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SPONTANEOUSLY BROKEN GAUGE SYMMETRIES AND SUPERSELECTION RULES (*)

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Abstract : Spontaneously broken gauge symmetries do not give rise to superselection rules so the superselection structure of the theory is determined by the subgroup of unbroken gauge symmetries . In the presence of broken gauge symmetries the observable algebra does not satisfy duality in the vacuum sector .

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A quantum field theory is usually described in terms of fields acting on a Hilbert space or in terms of some effective substitute such as the Wightman functions or Green's functions of the fields . Such a description should be regarded as incomplete unless supplemented by conditions which specify how the observables are constructed from the fields .[1)] In practice these conditions take the form of a principle of gauge invariance so that the observables are just the gauge-invariant functions of the fields .We are left with the problem of deciding which symmetries of the theory correspond to gauge transformations . There are arguments which lead one to conclude that certain symmetries must be gauge transformations . The principle of locality tells one that space like separated observables should commute and the particle spectrum of the theory may force one to postulate certain superlection rules to "explain" the stability of these particles . On the other hand it seems that any internal symmetry may be treated as a gauge symmetry without the resulting theory being in any way inconsistent . It may even be the case that all internal symmetries should be treated as gauge transformations . One may illustrate this point by considering the isospin group in strong interaction physics . For the isospin group to be an exact symmetry one must neglect electromagnetic interactions However in the absence of electromagnetic interactions there is no way of measuring the local electric charge . The consequent reduction in the number of observables may be summed up by saying that the isospin group is actually a gauge group in strong interaction physics .

Since by definition gauge transformations act trivially on any observable , they cannot be given a direct physical interpretation in the way that space-time translations can . One might even be tempted to think that gauge invariance has no physical interpretation because the theory can be described in terms of the observables alone . However a gauge group $\mathcal{G}$ of the the first kind does have an indirect physical interpretation because the superselection structure of the theory is governed by the representations of $\mathcal{G}$ [1]. In other words it is the "dual object" of $\mathcal{G}$ which is of direct physical significance . However even this interpretation of $\mathcal{G}$ is valid only when the gauge transformations are not spontaneously broken . Under certain hypotheses , we shall show here that , as might be expected , a spontaneously brokengauge

1) One might expect that, in realistic models, the observables should be determined by some self-consistency argument because the interactions present should tell us what measurementa can be performed within the theory . However , in our present state of ignorance we have little choice but to adopt the stand point in the text . Compare also the remarks on the isospin group in strong strong interaction physics .

symmetry does not give rise to superselection rules .

We begin by listing some of the most familiar examples of spontaneously broken symmetries in quantum field theory .

Example a: The free neutral scalar field ϕ of mass zero allows a spontaneously broken symmetry $\phi(x) \to \phi(x) + a$ [2] . Treating this as a gauge symmetry means that the derivatives $\partial_\mu \phi(x)$ generate the observables .

Example b: Consider the gauge transformations of the vector potential $A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$, in quantum electrodynamics . If one takes $\Lambda(x) = \tau . x$ so that $A_\mu(x) \to A_\mu(x) + \tau_\mu$ we have a spontaneously broken symmetry and the physical photon may be interpreted as the corresponding Goldstone boson [2,3] . From our point of view the interesting thing is that we have an example which is unquestionably a spontaneously broken gauge symmetry .

Example c : The σ- model ; the Lagrangian for this model can be written $\mathcal{L} = \mathcal{L}_0 + \varepsilon \mathcal{L}_1$ where $\mathcal{L}_0 = \bar{\psi}[i\gamma_\mu \partial^\mu - g(\sigma + i\, \underline{\pi} \cdot \underline{\tau}\, \gamma_5)]\psi + \frac{1}{2}[(\partial_\mu \sigma)^2 + (\partial_\mu \underline{\pi})^2] - \frac{\mu^2}{2}[\sigma^2 + \underline{\pi}^2] - \frac{\lambda^2}{4}[\sigma^2 + \underline{\pi}^2]^2$ and $\mathcal{L}_1 = c\sigma$. Here $\lambda^2 > 0$ and if $\mu^2 < 0$ we have in the limit $\varepsilon \to 0$, $\langle \sigma \rangle_0 \neq 0$ and the Goldstone mode of symmetry is realized . The theory is $SU(2) \times SU(2)$ invariant but the symmetry group of the vacuum is just the isospin symmetry subgroup . This example has been extensively discussed in the literature (see [4 - 6]) particularly in the case where the ψ- field is absent . It is the most important example for what follows because we have a compact gauge group and a non-trivial subgroup as the stability subgroup of the vacuum .

Example d : The non-linear realization of $SU(2) \times SU(2)$ which may be thought of as derived from example c by imposing the subsidiary condition $\underline{\pi}^2 + \sigma^2 = 1$ (see [7, 8]) . Although apparently little has changed as regards the symmetry of the theory because the Lagrangian is still $SU(2) \times SU(2)$ invariant and the isospin group is the stability subgroup of the vacuum , it is not clear that the theory admits an $SU(2) \times SU(2)$ symmetry in the sense used here . It is difficult to see how the non-linear chiral transformations can give rise to automorphisms of an irreducible field algebra .

2) This is not the case in a 2-dimensional space-time world

We next give a list of assumptions designed to treat the case where the observables are defined from the fields by using a principle of gauge invariance.

1) The field algebra $\mathfrak{F}$ is the global algebra of a net $\mathcal{O} \to \mathfrak{F}(\mathcal{O})$ of von Neumann algebras and acts irreducibly on a Hilbert space $\mathcal{H}$.

2) There is a strongly continuous unitary representation $L \to \mathcal{U}(L)$ of the covering group $\mathcal{P}$ of the Poincaré group on $\mathcal{H}$ inducing automorphisms α_L of the field algebra , $\alpha_L(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(L\mathcal{O})$. There is a unit vector $\Omega \in \mathcal{H}$, the vacuum vector , invariant under the $\mathcal{U}(L)$, $L \in \mathcal{P}$ and inducing the vacuum state ω_0 of $\mathfrak{F}$,

$$\omega_0(F) = (\Omega, F\Omega).$$

The energy-momentum spectrum is contained in the forward light cone .

3) Ω is a cyclic and separating vector for each $\mathfrak{F}(\mathcal{O})$, (Reeh-Schlieder property)

4) There is a faithful representation $g \to \beta_g$ of a compact group $\mathcal{G}$, the gauge group , by automorphisms of $\mathfrak{F}$.
β_g commutes with α_L and $\beta_g(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O})$. Further we suppose that if $F \in \mathfrak{F}(\mathcal{O})$ then $g \to \beta_g(F)$ is weakly continuous , [3)]

5) We assume that the field algebra has Bose-Fermi commutation relations . The easiest way of expressing this is to suppose there is a $k \in \mathcal{G}$ with $k^2 = e$ so that if we set $F_+ = \frac{1}{2}(F + \beta_k(F))$ and $F_- = \frac{1}{2}(F - \beta_k(F))$ then

$$\begin{aligned} F_+ F'_+ - F'_+ F_+ &= 0 \\ F_+ F'_- - F'_- F_+ &= 0 \qquad F \in \mathfrak{F}(\mathcal{O}_1),\ F' \in \mathfrak{F}(\mathcal{O}_2),\ \mathcal{O}_1 \subset \mathcal{O}_2'. \\ F_- F'_- + F'_- F_- &= 0 \end{aligned}$$

k is automatically in the centre of $\mathcal{G}$.

3) We use the term weak topology for a von Neumann algebra to mean the weak topology induced by the normal linear forms . This topology is often referred to as the ultraweak topology .

These assumptions have the following main drawbacks . They are not designed to cope with theories like quantum electrodynamics where gauge invariance of the second kind and the indefinite metric play a role . Furthermore , when dealing with spontaneously broken gauge symmetries there is no reason ,as example a) shows , to suppose that $\mathcal{G}$ is compact . However , there is as yet no indication that the general qualitative conclusion that the superselection rules are determined solely by the unbroken part of the gauge group of the first kind needs any modification .
Our assumption that the observables are defined from the fields by using a principle of gauge invariance and that $\mathcal{G}$ is the gauge group of the theory may be made precise by saying that the net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ of local observables is defined by

$$\mathfrak{A}(\mathcal{O}) = \{ A \in \mathfrak{F}(\mathcal{O}) : \beta_g(A) = A , g \in \mathcal{G} \} . \mathcal{G}_0 \quad \text{is closed}$$

Since $g \to \beta_g$ is locally weakly continuous , $\mathfrak{A}(\mathcal{O})$ is a von Neumann algebra . To determine the superselection rules of the theory we must identify the inequivalent irreducible representations of the net $\mathfrak{A}$ contained in the defining representation on $\mathcal{H}$; these are the superselection rules of the theory .[4)]

Let $\mathcal{G}_0 \subset \mathcal{G}$ be the stability subgroup of the vacuum state

$$\mathcal{G}_0 = \{ g \in \mathcal{G} : \omega_0 \circ \beta_g = \omega_0 \} ;$$

since $\mathcal{G}_0 = \bigcap_{F \in \bigcup_{\mathcal{O}} \mathfrak{F}(\mathcal{O})} \{ g \in \mathcal{G} : \omega_0 \circ \beta_g(F) = \omega_0(F) \}$ and $g \to \beta_g$ is locally weakly continuous . We shall see that it is only the subgroup $\mathcal{G}_0$ which gives rise to superselection structure .

This result does not depend on the full force of the assumptions but only on assumption 4) and the cluster property of the vacuum , a standard result whose proof we include for completeness .

<u>Proposition 1</u> If $F \in \mathfrak{F}$ then $\alpha_x(F)$ tends weakly to $\omega_0(F) I$ as x tends spacelike to infinity .
<u>Proof</u> Suppose first $\beta_k(F) = F$, then any weak limit point G of $\alpha_x(F) - \omega_0(F) I$ as x tends spacelike to infinity commutes with $\mathfrak{F}$. Hence G is a

4) In principle there might be other superselection sectors of $\mathfrak{A}$ not contained in the defining representation on $\mathcal{H}$; this would mean that we had not used a complete set of fields in the first place .

multiple of the identity ; however $\omega_o(G) = 0$ so $G = 0$. Hence $\alpha_x(F)$ tends weakly to $\omega_o(F)I$. Next suppose that $\beta_k(F) = -F$ and let G be a weak limit point of $\alpha_x(F)$ as x tends spacelike to infinity . The Bose-Fermi commutation relations now give

$$G F' = \beta_k(F') G , \qquad F' \in \mathfrak{F} .$$

In particular
$$G\alpha_x(F)^* = \beta_k\alpha_x(F)^* G = -\alpha_x(F)^* G .$$

However we now deduce for the weak limit point G that $GG^* = -G^*G$. Hence $G = 0$ and $\alpha_x(F)$ tends weakly to zero . The result now follows for general F by writing $F = F_+ + F_-$.

Notice that in proving the Proposition we have also shown that $k \in \mathcal{G}_o$. Using $\mathcal{G}_o$ we can also define another net $\mathcal{O} \to \mathfrak{B}(\mathcal{O})$, setting

$$\mathfrak{B}(\mathcal{O}) = \{ B \in \mathfrak{F}(\mathcal{O}) : \beta_g(B) = B , \ g \in \mathcal{G}_o \}$$

Since $k \in \mathcal{G}_o$, $\mathcal{O} \to \mathfrak{B}(\mathcal{O})$ is also a local net .

To emphasize the basic simplicity of the argument we begin by assuming that $\mathcal{G}$ is a compact Abelian group . Let $\hat{\mathcal{G}}$ denote the dual group (character group) of $\mathcal{G}$, and given $\sigma \in \hat{\mathcal{G}}$, set

$$\mathfrak{F}_\sigma(\mathcal{O}) = \{ F \in \mathfrak{F}(\mathcal{O}) : \beta_g(F) = F\sigma(g) , \ g \in \mathcal{G} \} ; \quad \mathfrak{F}_\sigma = \{ F \in \mathfrak{F} : \beta_g(F) = F\sigma(g) , \ g \in \mathcal{G} \} .$$

Since it is $\hat{\mathcal{G}}$ rather than $\mathcal{G}$ itself which is of physical interest we try and replace $\mathcal{G}_o$ by something related to $\hat{\mathcal{G}}$. To this end we set

$$\Sigma = \{ \sigma \in \hat{\mathcal{G}} : \text{there exists } F \in \mathfrak{F}_\sigma \text{ with } \omega_o(F) \neq 0 \} .$$

The first result establishes the connexion between Σ and $\mathcal{G}_o$.

<u>Proposition 2</u> $\mathcal{G}_o = \Sigma^\perp \equiv \{ g \in \mathcal{G} : \sigma(g) = 1, \ \sigma \in \Sigma \}$ and conversely

$$\Sigma = \mathcal{G}_o^\perp \equiv \{ \sigma \in \hat{\mathcal{G}} : \sigma(g) = 1 , \ g \in \mathcal{G}_o \} .$$

<u>Proof</u> If $g \in \mathcal{G}_o$, $\sigma \in \Sigma$, pick $F \in \mathfrak{F}_\sigma$ with $\omega_o(F) \neq 0$. Then $\omega_o(F) = \omega_o(\beta_g(F)) = \omega_o(F)\sigma(g)$, so $\sigma(g) = 1$ and $\mathcal{G}_o \subset \Sigma^\perp$. Now suppose $g \in \Sigma^\perp$ then $\omega_o(\beta_g(F)) = \omega_o(F)$, $F \in \mathfrak{F}_\sigma$, $\sigma \in \hat{\mathcal{G}}$.

However $\mathfrak{F}(\mathcal{O})$ is the weakly closed linear span of $\bigcup_{\sigma\in\hat{\mathcal{G}}} \mathfrak{F}_\sigma(\mathcal{O})$,(see Proposition 11) . Thus $\omega_0(\beta_g(F)) = \omega_0(F)$ for $F \in \mathfrak{F}(\mathcal{O})$ and taking norm limits we deduce that $\omega_0\circ\beta_g = \omega_0$, so $\mathcal{G}_0 \supset \Sigma^\perp$.
To show that $\Sigma = \mathcal{G}_0^\perp = \Sigma^{\perp\perp}$ it suffices to show that Σ is a subgroup of $\hat{\mathcal{G}}$ (compare Proposition 10) . However if $F \in \mathfrak{F}_\sigma$ then $F^* \in \mathfrak{F}_{\bar{\sigma}}$, hence $\sigma \in \Sigma$ implies $\bar{\sigma} \in \Sigma$
Also if $\sigma, \sigma' \in \Sigma$, $F \in \mathfrak{F}_\sigma$ and $F' \in \mathfrak{F}_{\sigma'}$ then $\alpha_x(F)F' \in \mathfrak{F}_{\sigma\sigma'}$,
However as x tends spacelike to infinity , $\omega_0(\alpha_x(F)F') \to \omega_0(F)\omega_0(F')$.
Hence $\sigma\sigma' \in \Sigma$ and Σ is a subgroup of $\hat{\mathcal{G}}$.

The next result shows how the net $\mathcal{O} \to \mathcal{B}(\mathcal{O})$ may be defined using Σ instead of $\mathcal{G}_0$.

<u>Proposition 3</u> $\mathcal{B}(\mathcal{O})$ is the von Neumann algebra generated by $\bigcup_{\sigma\in\Sigma} \mathfrak{F}_\sigma(\mathcal{O})$.

<u>Proof</u> Since $\Sigma = \mathcal{G}_0^\perp$, $\mathfrak{F}_\sigma(\mathcal{O}) \subset \mathcal{B}(\mathcal{O})$ for $\sigma \in \Sigma$.
Given $\sigma \in \hat{\mathcal{G}}$ set $m_\sigma(F) = \int \bar{\sigma}(g)\, \beta_g(F)\, d\mu(g)$ where μ is the normalized Haar measure on $\mathcal{G}$. Since $\mathcal{G}$ is Abelian and $\mathcal{B}(\mathcal{O})$ is weakly closed , $F \in \mathcal{B}(\mathcal{O})$ implies $m_\sigma(F) \in \mathcal{B}(\mathcal{O}) \cap \mathfrak{F}_\sigma(\mathcal{O})$.
But $\Sigma = \mathcal{G}_0^\perp$, so $m_\sigma(F) = 0$ unless $\sigma \in \Sigma$. However F is in the weakly closed linear span of the $m_\sigma(F)$ for $\sigma \in \hat{\mathcal{G}}$ (see Proposition 11) . So $\mathcal{B}(\mathcal{O})$ is just the von Neumann algebra generated by the $\mathfrak{F}_\sigma(\mathcal{O})$ with $\sigma \in \Sigma$.

We now show that we may use the net $\mathcal{O} \to \mathcal{B}(\mathcal{O})$ instead of $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ to determine the superselection structure .

<u>Theorem 4</u> If $\mathcal{G}$ is compact Abelian , $\mathfrak{A}^- = \mathcal{B}^-$.
<u>Proof</u> :It suffices to show by proposition 3 that if $\sigma \in \Sigma$ then $\mathfrak{F}_\sigma \subset \mathfrak{A}^-$.
However if $F, G \in \mathfrak{F}_\sigma$ then $F\alpha_x(G)^* \in \mathfrak{A}$ and as x tends spacelike to infinity $F\alpha_x(G)^* \to F\omega_0(G^*)$.
Picking $\omega_0(G) \neq 0$ we conclude that $F \in \mathfrak{A}^-$ as required .

At this point we drop the hypothesis that $\mathcal{G}$ is Abelian and reprove the last three results for the case of an arbitrary compact group $\mathcal{G}$. Instead of dealing with characters , σ will now denote an arbitrary finite-dimensional continuous unitary representation of $\mathcal{G}$ on a Hilbert space H_σ. The notion of an operator of $\mathfrak{F}$ transforming like a character is replaced by that of a multiplet of operators transforming like the basis vectors of H_σ. The precise definitions and a few elementary results on the harmonic analysis of the action of compact groups on von Neumann algebras are given in the appendix. The set Σ of characters is replaced by a subspace $K_\sigma \subset H_\sigma$ for each σ. We define

$$K_\sigma = \omega_0(\mathfrak{F}_\sigma^{*}) \subset H_\sigma .$$

The statement that $\mathcal{G}_0 = \Sigma^{\perp}$ is now replaced by

<u>Proposition 5</u> $\mathcal{G}_0 = \{ g \in \mathcal{G} : \sigma(g) b = b$ for all $b \in K_\sigma$ and all $\sigma \}$.

<u>Proof</u> Let $g \in \mathcal{G}_0$ and $\underline{F} \in \mathfrak{F}_\sigma$ then by Eq(5) of the Appendix

$$\omega_0(\underline{F}^{*}) = \omega_0(\beta_g(\underline{F})^{*}) = \sigma(g^{-1})\,\omega_0(\underline{F}^{*}) .$$

Hence $\sigma(g)b = b$ for all $b \in K_\sigma$ and all $g \in \mathcal{G}_0$. Now suppose $g \in \mathcal{G}$ and $\sigma(g)b = b$ for all $b \in K_\sigma$ and all σ then if $\underline{F} \in \mathfrak{F}_\sigma$

$$\omega_0(\underline{F}^{*}) = \sigma(g^{-1})\,\omega_0(\underline{F}^{*}) = \omega_0(\beta_g(\underline{F})^{*})$$

However since $\mathfrak{F}(\mathcal{O})$ is the weakly closed linear span of $\bigcup_\sigma \mathfrak{F}_\sigma(\mathcal{O}) H_\sigma$ (see Proposition 11), we deduce $\omega_0(F) = \omega_0(\beta_g(F))$ for $F \in \mathfrak{F}(\mathcal{O})$ or by norm continuity $\omega_0 \circ \beta_g = \omega_0$. Thus $g \in \mathcal{G}_0$ as required.

The statement that $\Sigma = \mathcal{G}_0^{\perp}$ is now replaced by

<u>Proposition 6</u> $K_\sigma = \{ b \in H_\sigma : \sigma(g) b = b,\ g \in \mathcal{G}_0 \}$

<u>Proof</u> In view of Propositions 5 and 10 and Eq(6) it suffices to show that $K_{\bar\sigma} = \overline{K_\sigma}$ and $K_\sigma \otimes K_\tau \subset K_{\sigma\otimes\tau}$.
However $\mathfrak{F}_\sigma^{*} = \mathfrak{F}_{\bar\sigma}$ so $K_{\bar\sigma} = \omega_0(\mathfrak{F}_{\bar\sigma}^{*}) = \omega_0(\mathfrak{F}_\sigma) = \overline{\omega_0(\mathfrak{F}_\sigma^{*})} = \overline{K_\sigma}$.
Now if $\underline{F} \in \mathfrak{F}_\sigma$, $\underline{G} \in \mathfrak{F}_\tau$ then $\underline{F} \otimes \alpha_x(\underline{G}) \in \mathfrak{F}_{\sigma\otimes\tau}$. As x tends spacelike to infinity $\omega_0(\underline{F} \otimes \alpha_x(\underline{G})) \to \omega_0(\underline{F}) \otimes \omega_0(\underline{G})$
But $K_{\bar\sigma\otimes\bar\tau}$ being a finite-dimensional vector space is weakly closed thus $\omega_0(\underline{F}) \otimes \omega_0(\underline{G}) \in K_{\bar\sigma\otimes\bar\tau}$, so $K_{\bar\sigma\otimes\bar\tau} \supset K_{\bar\sigma} \otimes K_{\bar\tau}$.

Proposition 7 $\mathcal{B}(\mathcal{O})$ is the von Neumann algebra generated by $\bigcup_\sigma \mathfrak{F}_\sigma(\mathcal{O}) K_\sigma$.

Proof Let $m_0(F) = \int \beta_g(F) d\mu_0(g)$ where μ_0 is the normalized Haar measure of $\mathcal{G}_0$. Since $\mathfrak{F}(\mathcal{O})$ is the weakly closed linear span of $\bigcup_\sigma \mathfrak{F}_\sigma(\mathcal{O}) H_\sigma$ and m_0 is weakly continuous, $\mathcal{B}(\mathcal{O})$ is the weakly closed linear span of $\bigcup_\sigma m_0(\mathfrak{F}_\sigma(\mathcal{O}) H_\sigma)$.
If $\underset{\sim}{F} \in \mathfrak{F}_\sigma(\mathcal{O})$, $b \in H_\sigma$ then $\beta_g(\underset{\sim}{F} b) = \underset{\sim}{F} \sigma(g) b$.
Hence $m_0(\mathfrak{F}_\sigma(\mathcal{O}) H_\sigma) = \mathfrak{F}_\sigma(\mathcal{O}) K_\sigma$ from which the result follows.

We can now extend Theorem 4 to the non-Abelian case.

Theorem 8 $\mathfrak{A}^- = \mathcal{B}^-$.

Proof Given $b \in K_\sigma$ let $b = \omega_0(\underset{\sim}{G})$ with $\underset{\sim}{G} \in \mathfrak{F}_{\bar{\sigma}}$. Let $\underset{\sim}{F} \in \mathfrak{F}_\sigma$ then $\underset{\sim}{F} \alpha_x(\underset{\sim}{G})^t \in \mathfrak{A}$. However as x tends spacelike to infinity, $\underset{\sim}{F} \alpha_x(\underset{\sim}{G})^t \to \underset{\sim}{F} b \in \mathfrak{A}^-$.
The result now follows from Proposition 7.

Hence we have reduced the problem of computing the superselection sectors to the case where $\mathcal{G}_0$ is the gauge group and the vacuum state is gauge invariant. Here we define

$$V(g) F \Omega = \beta_g(F) \Omega, \qquad F \in \mathfrak{F}$$

and verify that $g \to V(g)$ gives a strongly continuous unitary representation of $\mathcal{G}_0$, such that $\beta_g(F) = V(g) F V(g)^{-1}$.
This situation has been dealt with in [1 ; Section III] ; applying these results here one shows that $\mathfrak{A}^- = \mathcal{B}^- = V(\mathcal{G}_0)'$
so that we may decompose the defining representation π (say) of $\mathfrak{A}$ on $\mathcal{H}$ into irreducible components by decomposing the representation V of $\mathcal{G}_0$.
Further one shows that there is a 1-1 correspondence $\tau \to \pi_\tau$ between equivalence classes of irreducible continuous unitary representations of $\mathcal{G}_0$ and equivalence classes of irreducible representations of $\mathfrak{A}$ which are subrepresentations of π. The multiplicity of π_τ in π is equal to the dimension of τ.

This has solved the problem of computing the sectors in the case of spontaneously broken gauge symmetries provided the gauge group $\mathcal{G}$ is compact. In certain cases, for example if $\mathcal{G}$ is locally compact Abelian and if we know that each $\mathfrak{F}(\mathcal{O})$ is generated as a von Neumann algebra by those elements which transform like characters under $\mathcal{G}$, our arguments still allow us to conclude that $\mathfrak{A}^- = \mathfrak{B}^-$. This remark applies in particular to the free neutral scalar field of mass zero , example a) , where the Weyl operators transform like characters under the gauge group .

Because we are dealing with spontaneously broken gauge symmetries there is no associated degeneracy of the vacuum state as a state over the observable algebra and one may ask what intrinsic features of the observable algebra correspond to the spontaneously broken part of the gauge symmetry .[5] A partial answer is provided by noting that the observable net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ allows a non-trivial extension $\mathcal{O} \to \mathfrak{B}(\mathcal{O})$ which still satisfies locality . Consequently , when $\mathcal{G}_0 \neq \mathcal{G}$, the observable algebra does not satisfy duality in the vacuum representation π_0: $\pi_0(\mathfrak{A}(\mathcal{O})) \subsetneq \pi_0(\mathfrak{A}(\mathcal{O}'))'$ (at least for sufficiently large $\mathcal{O}$) . As duality is the characteristic assumption of the approach to superselection structure based on the observable algebra given in [9] , this approach must be modified somewhat to allow for the possibility of spontaneously broken gauge symmetries . We shall not attempt this in detail here but confine ourselves to deriving the structural assumptions which may be used to replace duality .

Proposition 9 a) $\mathfrak{A}(\mathcal{O}')^- = \mathfrak{B}(\mathcal{O}')^-$.

b) If $\mathcal{O} \subset \mathcal{O}_1' \cap \mathcal{O}_2'$ then $\mathfrak{B}(\mathcal{O}) \subset \{\mathfrak{A}(\mathcal{O}_1') \cap \mathfrak{A}(\mathcal{O}_2')\}''$.

Proof a) is just a special case of b) ; we take $\mathcal{O}_1 = \mathcal{O}_2$ and recall that $\mathfrak{B}(\mathcal{O}_1')$ is by definition the C^* - algebra spanned by all $\mathfrak{B}(\mathcal{O})$ with $\mathcal{O} \subset \mathcal{O}_1'$. To prove b) it suffices , by proposition 7 , to show that for all σ, $\mathfrak{F}_\sigma(\mathcal{O}) K_\sigma \subset \{\mathfrak{A}(\mathcal{O}_1') \cap \mathfrak{A}(\mathcal{O}_2')\}''$.

However , if $\underline{F} \in \mathfrak{F}_\sigma(\mathcal{O})$ and $b \in K_\sigma$ we may pick a $\underline{G} \in \bigcup_{\mathcal{O}} \mathfrak{F}_{\bar{\sigma}}(\mathcal{O})$ with $\omega_0(\underline{G}) = b$ and hence there is a sequence $\{x_i\}$ such that x_i tends spacelike to infinity and such that $\underline{F}\alpha_{x_i}(\underline{G})^* \in \mathfrak{A}(\mathcal{O}_1') \cap \mathfrak{A}(\mathcal{O}_2')$. Taking the limit as $i \to \infty$ we deduce that $\underline{F}\omega_0(\underline{G}) = \underline{F}b \in \{\mathfrak{A}(\mathcal{O}_1') \cap \mathfrak{A}(\mathcal{O}_2')\}''$.

The evidence , such as it is [1 ; Section IV] , suggests that $\mathcal{O} \to \mathfrak{B}(\mathcal{O})$ will satisfy duality in the vacuum sector

$$\pi_0(\mathfrak{B}(\mathcal{O})) = \pi_0(\mathfrak{B}(\mathcal{O}'))' .$$

5) Of course the connexion between spontaneously broken symmetry and zero mass particles is equally valid for gauge symmetries .

Now by proposition 9 a) above, we deduce from this that

$$\pi_0(\mathfrak{B}(\mathcal{O})) = \pi_0(\mathfrak{A}(\mathcal{O}'))' .$$

Since π_0 is a faithful representation of $\mathfrak{B}$, we may use this equation to identify $\mathcal{O} \to \mathfrak{B}(\mathcal{O})$ if we are given the observable net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$. Thus we replace the duality assumption of [9] by the two assumptions on the <u>vacuum</u> representation of $\mathfrak{A}$

a) $\mathcal{O} \to \mathfrak{B}(\mathcal{O}) \equiv \mathfrak{A}(\mathcal{O}')'$ is a local net,

b) If $\mathcal{O} \subset \mathcal{O}_1' \cap \mathcal{O}_2'$ then $\mathfrak{B}(\mathcal{O}) \subset \{\mathfrak{A}(\mathcal{O}_1') \cap \mathfrak{A}(\mathcal{O}_2')\}''$.

Condition a) gives a net $\mathcal{O} \to \mathfrak{B}(\mathcal{O})$ satisfying duality and b) guarantees that a representation π satisfying the selection criterion 1.1 of [9] may be extended in a canonical way to a representation $\hat{\pi}$ of $\mathfrak{B}$ on the same Hilbert space satisfying the selection criterion with $\mathfrak{B}$ in place of $\mathfrak{A}$. The superselection structure is then determined as in [9] in terms of equivalence classes of localized morphisms of $\mathfrak{B}$. As example a) shows, there may be many inequivalent localized morphisms (even automorphisms) of $\mathfrak{A}$ which do not correspond to different sectors of $\mathfrak{A}$; these become equivalent when extended to localized morphisms of $\mathfrak{B}$.

We close with the remark that the situation in Statistical Mechanics is quite different from that in Quantum Field Theory. Although we may prove Theorem 8 as above if we assume $\mathfrak{F}^-$ is a factor (corresponding physically to a pure phase) the results of [1 ; Section III] no longer apply. Instead one can show that $\mathfrak{A}^- = \mathfrak{B}^-$ is also a factor so that gauge symmetries do not give rise to superselection sectors in Statistical Mechanics.

APPENDIX

We collect here the few elementary results on the harmonic analysis of the action of compact groups on von Neumann algebras which are used in the text.

Let σ be a finite-dimensional continuous unitary representation of a compact group $\mathcal{G}$ on a Hilbert space H_σ. We denote by $\bar{\sigma}$ the conjugate representation on the opposite Hilbert space $\bar{H}_\sigma = H_{\bar{\sigma}}$. We first establish a purely group-theoretical result needed in Proposition 6 which expresses the duality between a closed subgroup of $\mathcal{G}$ and an assignment K of a subspace $K_\sigma \subset H_\sigma$ to each σ satisfying the following three properties

$$\bar{K}_\sigma = K_{\bar{\sigma}} \tag{1}$$

$$K_\sigma \otimes K_\tau = K_{\sigma\otimes\tau} \, , \tag{2}$$

If t intertwines σ and τ , i.e . if t is a bounded linear operator mapping H_σ into H_τ such that $t\,\sigma(g) = \tau(g)\,t$, $g \in \mathcal{G}$ then

$$t(K_\sigma) \subset K_\tau \, . \tag{3}$$

If $\mathcal{L} \subset \mathcal{G}$, then setting $(\mathcal{L}^\perp)_\sigma = \{ b \in H_\sigma : \sigma(g)b = b,\ g \in \mathcal{L} \}$ we see that $\sigma \to (\mathcal{L}^\perp)_\sigma$ satisfies (1) , (2) and (3) . Conversely if $K_\sigma \subset H_\sigma$ for each σ then

$$K^\perp = \{ g \in \mathcal{G} : \text{for all } \sigma,\ \sigma(g)b = b,\ b \in K_\sigma \}$$

is a closed subgroup of $\mathcal{G}$.

<u>Proposition 10</u> [6] If $\sigma \to K_\sigma \subset H_\sigma$ satisfies (1),(2) and (3) above and $\mathcal{G}_0 = K^\perp$ then $K = \mathcal{G}_0^\perp = K^{\perp\perp}$,i.e. $K_\sigma = \{ b \in H_\sigma : \sigma(g)b = b,\ g \in \mathcal{G}_0 \}$.

<u>Proof</u> Let $\mathcal{F}$ denote the complex linear space of continuous functions on $\mathcal{G}$ spanned by functions f of the form $f(g) = (b', \sigma(g)b)$ where $b' \in H_\sigma$, $b \in K_\sigma$ and σ is some finite-dimensional continuous unitary representation . The space $\mathcal{G}/\mathcal{G}_0$ of left cosets equipped with the quotient topology is compact and we let $\mathcal{C}(\mathcal{G}/\mathcal{G}_0)$ denote the C*- algebra of complex continuous functions on $\mathcal{G}/\mathcal{G}_0$. Since an $f \in \mathcal{F}$ satisfies $f(gg') = f(g), g' \in \mathcal{G}_0$, we may consider $\mathcal{F}$ as a subspace of $\mathcal{C}(\mathcal{G}/\mathcal{G}_0)$. Conditions (1) and (2) ensure that $\mathcal{F}$ is a *-subalgebra of $\mathcal{C}(\mathcal{G}/\mathcal{G}_0)$. Now suppose that $g, g' \in \mathcal{G}$ and that for all σ

$$(b', \sigma(g)b) = (b', \sigma(g')b) \, , \quad b' \in H_\sigma \, , \ b \in K_\sigma$$

6) This is just a variant of Theorem (30-47) of [10]

then $\sigma(g)b = \sigma(g')b$, $b \in K_\sigma$. Hence $g^{-1}g' \in K^\perp = \mathcal{G}_0$. Thus $\mathcal{S}$ separates points of $\mathcal{C}(\mathcal{G}/\mathcal{G}_0)$ and the Stone-Weierstrass Theorem implies that $\mathcal{S}$ is dense in $\mathcal{C}(\mathcal{G}/\mathcal{G}_0)$. Now suppose $\sigma(g)c = c$ for all $g \in \mathcal{G}_0$ but $c \notin K_\sigma$ then we may suppose that c is orthogonal to K_σ. If $c' \in H_\sigma$ and $f(g) = (c', \sigma(g)c)$, $f \in \mathcal{C}(\mathcal{G}/\mathcal{G}_0)$. Pick $b \in K_\tau$ and $b' \in H_\tau$ then

$$\int (b', \tau(g)b)\overline{f(g)}\,d\mu(g) = (c, tb) \qquad \text{where} \quad t = \int \sigma(g)\, c' \otimes \overline{b'}\, \tau(g)^{-1} d\mu(g)$$

and $c' \otimes \overline{b'}$ denotes the rank one mapping from H_τ to H_σ taking $a \in H_\tau$ to $(b', a)\,c'$. Now t intertwines τ and σ so by (3) we have $tb \in K_\sigma$ and $(c, tb) = 0$. Hence f is orthogonal $\mathcal{S}$ in $L^2(\mathcal{G})$ but since $\mathcal{S}$ is dense in $\mathcal{C}(\mathcal{G}/\mathcal{G}_0)$ this implies that $f = 0$. Hence c=0 and this contradiction completes the proof.

Now let M be a $*$-algebra and $g \to \beta_g$ a representation of $\mathcal{G}$ by $*$-automorphisms of M. In order to describe elements of M which 'transform according to the representation σ', we consider the tensor product $M \otimes \overline{H}_\sigma$ as a space of row matrices with entries in M by picking a basis $b_1, b_2, \cdots b_d$, $d = \dim \sigma$ of H_σ and expressing $\underset{\sim}{F} \in M \otimes \overline{H}_\sigma$ in the form $\underset{\sim}{F} = \sum_{i=1}^{d} F_i \otimes \overline{b}_i$; $\underset{\sim}{F}$ is then regarded as the row matrix $\underset{\sim}{F} = (F_i)$. We may define an action (again denoted by β) of $\mathcal{G}$ on $M \otimes \overline{H}_\sigma$ by acting with β_g on the components of $\underset{\sim}{F}$. Set

$$M_\sigma = \{ \underset{\sim}{F} \in M \otimes \overline{H}_\sigma : \beta_g(\underset{\sim}{F}) = \underset{\sim}{F}\sigma(g),\ g \in \mathcal{G} \},$$

where $\underset{\sim}{F}\sigma(g)$ denotes the matrix multiplication of $\underset{\sim}{F}$ on the right by $\sigma(g)$. Another way of looking at M_σ is as the fixed points under the action $g \to \beta_g \otimes \overline{\sigma}(g)$ of $\mathcal{G}$ on $M \otimes \overline{H}_\sigma$.

There are a number of elementary operations we may define on the M_σ. Given $\underset{\sim}{F} = \sum F_i \otimes \overline{b}_i \in M \otimes \overline{H}_\sigma$ we define $\underset{\sim}{F}^* \in M \otimes H_\sigma$ by $\underset{\sim}{F}^* = \sum F_i^* \otimes b_i$. Clearly if $\underset{\sim}{F} \in M_\sigma$, $\underset{\sim}{F}^* \in M_{\overline{\sigma}}$. If $\underset{\sim}{G} = \sum G_j \otimes c_j \in M_\tau$ then we may define $\underset{\sim}{F} \otimes \underset{\sim}{G} = \sum_{i,j} F_i G_j \otimes (\overline{b}_i \otimes \overline{c}_j)$, an element of $M \otimes \overline{H}_{\sigma\otimes\tau}$, and verify that $\underset{\sim}{F} \otimes \underset{\sim}{G} \in M_{\sigma\otimes\tau}$.

If $\tau = \sigma$ then $\underset{\sim}{F}\,\underset{\sim}{G}^t \equiv \sum_{i=1}^{d} F_i G_i$ is a β-invariant element of M .
If $t : H_\sigma \to H_\tau$ intertwines σ and τ and $\underset{\sim}{F} \in M_\sigma$ then $\underset{\sim}{F} t^* \in M_\tau$ where $\underset{\sim}{F} t^*$ denotes the matrix multiplication of $\underset{\sim}{F}$ on the right by t^*; this operation may also be regarded as induced by $\iota \otimes \bar{t}$ on $M \otimes \bar{H}_\sigma$ where ι is the identity automorpnism of M and $\bar{t} : \bar{H}_\sigma \to \bar{H}_\tau$ denotes the conjugate intertwiner

If $\underset{\sim}{F} = \sum F_i \otimes \bar{b}_i \in M \otimes \bar{H}_\sigma$ we define for a state ω of M $\omega(\underset{\sim}{F}) \in \bar{H}_\sigma$ by $\omega(\underset{\sim}{F}) = \sum_i \omega(F_i)\bar{b}_i$, and see that

$$\omega(\underset{\sim}{F}^*) = \overline{\omega(\underset{\sim}{F})} \tag{4}$$

If $\underset{\sim}{F} \in M_\sigma$ then

$$\omega(\beta_g(\underset{\sim}{F})) = \bar{\sigma}(g^{-1})\,\omega(\underset{\sim}{F}) \tag{5}$$

and if t intertwines σ and τ as above

$$\omega(\underset{\sim}{F} t^*) = \bar{t}\,\omega(\underset{\sim}{F}) \tag{6}$$

if $\underset{\sim}{F} = \sum_i F_i \otimes \bar{b}_i \in M \otimes \bar{H}_\sigma$ and $b = \sum \lambda_i b_i \in H_\sigma$ we define

$$\underset{\sim}{F} b = \sum \lambda_i F_i \in M . \tag{7}$$

This is of course nothing but multiplying the row matrix $\underset{\sim}{F}$ on the right by the column matrix b . If $\underset{\sim}{F} \in M_\sigma$ we term $\underset{\sim}{F} b$ a tensor of type σ in M and denote by

$$M_\sigma H_\sigma = \{ \underset{\sim}{F} b : \underset{\sim}{F} \in M_\sigma ,\, b \in H_\sigma \} \tag{8}$$

the set of tensors of type σ in M . This set depends only on the equivalence class of σ

<u>Proposition II</u> Let M be a von Neumann algebra and suppose $g \to \omega(\beta_g(F))$ is continuous for each $F \in M$ and each normal state ω of M then M is the weakly closed linear span of $\bigcup_\sigma M_\sigma H_\sigma$ where σ runs over a complete set of irreducible continuous unitary representations of $\mathcal{G}$.

<u>Proof</u> : Given $b \in H_\sigma$ define for $F \in M$

$$m_{b_i, b}(F) = \int (\sigma(g)b, b_i)\, \beta_g(F)\, d\mu(g)$$

The integral exists in the weak topology of M , i.e. the topology generated by the normal linear forms of M , and an elementary computation shows that if $\underset{\sim}{F} = \sum_i M_{b_i, b}(F) \otimes \bar{b}_i$, then $\underset{\sim}{F} \in M_\sigma$. It follows that if $b, b' \in H_\sigma$ and $f(g) = (\sigma(g)b, b')$ then $\int f(g)\beta_g(F)d\mu(g) \in M_\sigma H_\sigma$. Now let $\mathcal{V}$ denote the linear space of functions of the form $g \to (\sigma(g)b, b')$ with $b, b' \in H_\sigma$ and σ any irreducible continuous unitary representation . $\mathcal{V}$ is uniformly dense in $\mathcal{C}(\mathcal{G})$. This may be proved using the Stone-Weierstrass Theorem (compare the proof of Proposition 10) . Hence we may approximate the Dirac measure at the identity weakly by functions from $\mathcal{V}$. In other words there is a net of functions $h_\alpha \in \mathcal{V}$ such that

$$f(e) = \lim_\alpha \int h_\alpha(g) f(g)\, d\mu(g) , \quad f \in \mathcal{C}(\mathcal{G}) .$$

Taking $f(g) = \omega(\beta_g(F))$ with ω a normal state of M we see that F is the weak limit of the net F_α , where

$$F_\alpha = \int h_\alpha(g)\, \beta_g(F) d\mu(g) \text{ is in the linear span of } \bigcup_\sigma M_\sigma H_\sigma .$$

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