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DETERMINATION OF RADIAL BEAM PROFILE BY MEASURING X-RAY INTENSITY VS POSITION OF A WIRE TARGET

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DETERMINATION OF RADIAL BEAM PROFILE BY MEASURING X-RAY INTENSITY VS POSITION ON A WIRE TARGET*

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ABSTRACT

This paper describe the physics and mathematics of determining the radial current density of an electron beam from its experimental X-ray intensify data. It is necessary to invert an Abel's integral equation in the appropriate variables. This method will also allow us to deduce the plasma electron density from interferometer phase shift data.

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INTRODUCTION

One of the very useful pieces of information to determine in the Beam Research Program is the radial profile of the experimental beam. This profile determines the amount of spread in the betatron frequency which in turn plays a dominant role in beam stability considerations. A technique has been developed in which the x-ray intensity from a wire target is measured.[1] In this paper it will be shown how this data in fact gives the required radial shape (assuming axial symmetry). The method involves inverting an Abel's integral equation in the appropriate variables. In the future this method will be used to deduce the plasma electron density from interferometer phase shift data. A computer code has been written which finds J(r), the radial current density, numerically from the given laboratory data. This code has been tested successfully on two special cases where we know the answers analytically. These cases are the uniform and the quadratic $[J(r) \sim 1 - r^2]$ profiles. It is found that the experimental beam is peaked on axis with a corresponding large spread in betatron frequency. The computer code described here was used in the reports on the experimental results of the Beam Research Group.

I. FORMULATION OF THE PROBLEM

The way the experimental x-ray data are obtained is shown in Fig. 1. The x-ray data give the integrated effect of J(r) along the wire. Call this integrated value I. Consider the diagram in Fig. 2 where we look at the upper right hand quarter of the beam. First note that as the wire is moved completely across the beam, I goes from zero to a maximum and then back to zero again. Hence we are able to determine where the center of the beam is and where its outer edge is. In actual practice, these two quantities may be known somewhat imprecisely, but theoretically they are well defined. In the diagram (Fig. 2) x is the distance of the wire W from the center of the beam and a is the beam radius. y, $\bar{y}$, and r are as indicated:

$$x^2 + y^2 = r^2, \qquad x^2 + \bar{y}^2 = a^2. \tag{1}$$

I is now a function of x only and is given by

$$I(x) = \int_{y=0}^{\bar{y}} J(r)\, dy = \int_{y=0}^{\bar{y}} J[r(x,y)]\, dy, \tag{2}$$

or

$$I(x) = \int_{y=0}^{\sqrt{a^2 - x^2}} J(\sqrt{x^2 + y^2})\, dy. \tag{3}$$

In order to get J(r) from I(x) it is convenient to make a change of variables in Eq. (3). The appropriate one is

$$u = x^2, \qquad v = r^2. \tag{4}$$

We will use v as the new variable of integration in Eq. (3). We see from Eq. (4) and Eq. (1) that for y = 0, v = u and for $y = \bar{y}$, $v = a^2$. We also define $J(r) = J_1(r^2) = J_1(v)$. From Eq. (1) we have $u + y^2 = v$ and hence, at const. u, 2y dy = dv. Thus

$$dy = \frac{dv}{2y} = \frac{dv}{2\sqrt{v - u}}.$$

Now Eq. (3) may be written as

$$I\ (x) = \int_{v=u}^{a^2} J_1(v)\ \frac{dv}{2\sqrt{v - u}}. \tag{5}$$

If we now define $I(x) = I_1(x^2) = I_1(u)$, we get finally

$$I_1(u) = \frac{1}{2}\int_{v=u}^{a^2} J_1(v)\ \frac{dv}{\sqrt{v - u}}. \tag{6}$$

Equation (6) is now in the form for which we were looking. This is an Abel integral equation.

II. THE SOLUTION OF THE PROBLEM AND SOME SAMPLE CASES

The solution of Eq. (6) is well known[2] to be

$$J_1(u) = -\frac{2}{\pi}\int_{v=u}^{a^2} \frac{I_1'(v)dv}{\sqrt{v - u}}. \tag{7}$$

In terms of I and J we easily find that

$$J(r) = -\frac{2}{\pi}\int_{t=r}^{a} \frac{I'(t)\ dt}{\sqrt{t^2 - r^2}}. \tag{8}$$

Hence we see that if I(x) is known experimentally, we are able to compute J(r).

In the appendix the integrals in Eqs. (3) and (7) are worked out for a uniform beam and for a quadratic shaped beam profile. These two cases are not only instructive but they are also valuable as tests for the computer code which numerically does the integrals. Figures 3-6 show I(v) and J(r) for the uniform beam and the quadratic beam respectively. The code is working accurately. This code is of some interest numerically since there is a weak singularity in the integrand at the lower limit, v = u. This however causes no major difficulties and is handled in an obvious manner.

III. COMPUTATION OF THE BETATRON SPREAD AND THE DISTRIBUTION FUNCTION

From the computed value of $J(r)$ we can calulate two other quantities of theoretical importance: the profile of betatron frequency and the distribution function $f(\omega_\beta)$. There are two limiting cases for which we will consider the betatron frequency. One is associated with circular orbits and the other with straight line orbits. The B_θ field consistent with $J(r)$ is given by

$$B_\theta(r) = \frac{4\pi}{c}\frac{1}{r}\int_{s=0}^{r} sJ(s)\, ds. \tag{9}$$

The betatron frequency for circular orbit of radius r associated with this field is well known to be

$$\omega_\beta^2(r) = \frac{e\beta}{\gamma m}\frac{B(r)}{r}. \tag{10}$$

The calculation of the betatron frequency for a straight line orbit through the center of the beam is not as easy for general $B_\theta(r)$. We must numerically integrate the equation of motion of a particle which at $t = 0$ has $x = r$ and $dx/dt = 0$. This equation is

$$\ddot{x} = -\frac{e\beta}{\gamma m} B(x). \tag{11}$$

The solution of Eq. (11), $x(t)$, is always periodic with period called T. The required frequency is $\omega_\beta = 2\pi/T$.

Since $\omega_\beta = \omega_\beta(r)$ is a monotone function for a nonhollow beam, the inversion $r = r(\omega_\beta)$ can always be performed. Hence $f(\omega_\beta)$ can be calculated from the relation

$$f(\omega_\beta) \sim J[r(\omega_\beta)]\, r(\omega_\beta)\, \frac{dr(\omega_\beta)}{d\omega_\beta}. \tag{12}$$

Figure 7 shows ω_β as a function of r for a quadratic beam ($J \sim 1 - r^2$).
Figure 8 shows the distribution function $f(\omega_\beta)$.

APPENDIX

THE SOLUTION OF ABEL'S EQUATION FOR THE UNIFORM BEAM AND THE QUADR[illegible] SHAPED BEAM PROFILE.

Let us consider a uniform beam of radius 1. Hence

$$J(r) = 1 \qquad r \leq 1,$$

$$J(r) = 0 \qquad r > 1.$$

From Eq. (3) we see this corresponds to

$$I(x) = \int_{y=0}^{\sqrt{1-x^2}} J\, dy = \int_{y=0}^{\sqrt{1-x^2}} dy = \sqrt{1-x^2}\,. \qquad \text{(A-1)}$$

From the definition of $I_1(v)$ we get

$$I_1(v) = \sqrt{1-v}\,. \qquad \text{(A-2)}$$

Hence,

$$I_1'(v) = -\frac{1}{2\sqrt{1-v}}\,.$$

We will now show that through Eq. (7) this $I_1(v)$ leads to the correct $J_1(u)$ or $J(r)$. By Eq. (7)

$$J_1(u) = -\frac{2}{\pi}\int_{v=u}^{1} \frac{-dv}{2\sqrt{1-v}\ \sqrt{v-u}}. \qquad \text{(A-3)}$$

We will change variables to do this integral. Letting $t = v - u$

$$J_1(u) = \frac{1}{\pi}\int_{t=0}^{1-u} \frac{dt}{\sqrt{1-(t+u)}\sqrt{t}}. \qquad \text{(A-3)}$$

Let $x = t/(1 - u)$. Hence

$$J_1(u) = \frac{1}{\pi} \int_{x=0}^{1} \frac{(1 - u)\,dx}{\sqrt{(1 - u) - x(1 - u)}\;\sqrt{x(1 - u)}}$$

$$= \frac{1}{\pi} \int_{x=0}^{1} \frac{dx}{\sqrt{1 - x}\;\sqrt{x}} \qquad \text{(A-4)}$$

Hence we see that $J_1(u)$ is constant as required. By using the trigonometric substitution $x = \sin^2 \theta$, we find $J_1(u) = 1$. Alternatively, we could look up the integral in Eq. (4) in a table[3].

We next consider a beam of radius 1 with a quadratic profile $J(r) = 1 - r^2$. Again from Eq. (3) we get

$$I(x) = \int_0^{\sqrt{1-x^2}} J(\sqrt{x^2 + y^2})\,dy = \int_0^{\sqrt{1-x^2}} [1 - (x^2 + y^2)]\,dy$$

$$= (y - x^2y - y^3/3)\Big|_{y=0}^{\sqrt{1-x^2}} = \frac{2}{3}(1 - x^2)^{3/2},$$

$$I(x) = \frac{2}{3}(1 - x^2)^{3/2}. \qquad \text{(A-5)}$$

Hence

$$I_1(v) = \frac{2}{3}(1 - v)^{3/2},$$

and

$$I_1'(v) = -\sqrt{1 - v}. \qquad \text{(A-6)}$$

We will now do the integral in Eq. (7) to recover $J(r)$ from the given $I_1(v)$:

$$J_1(u) = -\frac{2}{\pi} \int_{v=u}^{1} -\frac{\sqrt{1 - v}\,dv}{\sqrt{v - u}}$$

Let $t = v - u$. Hence

$$J_1(u) = \frac{2}{\pi} \int_{t=0}^{1-u} \frac{\sqrt{1 - (t + u)}\, dt}{\sqrt{t}}.$$

Let $x = t/(1 - u)$. Hence

$$J_1(u) = \frac{2}{\pi} \int_{x=0}^{1} \frac{\sqrt{(1 - u) - (1 - u)\, x}\, (1 - u)\, dx}{\sqrt{(1 - u)} \quad \sqrt{x}},$$

$$J_1(u) = (1 - u) \frac{2}{\pi} \int_{x=0}^{1} \frac{\sqrt{1 - x}\; dx}{\sqrt{x}}. \qquad \text{(A-7)}$$

Using the substitution $x = \sin^2 \theta$ we get

$$J_1(u) = 1 - u \quad \text{or } J(r) = 1 - r^2 \qquad \text{(A-8)}$$

as required.

REFERENCES

[1] D. S. Prono and E. J. Lauer, private communication.

[2] Courant-Hilbert, Methods of Mathematical Physics, Vol. I, pg. 158.

[3] Dwight, Tables of Integrals and Other Mathematical Data, pg. 211, 855.11.

$$J_1(u) = \frac{2}{\pi} \int_{t=0}^{1-u} \frac{\sqrt{1 - (t + u)}\,dt}{\sqrt{t}}.$$

Let $x = t/(1 - u)$. Hence

$$J_1(u) = \frac{2}{\pi} \int_{x=0}^{1} \frac{\sqrt{(1 - u) - (1 - u)\,x}\,(1 - u)\,dx}{\sqrt{(1 - u)}\,\sqrt{x}},$$

$$J_1(u) = (1 - u)\,\frac{2}{\pi} \int_{x=0}^{1} \frac{\sqrt{1 - x}\,dx}{\sqrt{x}}. \qquad \text{(A-7)}$$

Using the substitution $x = \sin^2 \theta$ we get

$$J_1(u) = 1 - u \quad \text{or} \quad J(r) = 1 - r^2 \qquad \text{(A-8)}$$

as required.

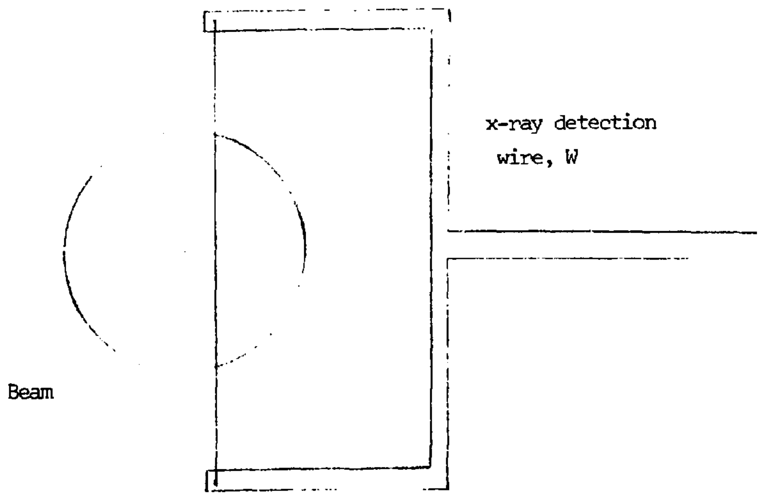

Figure 1

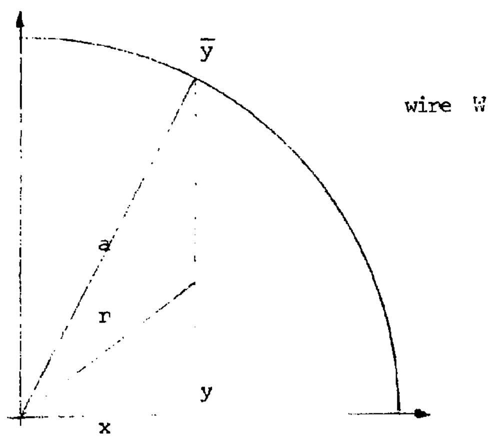

Figure 2

UNIFORM BEAM

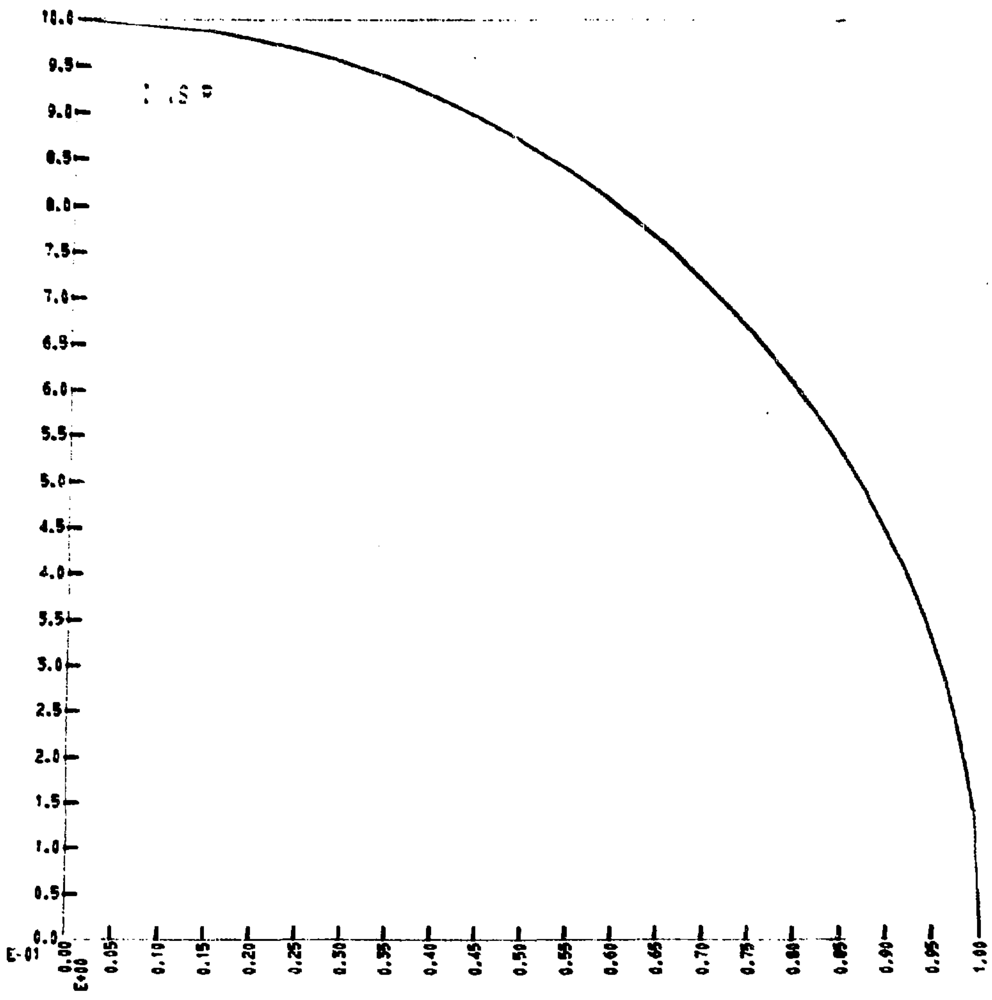

FIGURE 3

UNIFORM BEAM

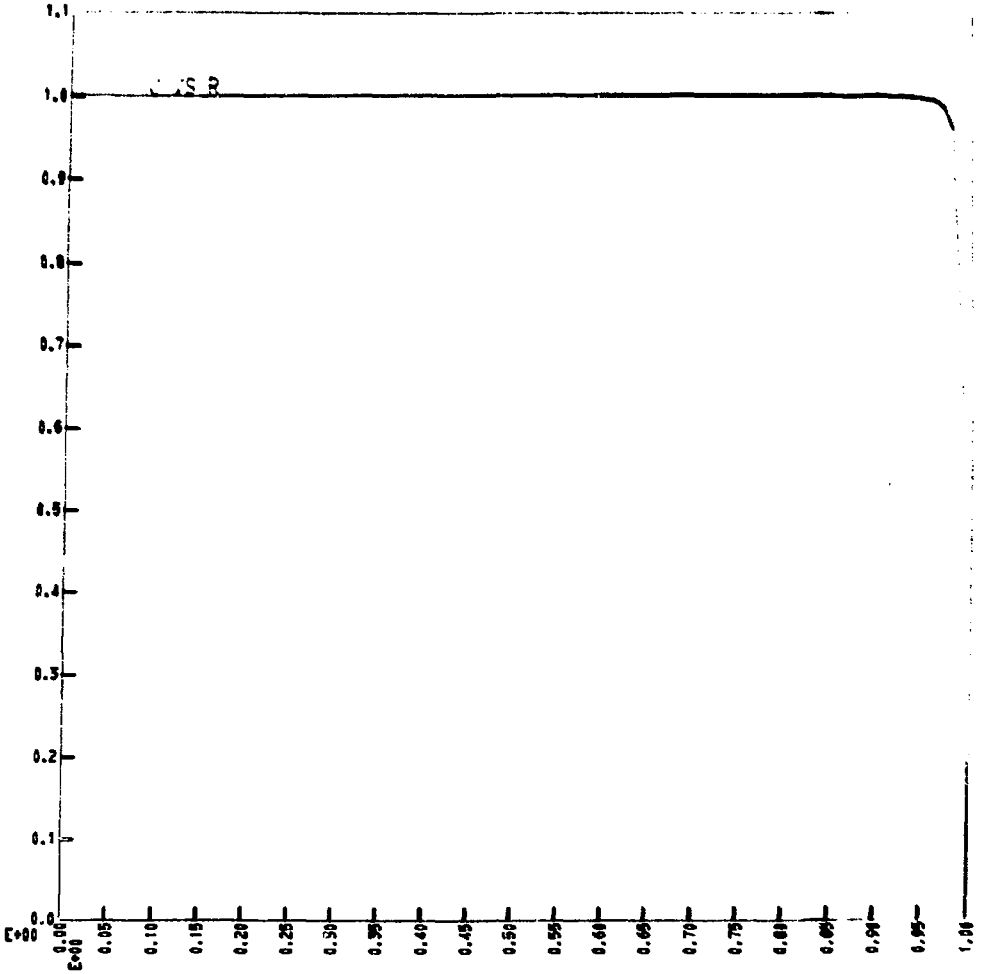

FIGURE 4

QUADRATIC BEAM

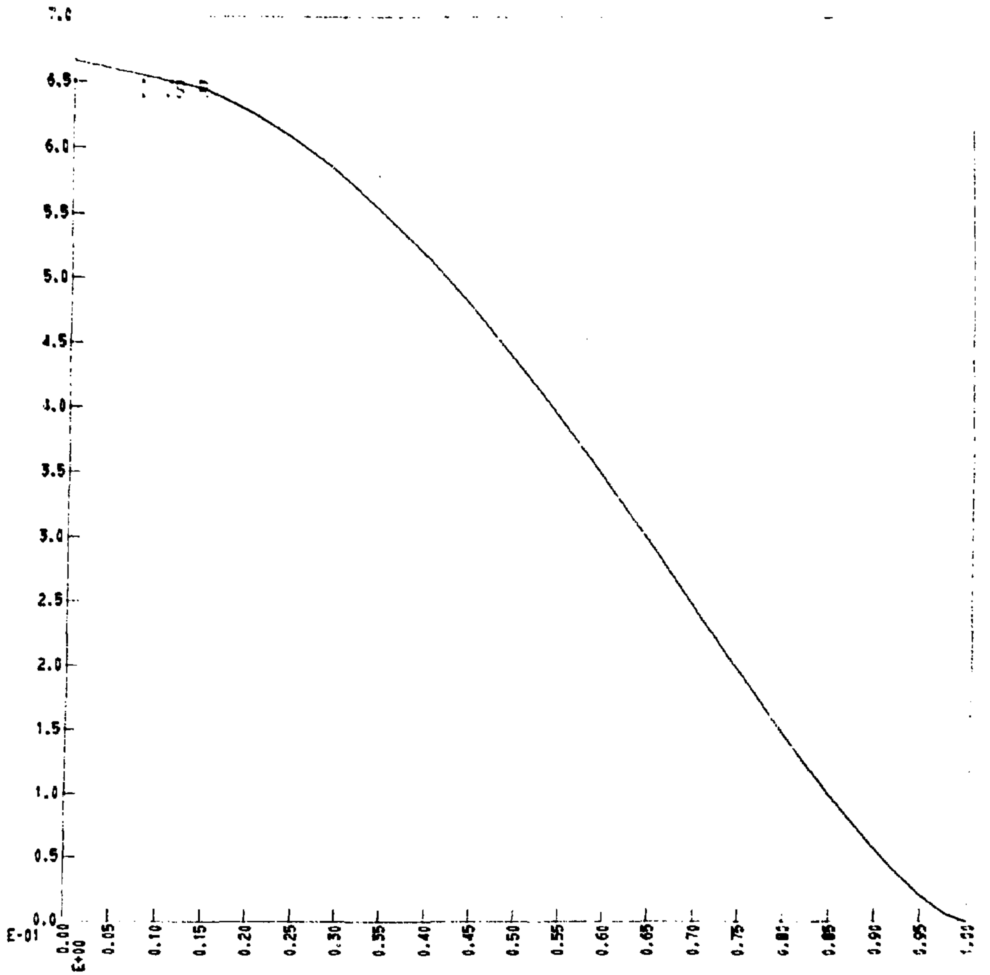

FIGURE 5

QUADRATIC BEAM

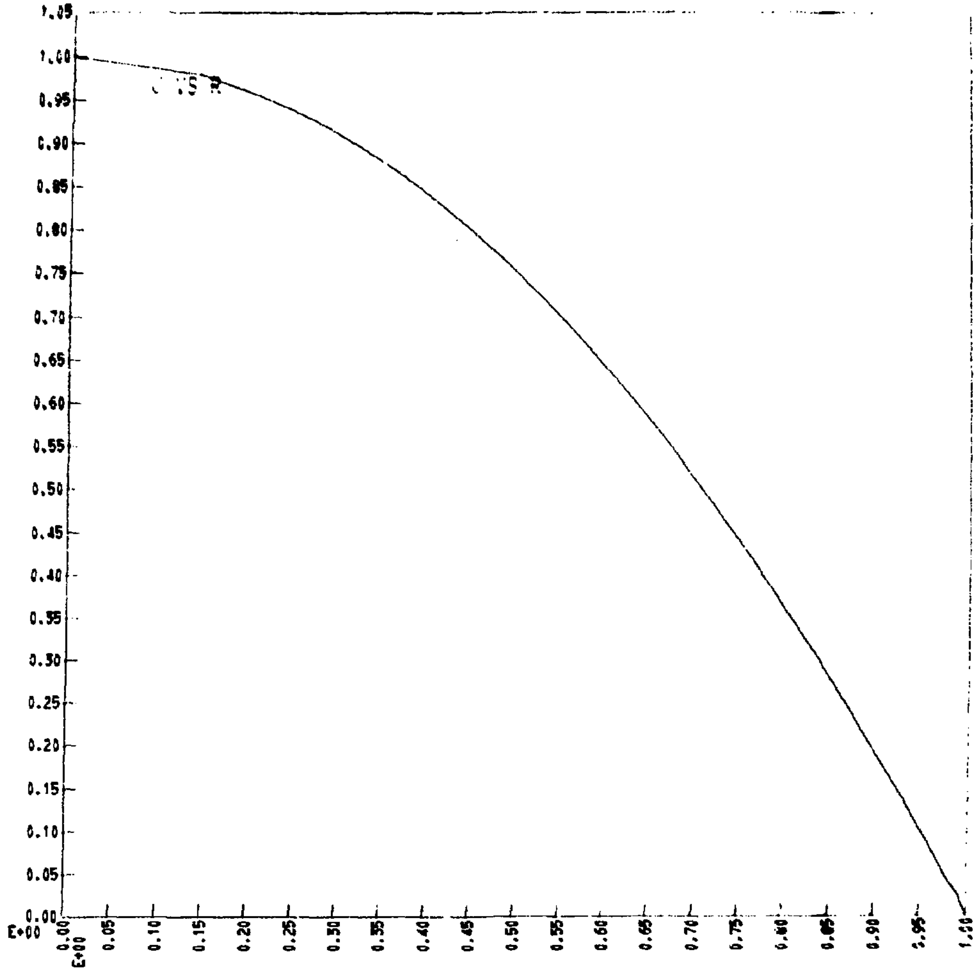

FIGURE 6

QUADRATIC BEAM

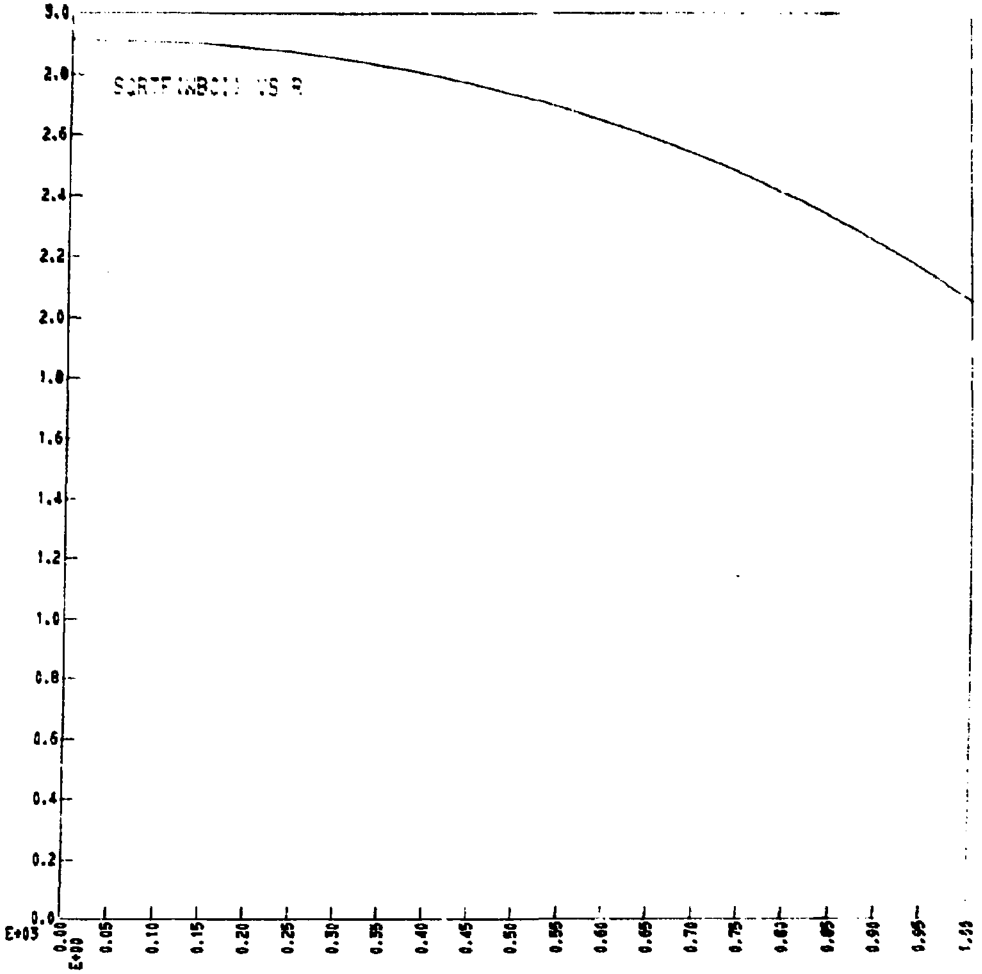

FIGURE 7

QUADRATIC BEAM

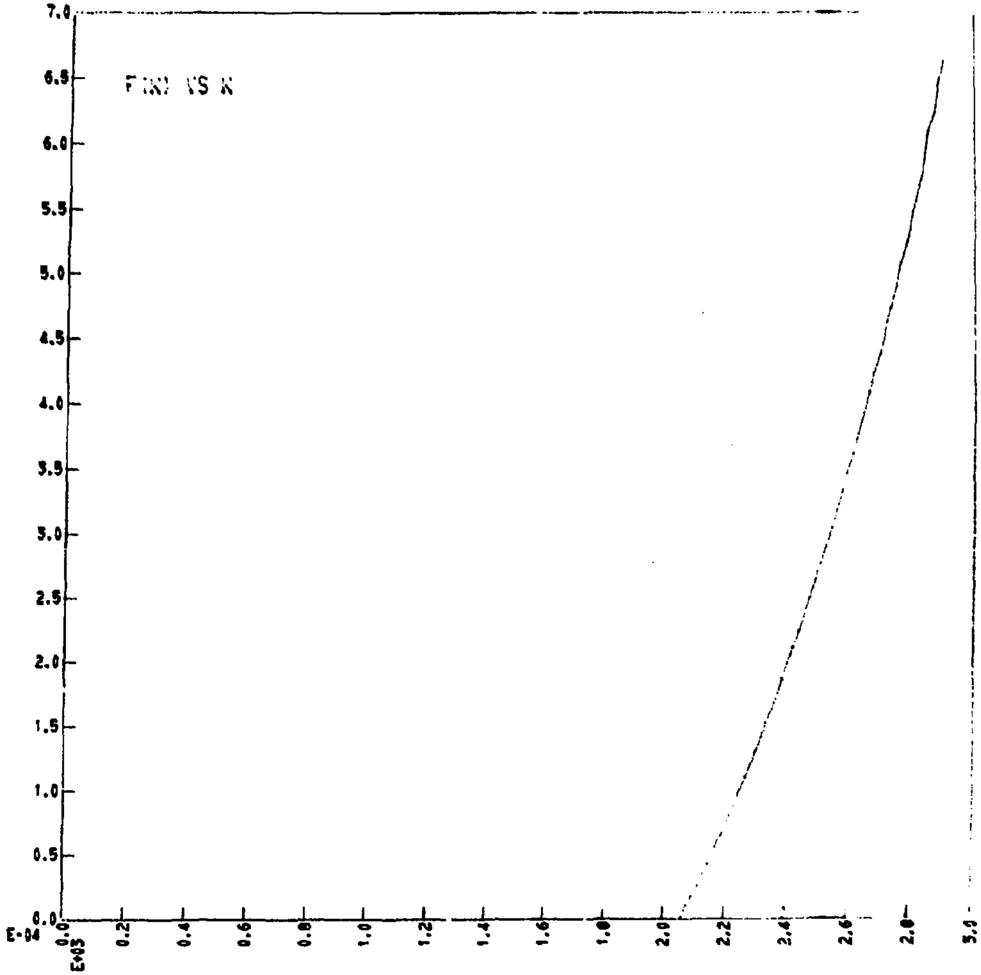

FIGURE 8