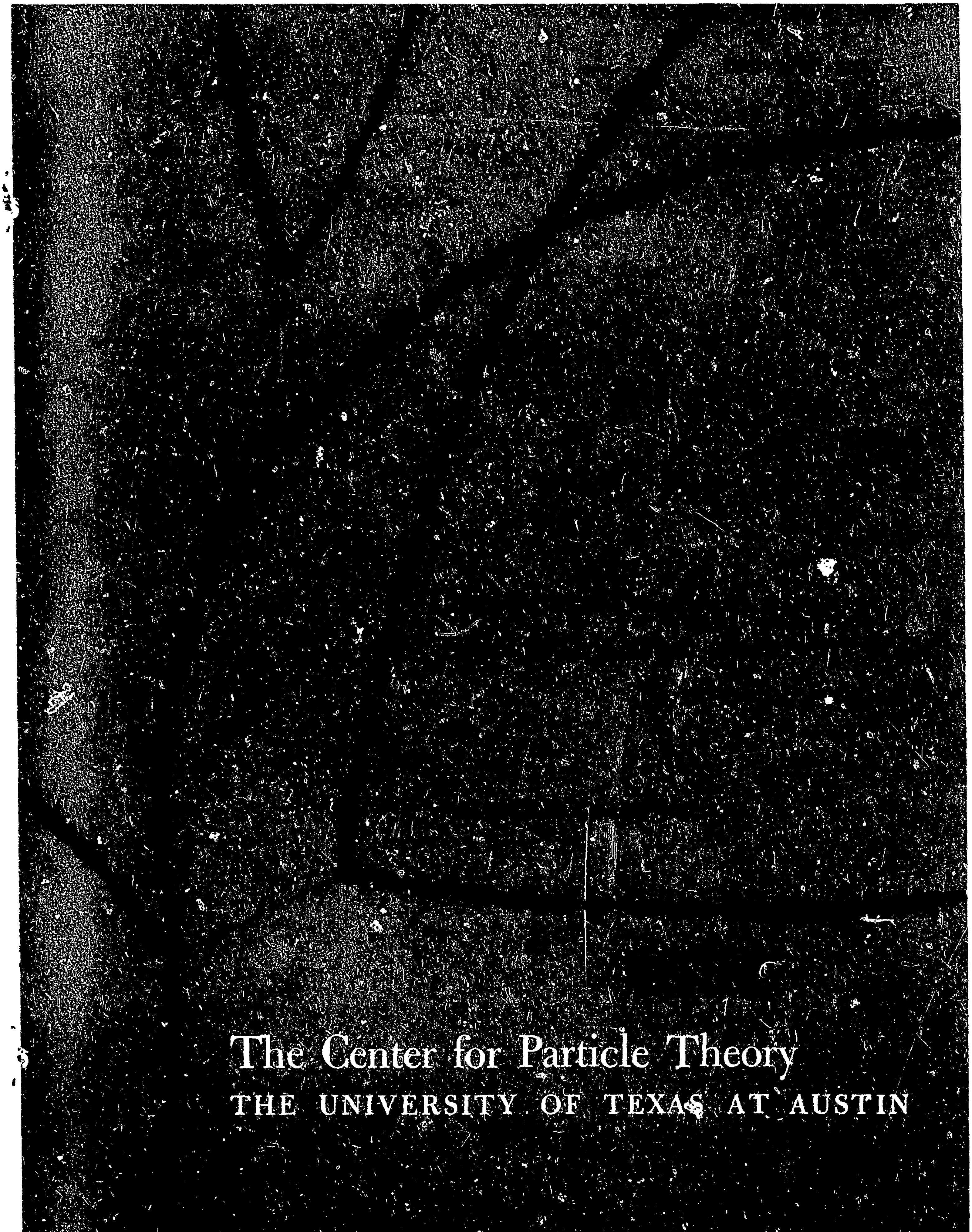

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EXACT SOLUTION TO GELL-MANN-LOW EQUATIONS IN A SIMPLE RELATIVISTIC MODEL *

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ABSTRACT

We study the exactly soluble relativistic model field theory proposed by Zachariasen. Renormalization of this model is carried out both in the conventional manner, and by choosing the point of renormalization at an arbitrary space-like momentum. The Gell-Mann-Low equations are then set up for the coupling constant, and the exact solution obtained. The asymptotic value of the coupling is seen to go to the bare constant. Some implications of the model are considered.

I INTRODUCTION

Renormalization group has been used in recent years in several important applications, especially in proving that Yang-Mills theories are asymptotically free [1]. The renormalization group method has also been employed by Adler [2] in a programme to determine the fine structure constant. The purpose of this paper is the modest one of setting up the renormalization group equations for a sample model which is exactly soluble. The model we employ is the one due to Zachariasen [3], and is similar to the Lee model in Nθ sector, except for relativistic kinematics. Different formulations of the model are presented by Deshpande and Bludman [4], and earlier references are cited therein. The Gell-Mann-Low[5] equations for the effective coupling constant are set up by choosing the renormalization point at an arbitrary space-like momenta. The equations can be exactly solved, and the effective coupling constant is seen to interpolate between the on-shell coupling and the bare coupling constant. The model thus provides an illustration of the context of renormalization group equations. Our treatment of the model parallels the lucid paper by Wilson [6] on the renormalization group. Several implications of the model are considered. In particular we find that finite Z_3 or bare coupling constant does not lead to an eigenvalue equation for the physical coupling as conjectured by Adler in quantum electro-dynamics. The relation of the renormalization group to the arbitrary choice of the substraction point in N/D method is also shown in context of this model.

II. CONVENTIONAL RENORMALIZATION OF THE ZACHARIASEN MODEL.

The version of the model we shall consider is the one with bare Yukawa coupling g_o $\bar{B}BA$, and only the process $A \to B + \bar{B}$ is allowed. Crossing is violated by the model in that $B \to B + A$ is not allowed. Thus $B\bar{B}$ scattering amplitude is given by a sum of bubble diagrams. The unrenormalized but exact Green's functions of the model are thus easily calculated.

A. Unrenormalized Green's Functions.

The unrenormalized A-particle propagator is

$$\Delta_u^{-1}(s) = s - \mu_0^2 - \Pi(s) \tag{2.1}$$

where μ_0^2 is the bare mass of A particle, and $\Pi(s)$ is the proper self-energy. The particles are all assumed spinless, so that $\Pi(s)$ is a logarithmically divergent integral if cut-off Λ is not employed. Thus

$$\Pi(s,\Lambda) = -g_0^2 \frac{1}{\pi}\int_1^{\Lambda} \frac{\rho(s')ds'}{s'-s-i\varepsilon} \tag{2.2}$$

where $\rho(s) = (1/16\pi)[(s-1)/s]^{1/2}$ is the two particle density of states, and we have assumed units such that the mass of B particle is $1/2$.

The $B\bar{B}$ scattering amplitude is given by

$$T(s) = g_0^2 \Delta_u(s) \tag{2.3}$$

B. Renormalization of the Model.

The physical mass μ^2 is defined through the mass equation

$$\mu^2 = \mu_0^2 + \Pi(\mu^2, \Lambda) \tag{2.4}$$

This equation ensures that Δ_u has a pole at $s=\mu^2$ the unrenormalized propagator can now be written as

$$\Delta_u^{-1} = s-\mu^2 + g_0^2 J(s) \qquad (2.5)$$

where

$$J(s) = \frac{(s-\mu^2)}{\pi}\int_1^{\Lambda} \frac{\rho(s')\,ds'}{(s'-\mu^2)(s'-s-i\epsilon)} \qquad (2.6)$$

notice that $\Lambda\to\infty$ limit can now be taken because $J(s)$ is a convergent integral.

The conventional renormalization now defines a renormalized propagator such that the residue at $s=\mu^2$ is unity

$$\Delta_R^c = \Delta_u Z_3^{-1} \qquad (2.7)$$

where

$$Z_3 = \lim_{s\to\mu^2} (s-\mu^2)\,\Delta_u \qquad (2.8)$$

Thus

$$Z_3^{-1} = 1 + \frac{g_0^2}{\pi}\int \frac{\rho(s')\,ds'}{(s'-\mu^2)^2} \qquad (2.9)$$

The physical renormalized coupling constant is

$$g^2 = Z_3 g_0^2 = \left[\frac{1}{g_0^2} + \frac{1}{\pi}\int \frac{\rho(s')\,ds'}{(s'-\mu^2)^2}\right]^{-1} \qquad (2.10)$$

It is possible to express Z_3 in terms of the coupling constant ,

$$Z_3 = 1 - \frac{g^2}{\pi}\int \frac{\rho(s')\,ds'}{(s'-\mu^2)^2} \qquad (2.11)$$

Equation 2.9 ensures the bounds $0 \leq Z_3 \leq 1$, while Eq 2.11 leads to upper bound on g^2. The conventional renormalized Green's function are then given by

$$\left(\Delta_R^c(s)\right)^{-1} = (s-\mu^2)\left[1 + g^2\frac{(s-\mu^2)}{\pi}\int_1^{\infty} \frac{\rho(s')\,ds'}{(s'-\mu^2)^2(s'-s-i\epsilon)}\right] \qquad (2.12)$$

and
$$T(s) = g^2 \Delta_R^c(s). \qquad (2.13)$$

a convenient form for T(s) which shows the unitarity explicitely is
$$T(s) = N(s)/D(s) \qquad (2.14)$$

with
$$N(s) = g^2/(s-\mu^2) \qquad (2.15)$$
$$D(s) = 1 + \frac{(s-\mu^2)}{\pi}\int \frac{\rho(s')N(s')ds'}{(s'-\mu^2)(s'-s-i\epsilon)} \qquad (2.16)$$

III. RENORMALIZATION AT THE ARBITRARY POINT.

We shall now define a renormalization program for the Zachariasen model which is unconventtial. The conventional definition of the physical mass, ie.,
$$\mu^2 = \mu_0^2 + \pi(\mu^2, \Lambda) \qquad (3.1)$$

is retained as in the Gell-Mann-Low approach. We choose the normalization point at λ for the renormalized Green's functions, instead of the mass-shell value μ^2. The Green's function so obtained are related to the renormalized functions in the conventional approach by finite constants. The renormalization group equation results from comparing renormalized theories for two different values of λ , and differential equations can then be set up for the coupling constants as in quantum electrodynamics.

The renormalized propagator is now defined through the normalization:
$$\Delta_R^\lambda(s) \xrightarrow[s\to\lambda]{} 1/(s-\mu^2) \qquad (3.2)$$

from the unrenormalized propagator, we see that
$$(s-\mu^2)\,\Delta_R^\lambda(s) = \left[1 + \frac{g_\lambda^2(s-\lambda)}{\pi}\int_1^\infty \frac{\rho(s')\,ds'}{(s'-\lambda)(s'-\mu^2)(s'-s-i\epsilon)}\right]^{-1} \qquad (3.3)$$

where we have defined
$$g_\lambda^2 = \left[\frac{1}{g_0^2} + \frac{1}{\pi}\int_1^\infty \frac{\rho(s')\,ds'}{(s'-\mu^2)(s'-\lambda)}\right]^{-1} \qquad (3.4)$$

the value of λ is restricted to be less then the lowest thresh-hold at $s = 4m_B^2 = 1$. For $\lambda > 1$ the integral develops imaginary parts, and the definition of coupling constant becomes meaningless.

The relation between the conventionally renormalized field and the field renormalized at λ is

$$A_R^\lambda(x) = (Z_3^\lambda)^{1/2} A_R^c(x) \qquad (3.5)$$

it is easy to see from the propagator,

$$\Delta_R^\lambda(s) = Z_3^\lambda \Delta_R^c(x) \qquad (3.6)$$

An argument similar to Wilson[6], then gives the renormalized coupling constant

$$g_\lambda^2 Z_3^\lambda = g^2 = Z_3 g_0^2 \qquad (3.7)$$

we have from Eq (3.6) two alternate expressions for Z_3^λ, depending on $s \to \mu^2$ or $s \to \lambda$. We thus find

$$g^2 = g_\lambda^2 \left[1 + \frac{(\mu^2-\lambda) g_\lambda^2}{\pi} \int_1^\infty \frac{\rho(s')\,ds'}{(s'-\lambda)(s'-\mu^2)^2} \right]^{-1} \qquad (3.8)$$

or

$$g_\lambda^2 = g^2 \left[1 + \frac{(\lambda-\mu^2) g^2}{\pi} \int_1^\infty \frac{\rho(s')\,ds'}{(s'-\lambda)(s'-\mu^2)^2} \right] \qquad (3.9)$$

From Eq (3.9), (2.10) and (2.11) one can establish Eq (3.4), thus proving that g_λ introduced before is indeen the renormalized coupling constant.

The scattering amplitude is given by

$$T(s) = g_\lambda^2 \Delta_R^\lambda(s) \qquad (3.10)$$

The equivalence of the two renormalization prescriptions can be shown after using equation (3.9) and a few algebraic manipulations. The scattering amplitude in Equation 3.10 corresponds to N/D amplitude with a different substraction point for the D function:

$$T(s) = N^\lambda(s)/D^\lambda(s)$$

where
$$N^\lambda(s) = g_\lambda^2/(s-\mu^2) \qquad (3.11)$$

and
$$D^\lambda(s) = 1 + \frac{(s-\lambda)}{\pi}\int\frac{\rho(s')\,N^\lambda(s')\,ds'}{(s'-\lambda)(s'-s-i\epsilon)} \qquad (3.12)$$

IV. RENORMALIZATION GROUP.

We now look at formulae connecting renormalized fields at two different values of λ, say, λ and λ'. The two sets of fields are connected by renormalization constants

$$A_R^{\lambda'} = Z_3^{1/2}(\lambda',\lambda)\,A_R^\lambda \qquad (4.1)$$

since
$$A_R^{\lambda'} = (Z_3^{\lambda'})^{1/2} A_R^c \qquad (4.2)$$
$$A_R^{\lambda} = (Z_3^{\lambda})^{1/2} A_R^c \qquad (4.3)$$

therefore
$$Z_3(\lambda',\lambda) = Z_3^{\lambda'}/Z_3^\lambda \qquad (4.4)$$

We further have
$$g_\lambda^2 = Z_3(\lambda,\lambda')\,g_{\lambda'}^2 \qquad (4.5)$$

it then follows
$$g_\lambda^2 Z_3^\lambda = g_{\lambda'}^2 Z_3^{\lambda'} = g^2 \qquad (4.6)$$

or equivalently,
$$g_{\lambda'}^2 = Z_3(\lambda,\lambda')\,g_\lambda^2 \qquad (4.7)$$

We can now set-up the Gell-Mann-Low equations for g_λ^2. Note

$$Z_3(\lambda,\lambda') = Z_3^\lambda/Z_3^{\lambda'} = (\Delta_R^\lambda/\Delta_R^c)(\Delta_R^c/\Delta_R^{\lambda'}) \qquad (4.8)$$

choosing $s = \lambda$ we obtain

$$g_{\lambda'}^2 = g_\lambda^2\left[1 + \frac{(\lambda'-\lambda)}{\pi}g_\lambda^2\int\frac{\rho(s')\,ds'}{(s'-\lambda)(s'-\mu^2)(s'-\lambda')}\right]^{-1} \qquad (4.9)$$

Taking an infinitesimal change $\lambda' - \lambda = \partial\lambda$, we have

$$\frac{\partial g_\lambda^2}{\partial \lambda} = -\frac{g_\lambda^4}{\pi}\int \frac{\rho(s')\,ds'}{(s'-\lambda)^2(s'-\mu^2)} \qquad (4.10)$$

This is the Gell-Mann-Low Equation for our model. A simpler form is

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{g_\lambda^2}\right) = \frac{1}{\pi}\int \frac{\rho(s')\,ds'}{(s'-\lambda)^2(s'-\mu^2)} \qquad (4.11)$$

The solution is obvious (boundary condition $g_\lambda \to g$ as $\lambda \to \mu^2$ is assumed)

$$\frac{1}{g_\lambda^2} = \frac{1}{g^2} + \frac{(\lambda-\mu^2)}{\pi}\int \frac{\rho(s')\,ds'}{(s'-\lambda)(s'-\mu^2)^2} \qquad (4.12)$$

The asymptotic value of g_λ^2 as $\lambda \to -\infty$ is

$$\frac{1}{g_{-\infty}^2} = \frac{1}{g^2} - \frac{1}{\pi}\int \frac{\rho(s')\,ds'}{(s'-\mu^2)^2} \qquad (4.13)$$

from Eqs (2.10) and (2.11) we have $g_0^2 = g^2/Z_3(g^2)$
or $g_{-\infty}^2 = g_0^2$ (the bare coupling constant)

Note that just as in quantum electrodynamics, since $g_0^2 > g^2$ which follows from $1 \geq Z_3 \geq 0$, such a theory can never be asymptotically free. Same conclusion is also true for the Lee model.

V. CONCLUSIONS.

The model we have considered provides a simple illustration of the methods used in Renormalization Group. In particular we see that the Gell-Mann-Low equation can be solved exactly and yields a coupling g_λ which interpolates between g_0 and g. It is to be noted that since Z_3 is finite in this theory, both g_0 and g are simultaneously finite. Adler has conjectured that finite Z_3 or finite bare charge will enable one to set up an eigenvalue equation for the physical charge. This model is seen to be a counterexample, since finiteness of g_0^2 does not lead to eigenvalue condition for the physical charge. Nevertheless the program may succeed in quantum-electrodynamics, which is an extremely complicated theory.

The model also provides an interesting connection between N/D method and Renormalization Group methods. In a forthcoming article we shall explore the constraints on N/D method arising from asymptotic freedom.

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