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APPLICATION OF THE RENORMALIZATION GROUP TECHNIQUE TO THE PROBLEM OF PHASE TRANSITION IN ONE-DIMENSIONAL METALLIC SYSTEMS

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I. INVARIANT COUPLINGS, VERTEX AND ONE-PARTICLE GREEN'S FUNCTION

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ABSTRACT

A one-dimensional system of electrons interacting via a BCS-type interaction is investigated, by renormalization group technique, in two successive approximations at $T = 0$, keeping only a single energy variable ω. The first approximation is equivalent to the summation of leading logarithmic terms carried out by Bychkov et al. and correspondingly the vertex function displays a singularity at a finite value of ω. The second approximation accounts for the next leading logarithmic terms as well, and by this means the singularity is shown to be pushed down to $\omega = 0$. Due to important self energy contributions, however, the invariant couplings behave differently and tend to a saturation value at $\omega = 0$.

РЕЗЮМЕ

Методом ренормализационной группы была исследована одномерная взаимодействующая система электронов при $T = 0^{o}K$ температура, в двух последовательных приближениях с учётом только одного параметра энергии ω и взаимодействия типа БКШ. Первое приближение оказалось эквивалентным приближением Бычкова - Горькова - Дзялошинского, основанным на суммировании старших логарифмических членов; следовательно, вершинная функция при конечном значении ω имеет сингулярность. Во втором приближении учитываются и непосредственно следующие логарифмические члены. Показано, что их учет приведет к перемещению сингулярности к $\omega=0$. Однако, из-за существенных собственно энергетических поправок инвариантные постоянные связи имеют другое поведение: при $\omega=0$ они имеют ограниченный максимум.

KIVONAT

Egydimenziós, kölcsönható elektron-gáz rendszert vizsgáltunk BCS tipusu kölcsönhatás esetén két egymásutáni közelitésben zérus hőmérsékleten a renormálási csoport módszer segitségével, egyetlen energiaváltozóra szoritkozva. Az első közelités ekvivalensnek bizonyult a vezető logaritmikus tagok Bicskov, Gorkov és Dzjalosinszkij által véghezvitt felösszegezésével; következésképpen a vertex függvény egy véges ω értéknél szinguláris viselkedést mutat. A második közelités az eggyel alacsonyabb rendü logaritmikus tagokról is számot ad. Megmutattuk, hogy ilyen módon a szingularitás $\omega = 0$-ra tolódott le. Lényeges sajátenergiás járulékok következtében az invariáns csatolások ettől eltérő viselkedésüek: $\omega = 0$-nál véges értékü maximumuk van.

1. Introduction

The article by Little[1] concerning the possibility of superconductivity at high temperatures aroused considerable interest in investigating one-dimensional metallic systems. It is well known that no phase exhibiting long-range order can exist in one dimension at finite temperatures provided the forces are of short range[2], thus the theorem does not apply to the BCS reduced Hamiltonian[3], where the interaction is of infinite range. A more realistic approach to the problem of superconducting type phase transitions is to use the Ginzburg-Landau[4] functional for the free energy. This has been done by Rice[5], who showed that, if the existence of superconducting order below some temperature T_c is supposed, the fluctuations in the phase of the order parameter act in such a way that the renormalized equilibrium value of the order parameter becomes zero in one and two dimensions. The applicability of the Ginzburg-Landau theory in one dimension, however, is itself open to question[6]. Using quite general arguments, the Bogoliubov inequality and the f sum-rule, Hohenberg[7] has obtained the same result as Rice: the absence of superconducting type ODLRO[8] in one- and two-dimensional systems.

Bychkov, Gorkov and Dzyaloshinsky[6] /BGD in the following/, on the other hand, reached a different conclusion when they investigated possible singular behaviour of the vertex part in a one-dimensional system of electrons

interacting via an interaction of BCS type but of finite range. Besides the Cooper diagrams, BGD took into account zero-sound bubbles, too, both of which are logarithmic in one dimension. The vertex function resulting from the solution of the hierarchy of parquet equations pointed to the existence of a critical temperature in essentially the same manner as in the BCS case.

In a recent paper Dzyaloshinsky and Larkin[9] emphasize, however, that the pole of the scattering amplitude obtained by BGD in the parquet approximation does not indicate a phase transition at some finite temperature; it shows only that at low temperatures the effective interaction becomes strong and the parquet approximation is no longer applicable.

A comparable situation pertains in the Kondo problem[10], where Abrikosov's[11] solution of the parquet equations leads to a scattering amplitude which diverges at the Kondo energy /temperature/. Fowler and Zawadowski[12], using the renormalization group theory[13], have presented an improved treatment of this problem analogous to the self consistent treatment of the X-ray problem by Nozières et al.[14] Fowler and Zawadowski's second-order scaling approximation goes beyond the leading logarithmic approximation of Abrikosov by accounting for the next leading logarithmic terms by means of the Lie differential equation of the renormalization group. The resulting scattering amplitude is free from the non-physical singularity. A similar result has been obtained by Abrikosov and Migdal[15].

It can be expected therefore that an analogous improved treatment of a one-dimensional system of electrons interacting via a BCS type /but finite-range/ interaction would remove the BGD-singularity at finite temperatures. It is the aim of the present paper to show that this really is the case.

In Sec.2 we describe the model. The interaction matrix elements are specified in such a way that the interaction Hamiltonian contains only two independent coupling constants. The particular choice of the frequency and momentum variables for which the vertices will be calculated is also given here. In Sec.3 a survey of the renormalization group technique is presented and its most important ingredients the invariant couplings are defined and discussed.

Sec.4 is devoted to the calculation of the two invariant couplings and the vertex in leading logarithmic approximation /first order renormalization/. The result obtained corresponds exactly to that of BGD /see eq. (11) of their paper/. Second-order renormalization is carried out in Sec. 5. By collecting $g^3 \ln \omega/\omega_0$ type contributions to the vertex and consistently $g^2 \ln \omega/\omega_0$ type terms in the self energy, the invariant couplings are shown to satisfy differential equations with solutions which exhibit no singularity as a function of ω and behave just in the same way as the corresponding quantity does in the Kondo

problem. On calculating the imaginary parts of the perturbational expressions, too, it turns out that these play no role whatsoever in the disappearance of the spurious singularity, at least in so far as the invariant couplings are concerned. These latter are all real. Knowing the invariant couplings as functions of a scaling energy, the vertex function and the one-particle Green's function are then determined from the corresponding Lie equation in an approximation which, however, does not account for the imaginary parts. It is found that the solutions for $\omega \to 0$ show a power law behaviour as $\Gamma(\omega) \sim \omega^{-\alpha}$ and $G(\omega) \sim G^{(0)}(\omega)\, \omega^{\beta}$, where in our approximation $\alpha = 3/2$ and $\beta = 3/4$. Thus second-order renormalization shifts the infinite growth of Γ, which occurs at a finite value of ω in the parquet approximation, to $\omega = 0$. When the temperature is taken as variable, instead of ω, this shows the tendency of an ordered phase to form only at $T=0$, as would be expected. This point will be examined more closely in the second paper of this series, in which various response functions will be calculated in the framework of the renormalization group technique.

A discussion of our results is given in Sec. 6.

2. The model

Let us consider a one dimensional system of electrons described by the Hamiltonian

$$H = H_0 + H_{int} \quad , \qquad /2.1/$$

$$H_0 = \sum_{k,\alpha} \varepsilon_k \, c^+_{k\alpha} \, c_{k\alpha} \,, \qquad /2.2/$$

$$H_{int} = \frac{1}{2L} \sum_{k_i,\alpha,\beta} g(k_1,k_2,k_3,k_4) \, c^+_{k_1\alpha} \, c^+_{k_2\beta} \, c_{k_3\gamma} \, c_{k_4\delta} \,, \qquad /2.3/$$

where $c^+_{k\alpha}$ is the creation operator of an electron with momentum k and spin α ; ε_k is the kinetic energy of the electron; and $g(k_1, k_2, k_3, k_4)$ is the interaction matrix element, which is restricted to a narrow energy range - characterized by the cut-off energy ω_c -in the vicinity of the Fermi energy.

The vertex $\Gamma_{\alpha\beta\gamma\delta}(k_1, \omega_1, k_2, \omega_2; k_3, \omega_3, k_4, \omega_4)$ depends on three independent momenta and three independent frequencies, which are chosen as usual to be k_1+k_2, k_3-k_1, k_4-k_1 and $\omega_1+\omega_2$, $\omega_3-\omega_1$, $\omega_4-\omega_1$ /$k_1+k_2 = k_3+k_4$; $\omega_1+\omega_2 = \omega_3+\omega_4$/.

It will be seen later that for our purposes the vertex can be calculated for a special choice of the variables. We want to restrict our calculation to a single frequency variable ω and fix the momentum variables so that the most singular terms are picked up. This will be done as follows.

If only the second-order vertex corrections are taken into account, the three diagrams shown in Fig. 1 can be drawn /see Ref. 6/. Fig. 1a is the Cooper-

type bubble, it depends on k_1+k_2 and $\omega_1+\omega_2$ in a logarithmic fashion provided these variables are small compared with ω_p . Fig. 1b shows the zero-sound-type bubble, which depends on k_3-k_1 and $\omega_3-\omega_1$ and also shows a logarithmic dependence if $|k_3-k_1| \approx 2k_o$ /leading to a doubling of the period/ and $\omega_3-\omega_1 \approx 0$. If the variables are chosen so that the above relations are fulfilled and also the momenta lie close to the Fermi momentum, the $k_4-k_1 \approx 0$ and the contribution of Fig. 1c is negligible. Accordingly we shall choose the momentum variables of the vertex to be $k_1=-k_o$, $k_2=k_o$, $k_3=k_o$, $k_4=-k_o$ /k_o is the Fermi momentum/. Moreover, the energies on all four legs of the vertex will be of the same order of magnitude, e.g. $\omega_1=\frac{3}{2}\omega$, $\omega_2=-\frac{1}{2}\omega$, $\omega_3=\frac{1}{2}\omega$, $\omega_4=\frac{1}{2}\omega$ can be taken/see Fig. 2/.

The restriction to a single variable in calculating the vertex is sufficient if only the invariant coupling is to be determined. It does not matter whether his variable is energy or momentum. We work here with a single energy variable, but exactly the same results would be obtained by taking all the energies equal to zero and keeping a single momentum variable q with the choice $k_1=-k_o$, $k_2=k_o+q$, $k_3=k_o-q$, $k_4=-k_o+2q$. In both cases the same number of typical logarithmic terms appear.

The generalization to more variables does not encounter any basic difficulties, though the actual calculation becomes much more complicated, especially in higher orders.

Forgetting Umklapp processes, the vertices are singular only when they describe the scattering of an electron with momentum near to $+k_0$ on another electron with momentum near to $-k_0$. When both incoming electrons have momenta near to $+k_0$ or $-k_0$, the contribution of these vertices is negligible. Those processes will therefore be neglected from the interaction in Eq. /2.3/, in which both k_1 and k_2 and similarly k_3 and k_4 are in the vicinity of $+k_0$ or $-k_0$. Only the following interaction matrix elements will be retained:

$$g_j(k_1,k_2,k_3,k_4) = \begin{cases} g_1 \begin{cases} \text{if} & -k_0-k_D \le k_1,k_3 \le -k_0+k_D \quad \text{and} \quad +k_0-k_D \le k_2,k_4 \le +k_0+k_D \\ \text{or} & -k_0-k_D \le k_2,k_4 \le -k_0+k_D \quad \text{and} \quad +k_0-k_D \le k_1,k_3 \le +k_0+k_D \end{cases} \\ g_2 \begin{cases} \text{if} & -k_0-k_D \le k_1,k_4 \le -k_0+k_D \quad \text{and} \quad +k_0-k_D \le k_2,k_3 \le +k_0+k_D \\ \text{or} & -k_0-k_D \le k_2,k_3 \le -k_0+k_D \quad \text{and} \quad +k_0-k_D \le k_1,k_4 \le +k_0+k_D \end{cases} \\ 0 \quad \text{otherwise} \end{cases} \tag{2.4}$$

where k_D is a cut-off in momentum space corresponding to the cut-off ω_D in energy representation.

/ The effect of Umklapp processes has been investigated in the parquet approximation by Dzyaloshinsky and Larkin[9];

an improved approximation is to be presented by one of the present authors in a subsequent publication./

With this special choice of the matrix elements the elementary vertex is the following

$$\Gamma^{(0)}_{\alpha\beta,\gamma\delta} = g_1 \delta_{\alpha\gamma} \delta_{\beta\delta} - g_2 \delta_{\alpha\delta} \delta_{\beta\gamma} . \qquad /2.5/$$

Two special cases are worth mentioning here. $g_1=g_2=g$ corresponds to the Little[1] interaction investigated by BGD in Sec. II of their paper. The choice $g_2=0$ leads to the phonon-mediated electron-electron interaction of BCS: g_1 describes processes in which the momentum transfer is equal to $2k_0$, and it is this matrix element which is large and negative in the case of electron-phonon interaction. From the point of view of superconducting-type behaviour, then, g_1 would be expected to represent the most important part of the matrix elements. The results of this and the following paper bear out this expectation. One important point worth noting here is that even if we wanted to restrict our calculation to the special cases $g_1=g_2=g$ or $g_1\neq 0$, $g_2=0$ only, it would still have been necessary to introduce two different coupling constants at the beginning and to make the specialization only at the end of the calculation. The reason is that in both cases higher-order contributions to Γ yield, in general, a spin structure

$$\Gamma_{\alpha\beta,\gamma\delta} = \Gamma_1 \delta_{\alpha\gamma} \delta_{\beta\delta} - \Gamma_2 \delta_{\alpha\delta} \delta_{\beta\gamma} , \qquad /2.6/$$

with $\Gamma_1 \neq \Gamma_2$, different from that of the elementary vertex, and therefore, as will be seen, the two couplings are renormalized differently. Later on, when we use perturbational expressions of Γ on the right-hand side of the Lie equation, the prescription is to replace all bare coupling constants by the invariant ones, which can only be done properly if it is assumed already from the beginning that $g_1 \neq g_2$ and is different from zero.

The Hamiltonian /2.3/ can be rewritten, taking into account the above restrictions on the interaction matrix elements, in the form

$$H_{int} = \frac{g_1}{L} \sum a^+_{k_1\alpha} b^+_{k_2\beta} a_{k_3\beta} b_{k_4\alpha} + \frac{g_2}{L} \sum a^+_{k_1\alpha} b^+_{k_2\beta} b_{k_3\beta} a_{k_4\alpha}, \qquad /2.7/$$

where $a^+_{k\alpha}$ denotes the creation operator for electrons with momenta lying near to $+k_o$, and $b^+_{k\beta}$ stands for the creation operator of electrons with momenta near to $-k_o$. Diagrammatically these interactions will be represented as shown on Fig. 3. The solid and dashed lines stand for the Green's functions of electrons with momenta near $+k_o$ and $-k_o$, respectively. The two Green's functions actually have the same form

$$G^{(o)}_{\pm}(k,\omega) = G^{(o)}(k,\omega) = \frac{1}{\omega - \xi(k) + i\delta \operatorname{sign} \xi(k)}, \qquad /2.8/$$

where $\xi(k) = \varepsilon(k) - \varepsilon_o = v(|k| - k_o)$ is the kinetic energy relative to the Fermi energy.

3. Survey of the renormalization group technique

The concept of the renormalization group was introduced in quantum field theory to eliminate divergences. These divergences can be compensated by adding counter terms to the Lagrangian, which is equivalent to a multiplicative renormalization of the Green's functions, vertices and charges, with the multiplicative renormalizing factors forming a continuous group. Considering this group, the arbitrariness in the choice of multiplicative factors is in turn equivalent to the introduction of new variables in the Green's functions and vertices, so that the multiplicative renormalization is carried out by changing these variables. A disadvantage of this method is that there is some arbitrariness in the choice of the functions with an increased number of variables; the physical Green's functions and vertices correspond to a particular value of the additional variables.

An analogous multiplicative renormalization procedure was performed by Fowler and Zawadowski[12] for the Kondo problem. It would be possible to carry out a similar treatment in the present problem, too, but instead, we shall give a simplified, though less general formulation. Looking at the low order perturbational expression for the Green's function and vertices, it can be seen that, by changing the cut-off energy ω_D to ω_D' and simultaneously

the coupling constants in an appropriate manner, the Green's function and vertices are multiplied by constants, independent of the frequency variables. This means that the cut-off energy serves as a natural scaling parameter, at least in low orders of the perturbational expansion. The suggestion that the cut-off energy is a good scaling parameter in similar logarithmic problems comes from Anderson[16], who used such a scaling in a very simple way, without applying the renormalization group technique, in the Kondo problem. The drawback of Anderson's derivation of scaling laws, though, is that the effective coupling depends not just on the scaling energy but on other energy and momentum variables, too. We believe that, if scaling indeed exists, the properly defined invariant coupling does not depend on the energy variables of the vertices by which it is itself defined. Nevertheless, as this frequency independence was checked only for low order vertices, the justification of our procedure needs further investigation.

Supposing now that scaling really exists, in the present problem, i.e. a simultaneous and correlated change of the cut-off energy and coupling constants leads to a multiplicative renormalization of the total Green's function and total vertices, the renormalizing factors should be real and independent of frequencies. Therefore in determining them via the Green's function and vertices a very simple choice of variables, the one described in

the previous section, can be made.

It should be emphasized that the existence of scaling is an assumption and does not hold a priori for any problem. But one can check that in the present problem the same results are obtained if one goes back to the more sophisticated formulation of the renormalization group technique, closely following e.g. the prescriptions of Ref. 13.

Now let us write the Green's function and vertex function in the following way:

$$G(k_0,\omega) = d\left(\frac{\omega}{\omega_D}, g_1, g_2\right) G^{(0)}(k_0,\omega) , \qquad /3.1/$$

$$\Gamma_{\alpha\beta\gamma\delta}(\omega) = g_1 \tilde{\Gamma}_1(\omega)\,\delta_{\alpha\gamma}\delta_{\beta\delta} - g_2 \tilde{\Gamma}_2(\omega)\,\delta_{\alpha\delta}\delta_{\beta\gamma} . \qquad /3.2/$$

Changing ω_D to ω_D' and, simultaneously, the couplings g_1 and g_2 to g_1' and g_2', respectively, multiplicative renormalization means that

$$d\left(\frac{\omega}{\omega_D'}, g_1', g_2'\right) = Z_1\, d\left(\frac{\omega}{\omega_D}, g_1, g_2\right) , \qquad /3.3/$$

$$\tilde{\Gamma}_1\left(\frac{\omega}{\omega_D'}, g_1', g_2'\right) = Z_2^{-1}\, \tilde{\Gamma}_1\left(\frac{\omega}{\omega_D}, g_1, g_2\right) , \qquad /3.4/$$

$$\tilde{\Gamma}_2\left(\frac{\omega}{\omega_D'}, g_1', g_2'\right) = Z_3^{-1}\, \tilde{\Gamma}_2\left(\frac{\omega}{\omega_D}, g_1, g_2\right) , \qquad /3.5/$$

$$g_1' = Z_4 g_1 , \qquad g_2' = Z_5 g_2 , \qquad /3.6/$$

where according to our assumption the factors z_i are real and independent of ω .

By requiring the invariance of Dyson's equation under transformations /3.3/-/3.6/, which means that G has to transform in the same way as $G^{(0)}\Gamma^{(0)}GGG\Gamma G$, we get

$$\Xi_4 = Z_1^{-2} Z_2 \quad , \qquad \Xi_5 = Z_1^{-2} Z_3 \, , \qquad /3.7/$$

i.e. only three of the z's are independent. Eq. /3.7/ ensures the equivalence of the new and original states.

The z's can be determined from eqs. /3.3/-/3.5/ by taking $\omega = \omega_0'$. We get

$$Z_1^{-1} = \frac{d\left(\frac{\omega_0'}{\omega_0}, g_1, g_2\right)}{d\left(1, g_1', g_2'\right)} \, , \qquad /3.8/$$

$$Z_2 = \frac{\tilde{\Gamma}_1\left(\frac{\omega_0'}{\omega_0}, g_1, g_2\right)}{\tilde{\Gamma}_1\left(1, g_1', g_2'\right)} \, , \qquad /3.9/$$

$$Z_3 = \frac{\tilde{\Gamma}_2\left(\frac{\omega_0'}{\omega_0}, g_1, g_2\right)}{\tilde{\Gamma}_2\left(1, g_1', g_2'\right)} \, . \qquad /3.10/$$

Thus, using eqs. /3.7/-/3.10/, eq. /3.6/ yields

$$g_i' = g_i \frac{d^2\left(\frac{\omega_0'}{\omega_0}, g_1, g_2\right)}{d^2\left(1, g_1', g_2'\right)} \frac{\tilde{\Gamma}_i\left(\frac{\omega_0'}{\omega_0}, g_1, g_2\right)}{\tilde{\Gamma}_i\left(1, g_1', g_2'\right)} \, , \quad i = 1, 2. \qquad /3.11/$$

The coupling constants g_1' and g_2' are to be determined by the self-consistent solution of eq. /3.11/. They specify how the couplings should be varied simultaneously with the change of the energy scale. As the combination $g\Gamma d^2$ is clearly invariant under transformations /3.3/-/3.6/, the g_i' will be called invariant couplings in what follows. They are sometimes also called invariant charges, as the concept comes originally from quantum electrodynamics.

The multiplicative property of the Green's function, vertex and other physical quantities /e.g. certain response functions/ - which is supposed to hold in our case but does not necessarily hold in general and has to be proved in each case - allows one to improve upon perturbation theoretical results for these quantities by use of the Lie equation of the group. Thus, for any quantity obeying the condition

$$A\left(\frac{\omega}{\omega_b'}, g_1', g_2'\right) = z\, A\left(\frac{\omega}{\omega_b}, g_1, g_2\right), \qquad /3.12/$$

a differential equation of the form

$$\frac{\partial}{\partial x} \ln A(x, g_1, g_2) = \frac{1}{x} \frac{\partial}{\partial \xi} \left[\ln A\left(\xi, g_1'(x, g_1, g_2), g_2'(x, g_1, g_2)\right)\right]_{\xi=1} \qquad /3.13/$$

can be derived, where $x = \omega/\omega_b$. Naturally, eq. /3.13/ applies to the invariant couplings as well.

It must be stressed again that the invariant coupling depends only on ω_b'/ω_b . When it appears in a Lie equation, ω_b' is replaced by the frequency variable, as a function of

which the quantity A is to be determined. It is after this replacement only, that the invariant coupling becomes frequency dependent. It is because in practical calculations we always have to use a Lie equation, and hence a formally frequency dependent invariant coupling, that this procedure is often quoted in the literature as frequency-dependent scaling.

The prescription of the renormalization group technique is to calculate the right hand side of eq. /3.13/ by perturbation theory for $\omega = \omega_0 / \xi = 1/$. The resulting differential equation will produce the quantity A in the whole energy range. This perturbation series goes in powers of the invariant coupling, so that if the invariant coupling is small for an arbitrary change of the scaling energy, the quantity determined from the solution of the Lie equation using a few terms of the perturbation series will represent a good approximation in the whole energy range. If, however, the invariant coupling increases and becomes of the order of unity, while the scaling energy goes towards lower energies, a perturbation expansion of the right hand side of the Lie equation breaks down and only qualitative results can be obtained.

Let us notice that the most important quantity of the theory is the invariant coupling, because it enters the right hand side of eq. /3.13/ for any physical quantity. Thus, before any other quantities are determined, first the invariant coupling has to be known.

4. First-order renormalization

The Lie equation for the invariant couplings g_1' and g_2' are

$$\frac{\partial}{\partial x} \ln g_i'(x, g_1, g_2) = \frac{1}{x} \frac{\partial}{\partial \xi} \left[\ln g_i'(\xi, g_1'(x, g_1, g_2), g_2'(x, g_1, g_2)) \right]_{\xi=1}, \qquad /4.2/4.1/$$

As we have already mentioned, the right hand sides of these equations can be calculated by means of perturbation theory, provided the invariant couplings are small. In the present section the first-order corrections to d, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ /see eqs. /3.1/ and /3.2//will be considered.

The first-order self energy contribution, being independent of ω, can be incorporated into the chemical potential and thus $d/\omega/=1$. It follows from eqs. /3.11/ that

$$g_i'(x, g_1, g_2) = g_i \frac{\tilde{\Gamma}_i(x, g_1, g_2)}{\tilde{\Gamma}_i(1, g_1', g_2')}, \qquad i = 1, 2. \qquad /4.2/$$

The diagrams contributing to Γ in the second order /they give the first-order correction to $\tilde{\Gamma}$ / are shown in Fig.4. Fig. 4a shows the Cooper-type contributions to the vertex while Fig. 4b gives the zero-sound type diagrams. The respective contributions are:

a) $$\frac{1}{2\pi v}\left(\ln\frac{\omega}{\omega_0} - \frac{i\pi}{2}\right)\left[2g_1g_2\,\delta_{\alpha\gamma}\delta_{\beta\delta} - (g_1^2+g_2^2)\,\delta_{\alpha\delta}\delta_{\beta\gamma}\right], \qquad /4.3/$$

b) $$\frac{1}{2\pi v}\left(\ln\frac{\omega}{\omega_0} - \frac{i\pi}{2}\right)\left[(2g_1^2 - 2g_1g_2)\,\delta_{\alpha\gamma}\delta_{\beta\delta} + g_2^2\,\delta_{\alpha\delta}\delta_{\beta\gamma}\right]. \qquad /4.4/$$

Thus up to first order, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are given by the following expressions:

$$\tilde{\Gamma}_1^{(1)} = 1 + \frac{g_1}{\pi v}\left(\ln\frac{\omega}{\omega_D} - \frac{1}{2}i\pi\right), \qquad /4.5/$$

$$\tilde{\Gamma}_2^{(1)} = 1 + \frac{1}{2\pi v}\frac{g_1^2}{g_2}\left(\ln\frac{\omega}{\omega_D} - \frac{1}{2}i\pi\right) \qquad /4.6/$$

From eqs. /4.2/, using eqs. /4.5/ and /4.6/ and expanding the denominators up to first order, the perturbational expressions

$$g_1'^{(1)} = g_1 + \frac{g_1^2}{\pi v}\ln\frac{\omega_D'}{\omega_D}, \qquad /4.7/$$

$$g_2'^{(1)} = g_2 + \frac{g_1^2}{2\pi v}\ln\frac{\omega_D'}{\omega_D}, \qquad /4.8/$$

are obtained. These quantities are real, as the imaginary parts appearing in the numerator and denominator of eq. /4.2/ cancel each other. Using eqs. /4.7/ and /4.8/ to calculate the right hand sides of eq. /4.1/, two simple differential equations result:

$$\frac{\partial}{\partial x}\ln g_1'(x, g_1, g_2) = \frac{1}{x}\frac{1}{\pi v}g_1'(x, g_1, g_2), \qquad /4.9/$$

$$\frac{\partial}{\partial x}\ln g_2'(x, g_1, g_2) = \frac{1}{x}\frac{1}{2\pi v}\frac{g_1'^2(x, g_1, g_2)}{g_2'(x, g_1, g_2)}, \qquad /4.10/$$

which solve to yield

$$g_1'(x) = \frac{g_1}{1 - \frac{g_1}{\pi v}\ln x}, \qquad /4.11/$$

$$g_2'(x) = g_2 - \frac{1}{2}g_1 + \frac{1}{2}\frac{g_1}{1 - \frac{g_1}{\pi v}\ln x} \qquad /4.12/$$

Similar results can be obtained for the vertices in the two spin channels provided we neglect the imaginary parts in eqs. /4.5/ and /4.6/:

$$\Gamma_1(x) = q_1'(x) \quad , \qquad \Gamma_2(x) = q_2'(x) \qquad /4.13/$$

Taking $g_1=g_2=g$ /Little interaction/ our results are exactly the same as those obtained by BGD in the corresponding special case where their variables ξ and η coincide. Actually the more general case /i.e. a vertex with two energy variables/ can also be obtained in the framework of the present theory by treating two Lie equations for $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, one for each energy variable.

Comparison with the result of BGD shows that this first-order scaling is equivalent to summing up the most singular contributions in logarithmic approximation. As the imaginary part is lower order in the typical logarithmic term this first order scaling cannot account for it properly.

It is worth pointing out that only g_1 is important in determining the singular character of g_1' and g_2', since terms proportional to $g_1 g_2$ and g_2^2 in eq. /4.3/ cancel the corresponding terms in eq. /4.4/. If we take $g_1=0$, the result in this approximation is $g_1'=0$, $g_2'=g_2$ in the one-dimensional case and $g_1'=0$, $g_2'=g_2\left(1-g_2\frac{mk_0}{2\pi^2}\ln\frac{\omega_p'}{\omega_p}\right)^{-1}$ in the three-dimensional case. The different behaviour of

$\mathcal{L}_2'$ stems from the fact that the zero-sound-type bubble is non-logarithmic in three dimensions. Thus for this special choice of interaction already a first-order renormalization is sufficient to account for the disappearance of the phase transition when the dimensionality of the system is reduced from three to one; second order renormalization does not modify this result.

5. Second-order renormalization

In order to improve upon the approximation of the last section, let us calculate the perturbation expression of the vertex and invariant couplings to third order. This means second-order corrections in the renormalizing factors z_i and consequently the self energy graphs shown in Fig. 5 have to be considered as well. These graphs give a contribution

$$\sum^{(2)}(k,\omega) = \frac{1}{4\pi^2 v^2}\left(g_1^2 - g_1 g_2 + g_2^2\right)\left(\omega - \xi(k)\right)\left(\ln\frac{\omega}{\omega_D} - \frac{1}{2}i\pi\right), \quad /5.1/$$

from which $d/\omega/$ is given as

$$d^{(2)}(\omega) = 1 + \frac{1}{4\pi^2 v^2}\left(g_1^2 - g_1 g_2 + g_2^2\right)\left(\ln\frac{\omega}{\omega_D} - \frac{1}{2}i\pi\right). \quad /5.2/$$

Even if it is taken into account here, the imaginary part does not give any contribution to the renormalization. On inserting eq. /5.2/ into eq. /3.3/, z_1 turns out to be real

$$z_1^{-1} = 1 + \frac{1}{4\pi^2 v^2}\left(g_1^2 - g_1 g_2 + g_2^2\right)\ln\frac{\omega_D'}{\omega_D} + \ldots \quad /5.3/$$

The third-order vertex corrections are shown only schematically on Fig. 6. The contributions of the diagrams of the Cooper as well as the zero-sound channel /Fig. 6a and 6b/ consist of two typical logarithmic integrations: to logarithmic accuracy they give $g_i^3 \ln^2 \omega/\omega_D$ and $g_i^2 g_j \ln^2 \omega/\omega_D$-type contributions many of which cancel out. These $\ln^2 \omega/\omega_D$ terms are already accounted for in the first-order scaling; they constitute the $g_i^3 \ln^2 \omega/\omega_D$ terms in the expansion of the result of the last section. These graphs, however, also yield an imaginary contribution which is linear in $\ln \omega/\omega_D$. The graphs /Fig. 6c/ belonging to the third channel - which is non-logarithmic in lower orders /Fig. 1c/ - give also logarithmic contribution in this order. The sum of the contributions of Fig. 6a, 6b and 6c is

$$\frac{1}{2\pi^2 v^2}\left(\ln^2\frac{\omega}{\omega_D} - i\pi \ln\frac{\omega}{\omega_D}\right)\left[2g_1^3 \delta_{\alpha\gamma}\delta_{\beta\delta} - g_1^3 \delta_{\alpha\delta}\delta_{\beta\gamma}\right] \qquad /5.4/$$

$$+ \frac{1}{4\pi^2 v^2} \ln\frac{\omega}{\omega_D}\left[(2g_1^2 g_2 - 2g_1 g_2^2)\delta_{\alpha\gamma}\delta_{\beta\delta}\right.$$

$$\left. -(g_1^3 - 2g_1^2 g_2 + 2g_1 g_2^2 - 2g_2^3)\delta_{\alpha\delta}\delta_{\beta\gamma}\right] + C(g^3),$$

where $C/g^3/$ denotes the constants of order g^3, which have not been calculated in detail and are generally complex. Using eqs. /4.5/ and /4.6/ $\tilde{\Gamma}_1^{(2)}$ and $\tilde{\Gamma}_2^{(2)}$ can be obtained from /5.4/, in accordance with the definition /3.2/ as

$$\tilde{\Gamma}_1^{(2)} = 1 + \frac{g_1}{\pi v}\left(\ln\frac{\omega}{\omega_D} - \frac{1}{2}i\pi\right) + \frac{g_1^2}{\pi^2 v^2}\left(\ln^2\frac{\omega}{\omega_D} - i\pi\ln\frac{\omega}{\omega_D}\right) \quad /5.5/$$

$$+ \frac{1}{2\pi^2 v^2}\left(g_1 g_2 - g_2^2\right)\ln\frac{\omega}{\omega_D} + C_1(g^2),$$

$$\tilde{\Gamma}_2^{(2)} = 1 + \frac{1}{2\pi v}\frac{g_1^2}{g_2}\left(\ln\frac{\omega}{\omega_D} - \frac{1}{2}i\pi\right) + \frac{1}{2\pi^2 v^2}\frac{g_1^3}{g_2}\left(\ln^2\frac{\omega}{\omega_D} - i\pi\ln\frac{\omega}{\omega_D}\right) \quad /5.6/$$

$$+ \frac{1}{4\pi^2 v^2}\frac{1}{g_2}\left(g_1^3 - 2g_1^2 g_2 + 2g_1 g_2^2 - 2g_2^3\right)\ln\frac{\omega}{\omega_D} + C_2(g^2).$$

After inserting eqs. /5.5/, /5.6/ into eqs. /3.4/, /3.5/ utilising eqs. /3.6/, /3.7/ and /5.2/, we retrieve a system of coupled equations which has to be solved self-consistently. Up to second order in g we get for the renormalizing factors

$$Z_2 = 1 + \frac{g_1}{\pi v}\ln\frac{\omega_D'}{\omega_D} + \frac{g_1^2}{\pi^2 v^2}\ln^2\frac{\omega_D'}{\omega_D} + \frac{1}{2\pi^2 v^2}\left(g_1 g_2 - g_2^2\right)\ln\frac{\omega_D'}{\omega_D} + \ldots \quad /5.7/$$

$$Z_3 = 1 + \frac{1}{2\pi v}\frac{g_1^2}{g_2}\ln\frac{\omega_D'}{\omega_D} + \frac{1}{2\pi^2 v^2}\frac{g_1^3}{g_2}\ln^2\frac{\omega_D'}{\omega_D}$$

$$+ \frac{1}{4\pi^2 v^2}\frac{1}{g_2}\left(g_1^3 - 2g_1^2 g_2 + 2g_1 g_2^2 - 2g_2^3\right)\ln\frac{\omega_D'}{\omega_D} + \ldots \quad /5.8/$$

and therefore

$$g_1' = g_1 + \frac{g_1^2}{\pi v}\ln\frac{\omega_D'}{\omega_D} + \frac{g_1^3}{\pi^2 v^2}\ln^2\frac{\omega_D'}{\omega_D} + \frac{g_1^3}{2\pi^2 v^2}\ln\frac{\omega_D'}{\omega_D} + \ldots \quad /5.9/$$

$$g_2' = g_2 + \frac{g_1^2}{2\pi v}\ln\frac{\omega_D'}{\omega_D} + \frac{g_1^3}{2\pi^2 v^2}\ln^2\frac{\omega_D'}{\omega_D} + \frac{g_1^3}{4\pi^2 v^2}\ln\frac{\omega_D'}{\omega_D} + \ldots \quad /5.10/$$

The imaginary parts and the constants $C_i/g^2/$ have been cancelled everywhere in the renormalizing factors and in the invariant couplings.

In this approximation the Lie equations for the invariant couplings become

$$\frac{\partial}{\partial x}\ln g_1'(x) = \frac{1}{x}\left(\frac{g_1'(x)}{\pi v} + \frac{g_1'^2(x)}{2\pi^2 v^2}\right), \quad /5.11/$$

$$\frac{\partial}{\partial x} \ln g_2'(x) = \frac{1}{2x} \frac{g_1'(x)}{g_2'(x)} \left(\frac{g_1'(x)}{\pi v} + \frac{g_1'^2(x)}{2\pi^2 v^2} \right). \qquad /5.12/$$

Comparing these equations it is apparent that

$$g_2'(x) = \frac{1}{2} g_1'(x) + g_2 - \frac{1}{2} g_1 . \qquad /5.13./$$

The same relation is valid in the first-order scaling. The solution of eq. /5.11/ can be obtained only in an implicit form. As is shown on Fig. 7, $g_1'/x/$ has no singularity and the dimensionless invariant coupling $g_1'/\pi v$ tends to the finite value -2 if $x \to 0$ for an attractive $g_1 < 0$ interaction. As before g_2 continues to be unimportant. For attractive interaction the second term on the right hand side of eq. /5.11/ is positive, and when the singularity due to the first negative term starts to build up, it counteracts and stops the singular growth of the invariant coupling. Curve /a/ on Fig. 7 displays the resulting smooth behaviour. For repulsive interaction the invariant coupling remains small and the result of the second-order renormalization, as shown by curve /b/ on Fig. 7, is essentially the same as that of the first-order renormalization.

In this approximation the invariant coupling and the vertex function differ from each other due to the important self energy contributions. Using eqs. /5.5/ and /5.6/ and neglecting the imaginary parts, the Lie equations for $\tilde{f}_1'$, and $\tilde{f}_2'$ are

$$\frac{\partial}{\partial x} \ln \tilde{\Gamma}_1(x) = \frac{1}{x}\left[\frac{g_1'(x)}{\pi v} - \frac{1}{2\pi^2 v^2}\left(g_2'^2(x) - g_1'(x)\, g_2'(x)\right)\right], \quad /5.14/$$

$$\frac{\partial}{\partial x} \ln \tilde{\Gamma}_2(x) = \frac{1}{x}\,\frac{1}{g_2'(x)}\left[\frac{g_1'^2(x)}{2\pi v} + \frac{1}{4\pi^2 v^2}\left(g_1'^3(x)\right.\right. \quad /5.15/$$

$$\left.\left. - 2\, g_1'^2(x)\, g_2'(x) + 2\, g_1'(x)\, g_2'^2(x) - 2\, g_2'^3(x)\right)\right],$$

and, using eq. /5.2/, the equation determining d/x/ is

$$\frac{\partial}{\partial x} \ln d(x) = \frac{1}{x}\,\frac{1}{4\pi^2 v^2}\left[g_1'^2(x) - g_1'(x)\, g_2'(x) + g_2'^2(x)\right] \quad /5.16/$$

Knowing $g_1'(x=0) = -2\pi v$, $g_2'(x=0) = -\pi v + g_1 - \frac{1}{2} g_1$ and applying the weak coupling assumption $g_i/\pi v \ll 1$, $i=1,2$, the solution of eqs. /5.14/-/5.16/ can be found for the limiting case $x = \omega/\omega_D \to 0$

$$\Gamma_1(\omega) = \Gamma_2(\omega) \propto \left(\frac{\omega}{\omega_D}\right)^{-3/2}, \quad /5.17/$$

$$d(\omega) \propto \left(\frac{\omega}{\omega_D}\right)^{3/4} \quad /5.18/$$

Eq. /5.17/ shows that the effect of the second order terms in eqs. /5.14/ and /5.15/ is such as to shift the singularity found for a finite value of ω in the parquet approximation to $\omega=0$.

6. Conclusions

We have investigated a one-dimensional system of electrons interacting via a two-particle interaction of finite range which has been assumed to be constant, with a cut-off at ω_D around the Fermi energy.

As the simplest case, only those matrix elements of the interaction have been retained which describe the interaction of electrons on opposite sides of the Fermi surface. By the use of renormalization group technique, we have calculated the invariant couplings, the one-particle Green's function and the vertex as a function of a single energy variable in two successive approximations in order to see if they show any singular behaviour indicating a phase transition or not.

In the corresponding three-dimensional problem, both approximations would yield nothing more than the sum of the Cooper-type ladder diagrams. The one-dimensional character of the problem manifests itself in the fact that there are more elementary vertex diagrams depending logarithmically on the variable ω than in three dimensions. For the interaction considered here only the elementary Cooper-type bubble is logarithmic in the three-dimensional case, whereas in one dimension, for large momentum transfers, the zero-sound-type bubble /Fig. 1b/ and in the next order, the diagrams coming from the third channel /shown on Fig. 6c/ as well, have to be added on the same footing.

Our first approximation consists in taking the first logarithmic correction in the perturbation expansion of the vertex and invariant coupling. Inserting this into the right-hand side of the Lie differential equation, the result corresponds exactly to the summation of the parquet diagrams in logarithmic approximation. This result was obtained by

BGD[6] by solving a system of non-linear integral equations. The first-order renormalization gives invariant couplings and vertex functions which become singular at some finite value of the energy.

In the next step the perturbational expressions for the reduced vertices are calculated up to second order. In this order self energy contributions of the form $g^2 \omega \ln \omega/\omega_0$ have also to be considered, hence in this approximation we go beyond the parquet approximation. These corrections play an important role in reducing the Lie differential equations of the invariant coupling to the relatively simple form of eqs. /5.11/ and /5.12/. The solutions of these equations are free from the spurious singularity of the first approximation and tend to a saturation value $g_1'/\pi v \approx -2$, $g_2'/\pi v \approx -1$ for $\omega \to 0$. The equations determining Γ_1, Γ_2 and d, on the other hand, are more complicated and an analytic expression for g_1' would be necessary to find their solution in the whole range of ω values. By restricting ourselves to $\omega \approx 0$, however, the above limiting values of g_1' and g_2' can be used to determine $\Gamma(\omega)$ and $d/\omega/$ from the corresponding Lie equations, with the result $\Gamma(\omega) \propto \omega^{-3/2}$ and $d(\omega) \propto \omega^{3/4}$ for $\omega \to 0$. Taking the temperature as a variable instead of ω, it can be concluded that the finite critical temperature T_c predicted by the first approximation is

shifted to zero, indicating that there is no phase transition at finite temperatures.

It was emphasized in Sec. 3 that the improvement of the perturbational expansion by the renormalization group technique stems from the fact that the series expansion on the right hand side of the Lie equations progresses in powers of the invariant couplings. But if the invariant coupling becomes of the order of unity, this expansion breaks down. This is precisely what happens in the present problem for $g_1 < 0$, so that even our second-order scaling is not completely reliable. Nevertheless, we believe that this approximation shows correctly that there is no singularity in the invariant couplings and as a consequence there is no phase transition at finite temperatures. The saturation value of the invariant couplings and the exponents of the vertices as well as the Green's function are determined by higher-order corrections. In this respect the second-order renormalization gives only qualitative results.

In calculating the effective couplings the imaginary parts have been properly accounted for, and g_1' and g_2' have proven to be real quantities. As far as the vertex function and the one-particle Green's function are concerned, however, their imaginary contributions could not be determined in a consistent way in the present framework: to do this higher-order terms need to be taken into consideration.

The limited information provided by eqs. /5.14/ and /5.15/ determining $\Gamma_{\alpha\beta\gamma\delta}(-k_o, \frac{1}{2}\omega, k_o, -\frac{1}{2}\omega; k_o, \frac{1}{2}\omega, -k_o, \frac{1}{2}\omega)$ seems sufficient to indicate an instability of the system, but for calculating physical quantities like conductivity a knowledge of the vertex as a function of all of its variables is needed. In principle it is possible to extend the present calculation to many variables, the prescription for this has been given by Bogoliubov and Shirkov[13]. Nevertheless, there are certain response functions that can be calculated directly by the renormalization group theory using the expression for the invariant coupling obtained here. These response functions may indicate the character of the ordering which takes place in the system at T=0. The results of such a calculation are reported in the following paper.

Another problem is posed by retention of the interaction matrix elements neglected in the present work. These are processes in which both incoming electrons are on the same side of the Fermi surface, however they do not alter the result that there is no phase transition at finite temperatures, they only modify the ground state of the system. The effect of these processes will be investigated in a later publication.

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Figure captions

Fig. 1. Second order vertex diagrams. The numbers 1, 2, 3 and 4 stand for k_1, ω_1, α; k_2, ω_2, β; k_3, ω_3, γ and k_4, ω_4, δ respectively. a./ Cooper-type bubble, b./ zero-sound-type bubble, c./ second-order diagram of the "third channel".

Fig. 2. General vertex diagram showing our special choice of variables on the four legs.

Fig. 3. Diagrammatic representation of the two interaction matrix elements g_1 and g_2 considered in eq. /2.7/.

Fig. 4. Second-order vertex diagrams. a./ Cooper-type bubble diagrams, b./ zero-sound-type bubble diagrams.

Fig. 5. Second-order self energy diagrams.

Fig. 6. Third-order vertex diagrams a./ in the Cooper channel, b./ in the zero-sound channel and c./ in the third channel. Each interaction point can represent either g_1 or g_2.

Fig. 7. The invariant coupling constant g_1' as a function of $x=\omega/\omega_D$ /a/ for attractive g_1, /b/ for repulsive g_1. The dotted line shows the result of first order renormalization.

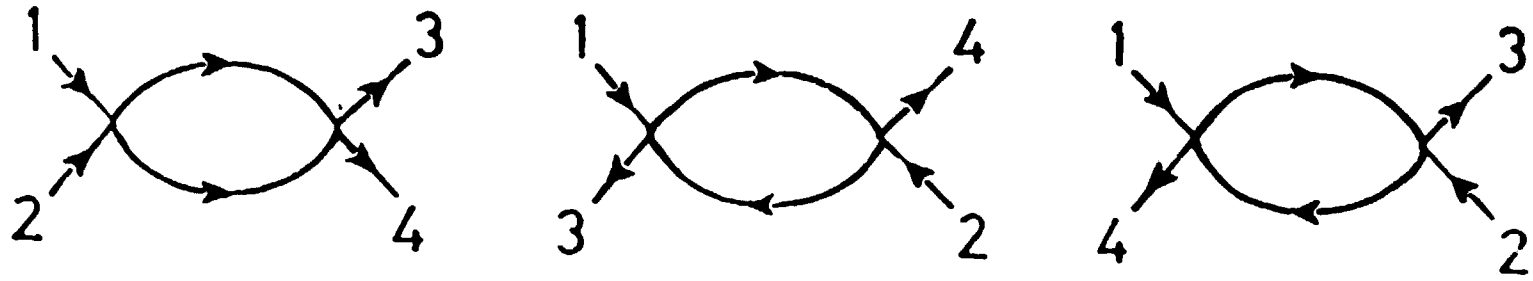


Fig. 1.

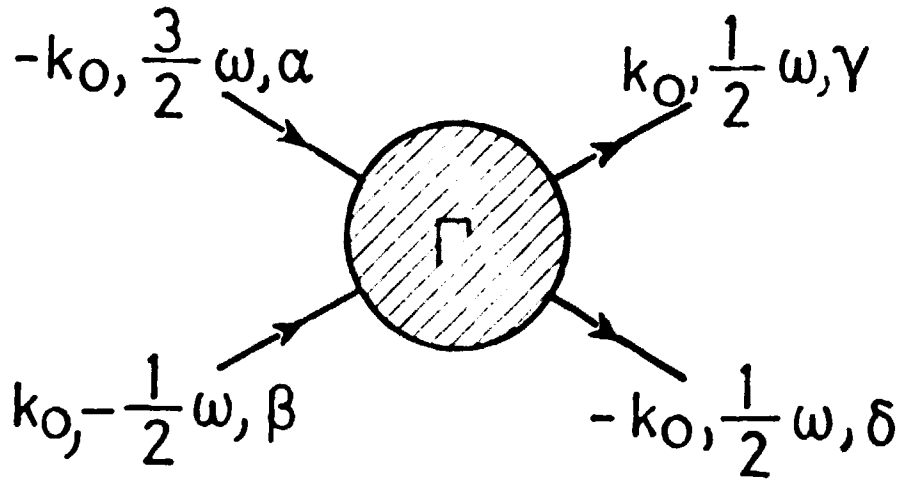
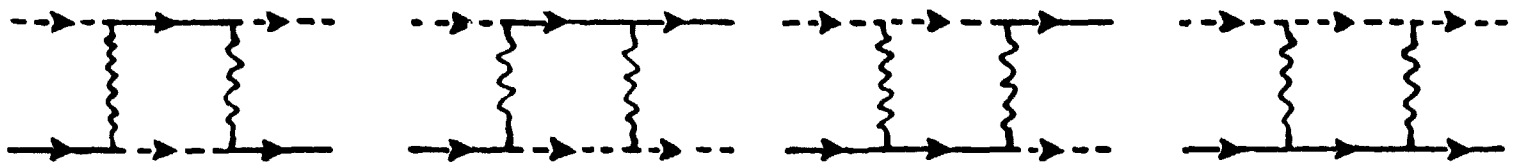
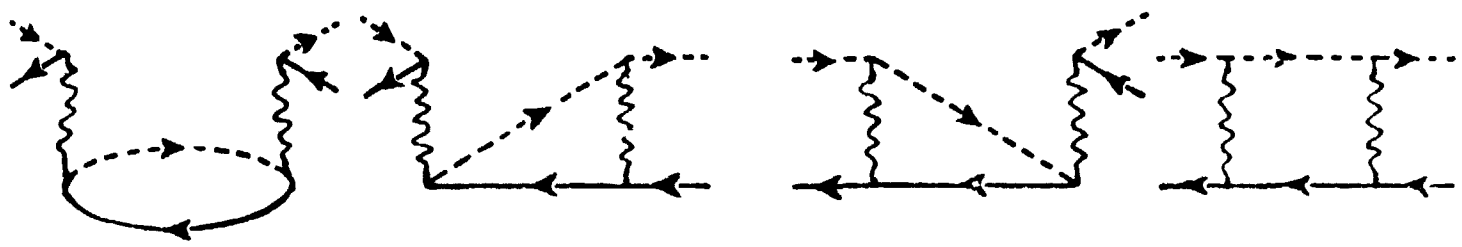


Fig. 2.

Fig. 3.



a)



b)

Fig.4.

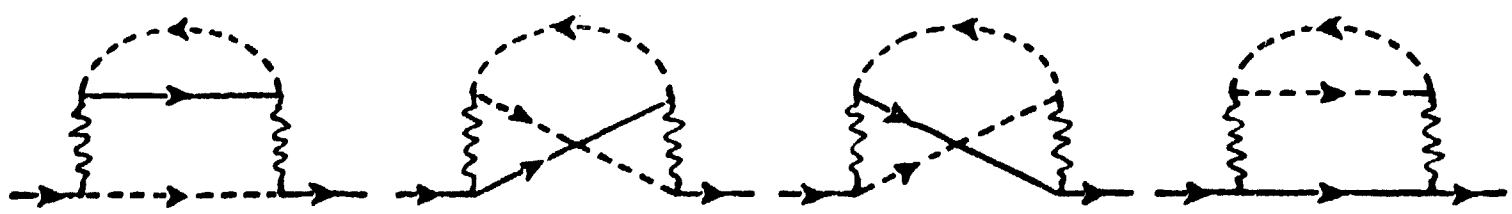
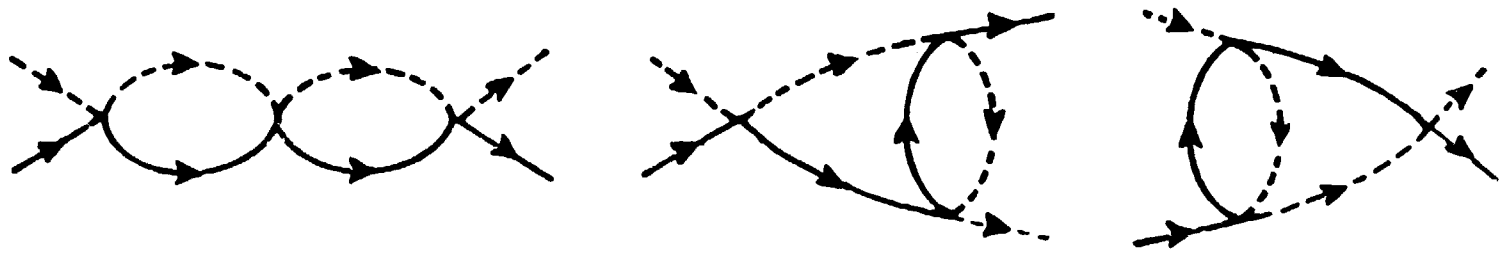


Fig.5.



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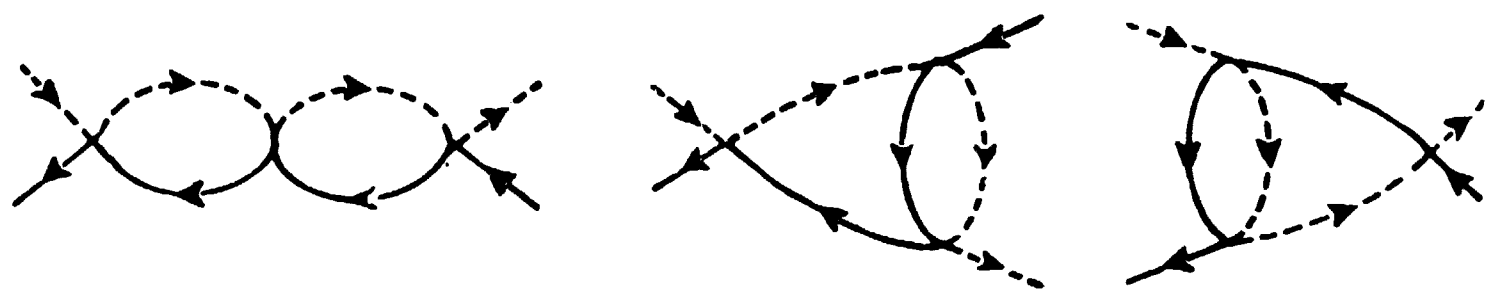

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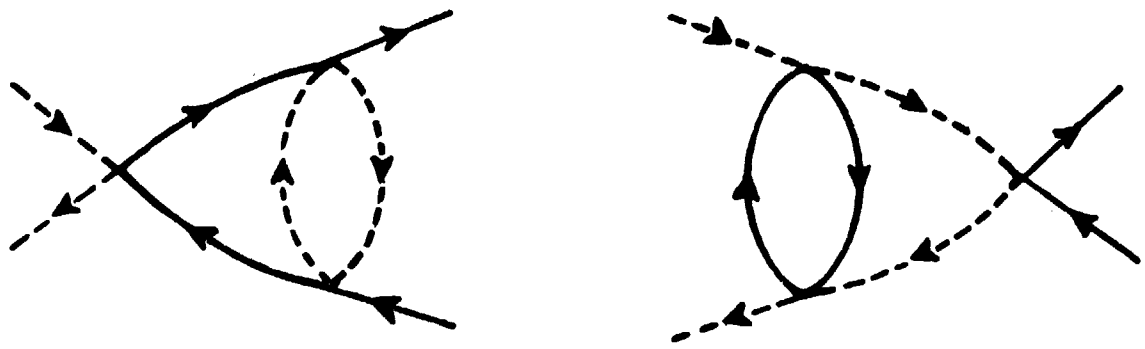

c)

Fig.6.

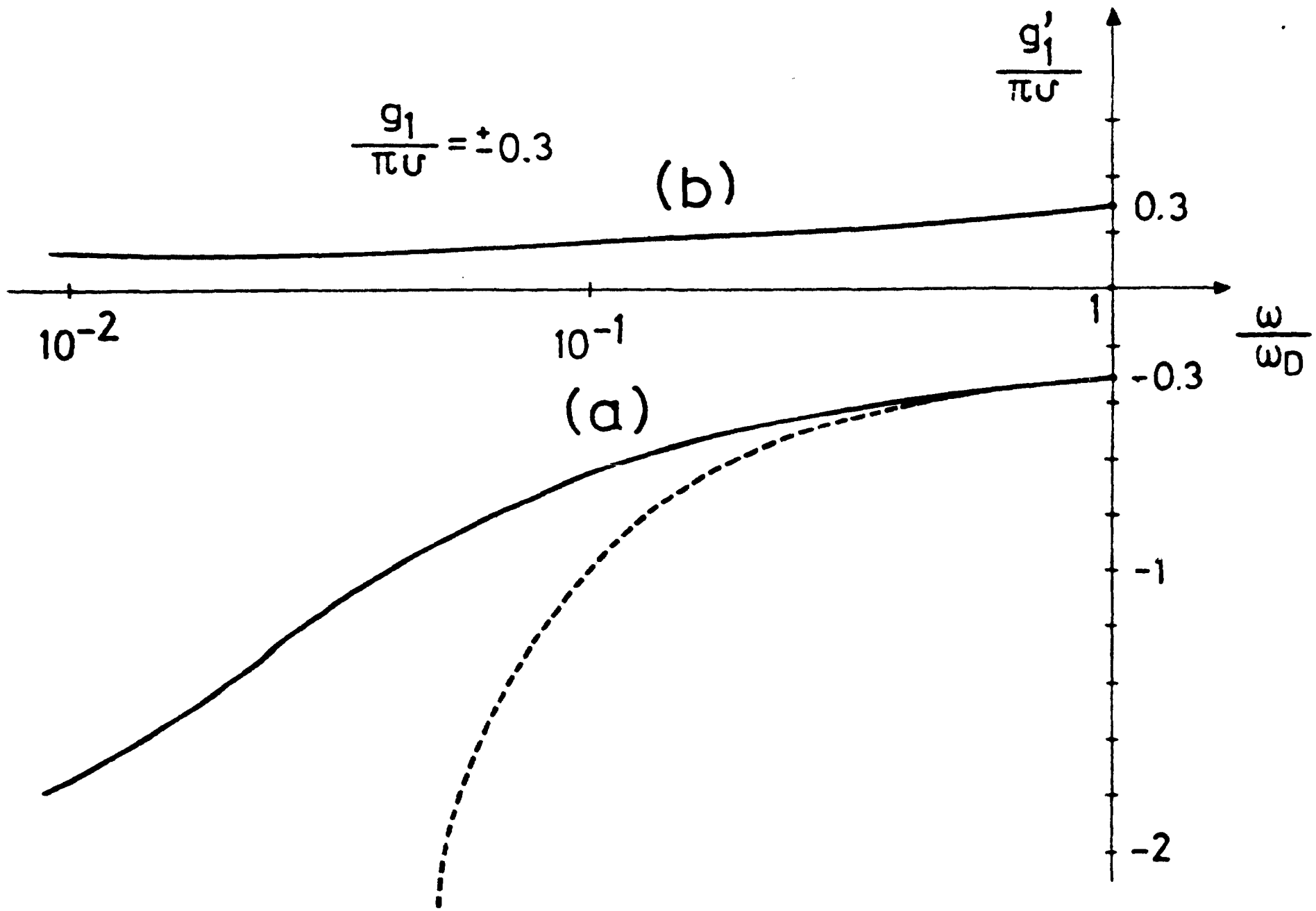
$\frac{g_1'}{\pi v}$
$\frac{g_1}{\pi v} = \pm 0.3$
(b)
0.3
10^{-2}
10^{-1}
1
$\frac{\omega}{\omega_D}$
-0.3
(a)
-1
-2

Fig. 7.

Kiadja a Központi Fizikai Kutató Intézet
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