(\alpha - \beta)\,|\delta| - 4\,\beta\,\tau\left(1 - e^{-|\delta|/\tau}\right)\right\} \\ &= \exp\left\{-2\,\rho_{tot}\,|\delta| + 4\,\beta\left(|\delta| - \tau + \tau \cdot e^{-|\delta|/\tau}\right)\right\} . \end{aligned} \tag{11}$$

A convenient graphical representation of the correlation function is for instance obtained by plotting the quantity $-\frac{1}{2} \cdot \ln R(\delta)$ as a function of the delay δ , which is according to (6) just μ_s (compare Fig. 1a). The experimental curve lies between the two straight lines representing the initial and the final slopes $\alpha \pm \beta$, respectively, and passes at $|\delta| = \tau$ through the point

$(\alpha + \beta)\tau - 2\beta\tau/e \approx (\alpha + 0.264\,\beta)\tau$.

A result equivalent to (11), apart from printing errors, has actually been obtained previously [10]. The much more elaborate method, however, has the drawback that it is not evident that R is entirely determined by the unpaired pulses alone, as is clearly shown by (4) or (6). Since in both this approach and the present one the poissonian nature of the (surviving) parent pulses is used in an essential way for the proof (by assuming an exponential interval density for any time origin), neither can be used to take dead-time effects into account in a rigorous way.

4. Constant time interval

In this model, an afterpulse is supposed to occur (if at all) at a fixed time lag τ after the genuine pulse. If the same notation is used as before, the experimental pair rate for a very large measuring interval is given by

$$\rho_p = \rho \cdot e_1 \cdot \theta e_2 = \rho\, \varepsilon_1 \varepsilon_2 = \beta \ . \tag{11a}$$

Applying (10), this leaves for the rate of the singles

$$\begin{aligned} \rho_s &= \rho_{tot} - 2\rho_p \\ &= \rho(\varepsilon_1 + \varepsilon_2 - 2\,\varepsilon_1 \varepsilon_2) + b = \alpha - \beta \ . \end{aligned} \tag{12b}$$

We now have to determine the distribution of the pairs in the interval δ.
This problem is very similar to the one considered in [8] for the exponential time distribution, but is actually simpler as the density corresponding to a constant distance τ is just the delta function $\delta(t - \tau)$. For the survival probability of a pair (with primary pulse at t) this yields

$$q(t) = \begin{cases} 1 & \text{if} \quad 0 < t < |\delta| - \tau \\ 0 & \text{"} \quad |\delta| - \tau < t < |\delta|. \end{cases}$$

The corresponding average probability is therefore

$$\begin{aligned} \bar{q} &= \frac{1}{|\delta|}\int_0^{|\delta|} q(t)\, dt = \begin{cases} \frac{1}{|\delta|}(|\delta| - \tau) & \text{for} \quad \tau \leq |\delta| \\ 0 & \text{"} \quad \tau \geq |\delta| \end{cases} \\ &= \text{Max}\left\{1 - \frac{\tau}{|\delta|},\ 0\right\} \ . \end{aligned} \tag{13}$$

This gives for the number of pairs within an interval δ the expectation

$$\mu_p = \varrho_p \bar{q} |\delta| = \varrho \theta e_1 e_2 \bar{q} |\delta| = \beta \bar{q} |\delta| . \tag{14}$$

The average number of uncorrelated pulses in δ is therefore

$$\mu_s = \varrho_{tot} |\delta| - 2 \mu_p ,$$

which after some elementary rearrangements can be brought into the form

$$\mu_s = \begin{cases} (\alpha + \beta) |\delta| & \text{for } |\delta| \leq \tau \\ (\alpha - \beta) |\delta| + 2 \beta \tau & \text{"} \ |\delta| \geq \tau \end{cases} \tag{15}$$

which is equivalent to

$$\mu_s = \begin{cases} \varrho_{tot} |\delta| & \text{for } |\delta| \leq \tau \\ \varrho_{tot} |\delta| - 2 \beta (|\delta| - \tau) & \text{"} \ |\delta| \geq \tau . \end{cases} \tag{15'}$$

Since we know that pairs as well as single pulses form an inhomogeneous Poisson process for any interval distribution [8], we now obtain the correlation function by simply inserting (15) into (6) as

$$R(\delta) = \begin{cases} \exp \left[-2 \varrho_{tot} |\delta| \right] & \text{for } |\delta| \leq \tau \\ \exp \left[-2 \varrho_{tot} |\delta| + 2 \beta (|\delta| - \tau) \right] & \text{"} \ |\delta| \geq \tau . \end{cases} \tag{16}$$

Fig. 1b shows that plotting $-\frac{1}{2} \cdot \ln R(\delta)$ versus δ is again a convenient method for determining τ as well as $\beta = \theta \cdot \varrho \ e_1 e_2$, which corresponds to the sudden change of slope occurring at $|\delta| = \tau$. If the other parameters can be assumed to be known, this therefore leads to a direct determination of the probability θ for the generation of afterpulses.

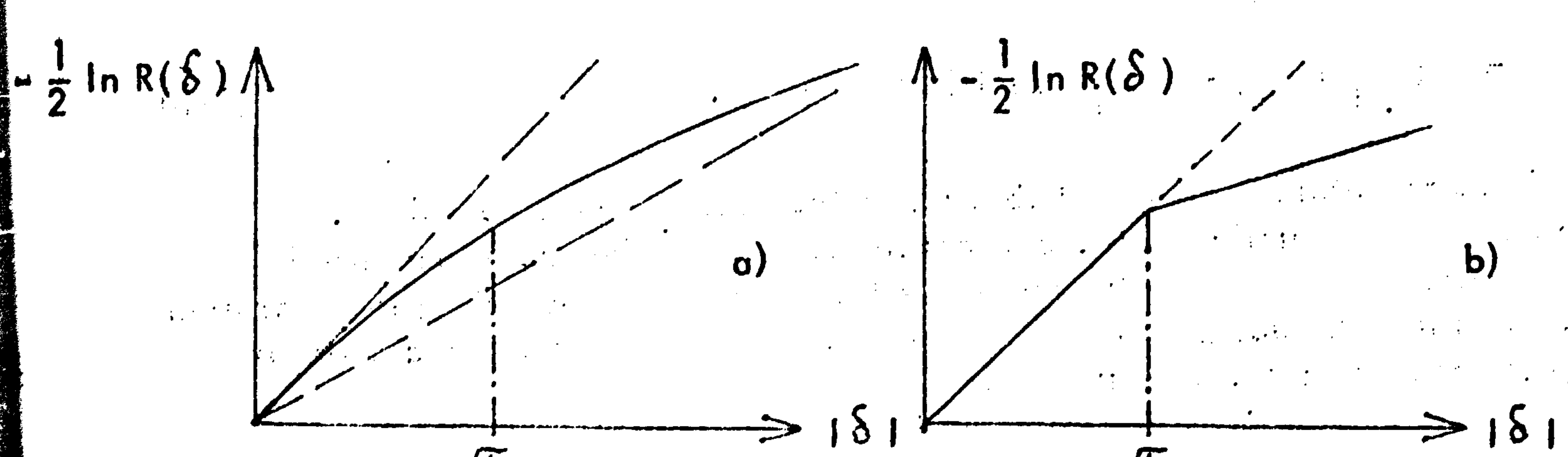

Fig. 1. Schematic behaviour of the correlation function $R(\delta)$ in the presence of afterpulses, if these have a) an exponential, b) a constant interval distribution with respect to the primary event, each time with mean τ. For details see text.

5. Final remarks

It may be interesting to note that in this method the quantity we are actually looking for, namely the pair rate β, is essentially obtained as the difference between two measured mean values (see Fig. 2), whereas in previous techniques the corresponding quantity had to be calculated from a difference of variances. Therefore a better precision might perhaps be expected for the new approach.

Finally, we may mention that the usefulness of this method should be largely independent of the presence of dead times. If the pairs are not too frequent, as will be the case for afterpulses, the "surviving" events form to a good approximation a dead-time-distorted Poisson process. Since the autocorrelation function is well known for this case [11], the corresponding influence can be taken into account.

It thus seems from what has been said above that such a correlator with only two possible states might offer itself as a nearly ideal instrument for distinguishing between paired and single events in a series of pulses. Nevertheless, some caution might well be in order here as no attempt has yet been made to check the feasibility of this idea experimentally.

A generalization of this method for determining quantitatively the occurrence of multiple pulses will be presented in another report.

APPENDICES

A. Direct derivation of (6) for a Poisson process

It has been shown previously ([12], eq. 12), that for the case of a parent-daughter process with parent pulses following the Poisson law, the probability for observing exactly k events is given by

$$W(k) = e^{-(\mu_s + \mu_p)} \sum_{j=0}^{K} \frac{\mu_p^{\,j}}{j!} \cdot \frac{\mu_s^{\,k-2j}}{(k-2j)!} , \qquad \text{(A1)}$$

where μ_s and μ_p are the expectations for the number of singles and pairs, respectively, and K is the largest integer below $(k+1)/2$.

Let us briefly consider two simple special cases of (A1).

a) $\mu_p = 0$, i.e. complete absence of all (true) pairs. As j in (A1) stands for the number of pairs, the sum reduces to the term $j=0$, thus

$$W(k) = e^{-\mu_s} \cdot \frac{\mu_s^{\,k}}{k!} ,$$

which is an ordinary Poisson distribution for k.

b) $\mu_s = 0$, i.e. there are only pairs. In this case, the only term remaining in the sum (A1) is for $j=k/2$, which requires that k be even. Hence

$$W(k) = \begin{cases} e^{-\mu_p} \cdot \dfrac{\mu_p^j}{j\,!} & \text{for } k = 2j \\ 0 & \text{" } k \text{ odd.} \end{cases}$$

We therefore arrive at a Poisson distribution for the pairs, as expected.

Let us now evaluate the probability for k even on the basis of (A1).

$$\text{Prob}(k \text{ even}) = \sum_{n=0}^{\infty} W(k=2n) \quad .$$

$$= e^{-(\mu_s+\mu_p)} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{\mu_p^j}{j\,!} \cdot \frac{\mu_s^{2n-2j}}{(2n-2j)\,!} \quad ,$$

where, applying its definition given in (A1), K has been replaced by n since k=2n.

Formally, the sum over j may be extended to infinity as $1/(n-j)\,! = 0$ for $j > n$. By reversing the order of the summations we get

$$\text{Prob}(k \text{ even}) = e^{-(\mu_s+\mu_p)} \sum_{j=0}^{\infty} \frac{\mu_p^j}{j\,!} \sum_{n=0}^{\infty} \frac{\mu_s^{2(n-j)}}{[2(n-j)]\,!} \quad .$$

But since [12], with $s = n-j$,

$$2 \sum_{s=0}^{\infty} \frac{\lambda^{2s}}{(2s)\,!} = e^{\lambda} + e^{-\lambda} \;, \tag{A2}$$

we may also write

$$\text{Prob (k even)} = e^{-(\mu_s+\mu_p)} \sum_{j=0}^{\infty} \frac{\mu_p^j}{j\,!} \cdot \frac{1}{2} (e^{\mu_s} + e^{-\mu_s})$$

$$= \frac{1}{2} (1 + e^{-2\mu_s}) \quad . \tag{A3}$$

As k can only be even or odd, there is obviously

$$\text{Prob}(k \text{ odd}) = 1 - \text{Prob}(k \text{ even}) = \frac{1}{2} (1 - e^{-2\mu_s}) \quad . \tag{A4}$$

Equation (2) then yields for the correlation function

$$R(\delta) = \frac{1}{2} \left[(1 + e^{-2\mu_s}) - (1 - e^{-2\mu_s}) \right] = e^{-2\mu_s} \;, \tag{6}$$

as we had expected.

B. Limiting values for μ_s and $R(\delta)$

For a sufficiently short time interval δ, we have only to consider the cases that 1 or 0 pulses arrive, as the probability for several events can be neglected for $|\delta| \ll \rho_{tot}^{-1}$. Thereby we obviously assume a "smooth" behaviour and in particular the absence of a delta function near the origin of the interval density. This therefore excludes the case where $\tau \longrightarrow 0$ (compare for example [9]). We then have

$$\begin{aligned} \text{Prob(k odd)} &= \text{Prob}(1) \approx \rho_{tot} \, |\delta| \; , \\ \text{Prob(k even)} &= \text{Prob}(0) \approx 1 - \rho_{tot} \, |\delta| \; . \end{aligned} \tag{B1}$$

This gives with (2) for the correlation function

$$\begin{aligned} R(\delta) &= \text{Prob(k even)} - \text{Prob(k odd)} \\ &\approx 1 - 2\,\rho_{tot}\,|\delta| \qquad \text{for } |\delta| \ll \rho_{tot}^{-1} \; . \end{aligned} \tag{B2}$$

The correlation function thus always starts for $\delta = 0$ at R=1 and then decreases linearly with the slope

$$-2\,\rho_{tot} \cdot \text{sign}\,(\delta) \; . \tag{B3}$$

This general feature has previously been used in a more complicated example [11].

It will be obvious that for $\delta \longrightarrow 0$

$$\mu_s = \rho_{tot}\,|\delta| = (\alpha + \beta)\,|\delta| \quad \text{and} \quad \mu_p = 0 \; , \tag{B4}$$

as pairs require a finite interval length to "survive" (Fig. 2).

The initial linear behaviour (B2) of the correlation function $R(\delta)$ is thus quite a general feature which is not restricted to a specific process or interval distribution.

In order to determine μ_s for the case of a very long delay, we restrict ourselves to a Poisson process for the original pulse sequence. For $|\delta| \gg \tau$, however, the relative contribution to μ_s from such (original) pairs where one of the partners happens to fall outside the beginning or the end of the measuring interval δ becomes negligible (edge effect). We therefore have for $|\delta| \gg \tau$

$$\mu_s = (\rho_{tot} - 2\,\rho\,\varepsilon_1\,\varepsilon_2)\,|\delta| = (\alpha - \beta)\,|\delta| \; , \tag{B5}$$

independently of the exact interval distribution for pairs (cf. Fig. 2), and the correlation function goes over into the simple exponential

$$R(\delta) = \exp\left\{-2(\alpha - \beta)\,|\delta|\right\} \; . \tag{B6}$$

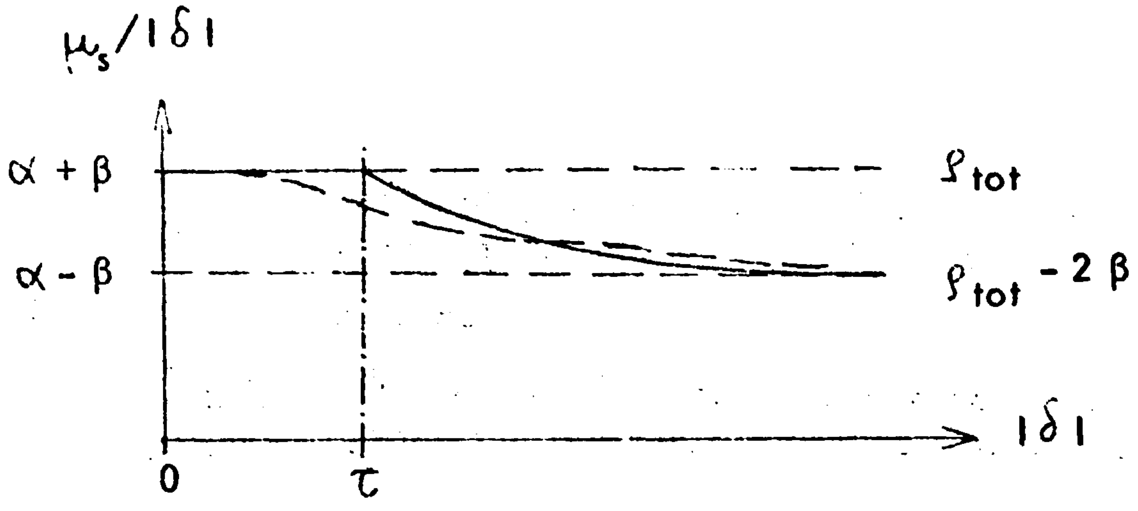

Fig. 2. Schematic plot of the effective count rate $\mu_s / |\delta|$ for single pulses as a function of the measuring interval δ

<u>References</u>

[1] "The detection and estimation of spurious pulses", (first draft of a planned BIPM monograph). Personal communication from Dr. P.J. Campion, NPL (June 1972)

[2] P.J. Campion: "Spurious pulses in proportional counters: a review"

[3] A. Williams, D. Smith: "Afterpulses in liquid scintillation counters"

[4] A. Williams, V.E. Lewis, D. Smith: "Correlation techniques in radioactivity measurement"

The papers 2, 3 and 4 have been presented at the "First International Summer School on Radionuclide Metrology" at Herceg Novi (August 1972) and will be published in the Proceedings.

[5] V.E. Lewis, D. Smith, A. Williams: "Correlation counting applied to the determination of absolute disintegration rates for nuclides with delayed states", to be published in Metrologia

[6] G. Landaud, C. Mabboux: "Analyse du coefficient de corrélation de la fonction aléatoire X(t) = $\pm$ 1. Application à l'étude de lois de désintégrations de radioéléments", J. Physique Rad. <u>21</u>, 615 (1960)

[7] J.W. Müller: "Etude de comptages à l'aide d'un corrélateur", Rapport BIPM-70/6 (July 1970)

[8] id.: "Some notes on pair statistics", Report BIPM-72/13 (December 1972)

[9] id.: "Intervalles entre impulsions corrélées", Rapport BIPM-72/3 (March 1972)

[10] G. Landaud, C. Mabboux: "Utilisation de la fonction aléatoire X(t) = $\pm$1 à l'étude des lois de désintégration de radioéléments à filiations", Comptes Rendus <u>250</u>, 3310 (1960)

[11] J.W. Müller: "Sur la corrélation d'une série d'impulsions avec temps mort", Rapport BIPM-70/8 (December 1970)

[12] id.: "Traitement statistique des résultats de mesure", Rapport BIPM-108 (May 1969 ff), p. 64/7

(December 1972) [redacted]