INSTITUTE OF PLASMA PHYSICS
CZECHOSLOVAK ACADEMY OF SCIENCES

ON THE BEAT INTERACTION OF PARTICLES WITH LARGE AMPLITUDE SPECTRUM IN THE DESCRIPTION OF MIXING SYSTEMS

L. Krlín

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ABSTRACT

We investigate on a simple model of particles in a discrete spectrum an influence of stochastic instability of particle motion on the nonlinear Landau damping. We show that as long as particles trajectory in the spectrum is stochastically unstable (at the beat resonances under consideration), diffusion of particles takes place. We discuss the influence of this effect on usually supposed nonlinear Landau mechanism and briefly the possibility of heating in this regime in the beam-plasma experiment.

1. INTRODUCTION

As follows from the literature (e.g. from Piffl V. et al., 1969; Seidl M., Šunka P., 1967), large amplitude hf field is generated in beam plasma experiment. The mechanism of its absorption has not yet been sufficiently clarified; the similar problem exists also in other plasma phenomena. It seems that parametric instabilities could be the reason (Kainer S. et al., 1972); nevertheless there are still other mechanisms that could be taken into the consideration.

The present paper discusses the absorption caused by the resonant interaction on beats; we suppose a plasma without magnetostatic field and the spectrum in which the resonant mechanism considered leads (due to nonlinearities), to the stochastic instability of particles.

2. NONLINEAR LANDAU DAMPING FOR WAVES OF THE SAME TYPE; POSSIBILITY OF THE EFFECT OF MIXING

In experiment (Piffl V. et al., 1971; Seidl M., Šunka P., 1967) two types of waves were identified - the waves with $\omega > \omega_{ce}$ which follow approximately the upper hybrid frequency $\omega_H = \sqrt{\omega_{pe}^2 + \omega_{ce}^2}$ and the waves with $\omega < \omega_{ce}$, $\omega \sim \omega_{pe}$. The temperature of plasma ($\omega \sim 10^4 k v_T$) is too small for absorption due to linear Landau damping and also for trapping of plasma particles into the wave (Dawson J.M., Shanny R., 1968). We shall not discuss already mentioned (Kainer S., et al., 1972) parametric instabilities with mobile ions, which lead to the generation of waves with lower phase velocity.

Because the Landau damping as well as the effect of trapped particles do not count the effect of the nonlinear Landau damping can be discussed for thermal velocities

$$v_T \sim \frac{\omega_2 - \omega_1}{k_2 - k_1}$$

with two waves ω_1, k_1 and ω_2, k_2. For a plasma with the small thermal velocity we can thus expect an effect caused by the particle resonance on the beat frequency of two waves.

However, it follows from the theory (e.g. Tsytovich V.N., 1970) that in the interaction of the same type of waves through the nonlinear Landau mechanism the energy transport appears only within the spectrum toward lower frequencies. This is caused by the symmetry properties of the scattering coefficient of the wave.

Let us try to reinterpret the damping mechanism on beats by including this system into so called mixing systems. As it is shown in (Zaslavskij G.M., 1970), time development of nonlinear systems can, under some circumstances, be described statistically together with interesting consequences following from this description. In the next chapter we will be concerned with this problem more closely.

3. NONLINEAR SYSTEMS WITH MIXING

Recently, a considerable interest has been devoted to the systems with mixing, especially in connection with questions of relaxation. Here, we shall follow Zaslavskij (Zaslavskij G.M., 1970); several effects are discussed there that are related closely to the questions of plasma physics.

The author introduces as a basic model the nonlinear oscillator in the field of external forces. Let us have a nonlinear oscillator with the Hamiltonian in the coordinates action-angle

$$H = H(J)$$

where J is the action corresponding to the angle $w = \int \omega \, dt$. Let us have, further, an external force acting on this oscillator in the form of δ - pulses and let the corresponding canonical equation for the time change of J of the form

$$\frac{dJ}{dt} = \varepsilon J \sum_{m=-\infty}^{+\infty} \delta(t - mT) \sin 2w \,; \qquad |\varepsilon| \ll 1$$

where T is the period of pulses. Define the quantity

$$\mathcal{K} = \frac{d\omega}{dJ} \frac{\Delta J}{\Omega}$$

where $\Omega = \frac{2\pi}{T}$ and ΔJ is the maximum change of J in the δ-pulse. Let the inequality $\mathcal{K} \gg 1$ be fulfilled. Then, one can show (for the proof see Zaslavskij G.M., 1970) that for times $t \gg \tau_{\mathcal{K}}$ where $\tau_{\mathcal{K}}$ is defined by

$$\tau_{\mathcal{K}} = (2 \Omega \lg \mathcal{K})^{-1}$$

the particle state can be described using the probability distribution function $f(J, w, t)$

$$f(J, w, t) = \sum_{n,m=-\infty}^{n,m=+\infty} f_n(J, t) e^{inw + m\Omega t}$$

satisfying the Fokker-Planck diffusion equation (e.g. for $f_0(J, t)$)

$$\frac{\partial f_0}{\partial t} = 8\pi\varepsilon^2 \frac{\partial}{\partial J} \sum_{n,m} |V_{n,m}| \delta(n\omega - m\Omega) \frac{\partial}{\partial J} (|V_{n,m}| f_0)$$

where

$$V(J, w, t) = J \sum \delta(t - mT) \sin 2w = \sum_{n,m} V_{n,m} \exp i (nw + m\Omega t).$$

We shall show that under some conditions the interaction of particles with wave beats can be described as a system with mixing; the diffusion of particles in the velocity space would follow (in complete analogy to the diffusion initiated by a stochastic field) with apparent possibility of particles heating. In the following, we determine conditions under which the nonlinear Landau damping can be considered to be a system with mixing. For this purpose let us discuss the dynamics of a particle in two waves and in the spectrum.

4. DYNAMICS OF A PARTICLE IN TWO WAVES

A spectrum measured usually appears within the region between two limiting models – – the case of one leading wave with large amplitude and a lower side band and the case of a rectangular spectrum. In order that we can use the perturbation method for both models, a partly different analytical apparatus must be applied; this is described in the following chapters 4.1 and 4.2.

4.1 Particle dynamics in the couple of waves with amplitudes $E_1,\ E_2;\quad E_2 = \varepsilon E_1;\ |\varepsilon| \ll 1$

First, consider the interaction of two waves with potentials φ_1 and φ_2 with $|\varphi_2| \ll |\varphi_1|$; the corresponding Hamiltonian of a particle is

$$H = \frac{1}{2m} p_x^2 + e\varphi_1 + e\varphi_2 \tag{1}$$

where, in one dimensional case

$$\varphi_1 = \varphi_{10} \exp i(k_1 x - \omega_1 t + \psi_1), \quad \varphi_2 = \varepsilon \varphi_{10} \exp i(k_2 x - \omega_2 t + \psi_2); \qquad |\varepsilon| \ll 1 .$$

Let us introduce the following notation

$$\mathcal{H} = \frac{1}{e\varphi_{10}} H; \qquad \varkappa = k_2/k_1; \qquad \alpha = k_2\left(\frac{\omega_1}{k_1} - \frac{\omega_2}{k_2}\right);$$

$$V = \tau_0 \frac{dy}{dt}; \qquad y = k_1 x - \omega_1 t; \qquad \omega_B = \tau_0^{-1} = \sqrt{\frac{e k_1 E_1}{m}}; \qquad \vec{E} = -\nabla \varphi .$$

We can thus express the Hamiltonian (1) in the coordinate system moving with the velocity $v_\varphi = \omega_1/k_1$ in the form

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1 \tag{2}$$

where

$$\mathcal{H}_0 = \frac{1}{2} V^2 - \cos y$$

$$\varepsilon \mathcal{H}_1 = -\varepsilon \cos(\varkappa y + \alpha t + \psi).$$

The motion given by the Hamiltonian (2) can be solved by means of the perturbation method. Before doing so let us perform the canonical transformation into the action--angle coordinates and express $\mathcal{H}_0$ in the cyclic form. Denote further

$$q^2 = \frac{1}{2}\left(1 + \mathcal{H}_0\right)$$

and consider the case $\mathcal{H}_0 > 1$ (i.e. untrapped particles). If we use the generating function

$$S = \frac{1}{2\pi}\int V dy = \frac{2}{\pi} q E\left(\frac{\chi}{2}, \frac{1}{q}\right)$$

where E is the total elliptical integral of the second king we have

$$\mathcal{H}_0 = \mathcal{H}_0(J)\ ; \qquad J = \frac{4}{\pi} q E\left(\frac{\pi}{2}, \frac{1}{q}\right); \qquad w = \frac{\partial S}{\partial J} .$$

We obtain for the first approximation

$$\frac{dJ^{(1)}}{dt} = -\frac{\varepsilon}{\omega \tau_0^2} V^{(0)} \sin\left(x y^{(0)} + \alpha t + \psi\right) \qquad (3)$$

where[1] $V^{(0)}, y^{(0)}$ are functions corresponding to $\mathcal{H}_0$. Let us denote $\mu = \frac{q \pi x}{\tau_0 F}$. (For the sake of clarity we denote the eliptic integral $K\left(\frac{\pi}{2}, \frac{1}{q}\right) \to F$). Then as follows from simple calculation, eq. (3) has the reсonant solution at

$$\alpha + \mu + 2 s \omega^{(0)} = 0\ ; \qquad \omega^{(0)} = \frac{\pi q}{\tau_0 F} \qquad (4)$$

where s is integer. Similarly, it follows that the value $s = -1$ determines the usual condition of the nonlinear Landau damping

$$v = \frac{\omega_2 - \omega_1}{k_2 - k_1} .$$

4.2 The particle dynamics in the pair of waves with amplitudes $E_1, E_2\ ;\ E_1 \sim E_2$

For purposes of use in the further model of spectrum let us consider again the Hamiltonian (1) now with $\varphi_{10} \sim \varphi_{20}$. Let us assume

$$v \ll \frac{\omega_i}{k_i}\ ; \qquad \frac{1}{2} m \frac{\omega_i^2}{k_i^2} \gg e\varphi_{i0} .$$

Then, again the notation appropriate for the perturbation solution is possible if we transform coordinates into the system moving together with one of waves. Denoting $m\frac{dx}{dt} = P_3 + m\frac{\omega_1}{k_1}$, $x - \frac{\omega_1}{k_1} t = Q_3$ we have

$$H = \frac{1}{2m} P_3^2 - e\varphi_{10} \cos k_1 Q_3 - e\varphi_{20} \cos\left(x k_1 Q_3 + \alpha t + \psi\right) \qquad (5)$$

and in the action-angle coordinates (where we use $S = k_1 Q_3 J$)

$$H = \frac{1}{2m} k_1^2 J^2 - e\,\varphi_{10} \cos w_3 - e\varphi_{20} \cos\left(x w + \alpha t + \psi\right)$$

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1) $V^{(0)} = 2q\, dn\left(\frac{t}{\tau_0}\right)$, where $dn\left(\frac{t}{\tau_0}\right)$ is Jacobi function of argument $\frac{t}{\tau_0}$

Here $w_3 = k_1 Q_3 = k_1 z - \omega_1 t$; $P_3 = \frac{\partial S}{\partial Q_3} = k_1 J$.

Hamiltonian (5) gives the nonlinear resonances in the second approximation; here

$$\frac{dJ^{(2)}}{dt} = -e\varphi_{10} \sin w_3^{(1)} - e\varphi_{20} \varkappa \sin(\varkappa w_3^{(1)} + \alpha t + \psi)$$

where

$$w^{(1)} = \frac{1}{m} k_1^2 \frac{e\varphi_{10}}{(k_1 v - \omega_1)^2} \sin(k_1 v - \omega_1)t +$$

$$+ \frac{1}{m} k_1^2 \frac{e\varphi_{20}\varkappa}{[\varkappa(k_1 v - \omega_1) + \alpha]^2} \sin[\varkappa(k_1 v - \omega_1)t + \alpha t + \psi]$$

and where $w^{(0)} = k_1 v^{(0)} - \omega_1 t$.

The resonant condition for $\frac{dJ^{(2)}}{dt}$ is here fulfilled through (4).

5. CHOICE OF THE SPECTRUM; THE RESONANT MOTION AS THE MIXING PROCESS

In beam-plasma experiments (see e.g. Piffl V., 1971; Seidl M., Šunka P., (1967) the authors measured in some phases always the spectrum of some effective width. Because of more extensive applications let us observe effect of two types of spectra (spectrum a and spectrum b, defined as following).

Let us first study the effect of the following model of the spectrum: let the central spectrum line (leading wave) has the amplitude E_1, the side lines have the amplitude εE_1 and let the beats of the side lines together with leading wave form very dense resonant structure around thermal velocity v_T. Let us suppose that $|\varepsilon| \ll 1$, $N \cdot |\varepsilon| \ll 1$, where N is the number of discrete lines. Hence, under these circumstances the perturbation method can be used. The Hamiltonian of a particle under the influence of this spectrum (further spectrum of the type a) in the form (2) is

$$\mathcal{H} = \frac{1}{2} V^2 - \cos y - \varepsilon \sum_{i=1}^{N} \cos(\varkappa_i y + \alpha_i t + \psi_i)$$

and the variation of J will be in the first approximation

$$\frac{dJ^{(1)}}{dt} = -\frac{\varepsilon}{\omega \tau_0^2} \sum_{i=1}^{N} V^{(0)} \sin(\varkappa_i y^{(0)} + \alpha_i t + \psi_i).$$

The above canonical equation and the resonant condition give the monotonic growth of J. From the entire spectrum of resonances let us consider only the basic nonlinear resonance

$$\mu + \alpha - 2\omega = 0.$$

Let $\overset{*}{q} = \exp\left(-\frac{\pi K'}{K}\right)$; $K' = F\left(\frac{\pi}{2}, \frac{1}{q'}\right)$; $\frac{1}{q'} = \sqrt{1 - \frac{1}{q^2}}$.

Considering $\frac{x q}{\tau_0 F} \cdot \frac{\varphi}{1+\varphi} \cdot \frac{1}{\omega} \ll 1$ the effect of higher harmonic expansion (and thus also higher resonances) is weaker than the influence of basic terms. Neglecting foregoing, we get stronger condition for stochastic instability.

Thus, let us retain in the relation for $\frac{dJ}{dt}$ only harmonics with the basic argument

$$\mu_i + \alpha_i - 2\omega_i \ .$$

We get
$$\frac{dJ}{dt} = -\frac{\varepsilon}{\omega \tau_0^2} \cdot \frac{2\pi}{F} \frac{\varphi}{1+\varphi} \sum_i \sin\left(\frac{x_i q \pi}{\tau_0 F} t + \alpha_i t + \psi_i - 2\omega t\right)(1+x_i)$$
where the right hand side is the sum of harmonics with the basic difference Ω .

We use the last relation for determination of the stochasticity condition. The motion of a particle will be stochastically unstable if the beat interaction of two waves causes such maximum change in the action ΔJ that a simultaneous change of ω makes possible the particle transition to other resonant beats. The following inequality must hold

$$\left(\frac{d\omega}{dJ} \ \frac{\Delta J}{\Omega}\right)^{1/2} \gg 1 \qquad (7)$$

(due to the resonant interaction of one resonant combination the particle goes through the whole series of other resonances). As follows from the above exposition the frequency ω is given by

$$\omega = \frac{\pi q}{\tau_0 F}$$

and it is - apart of intensity of the leading wave - also the function of particle energy. The velocity of a particle and thus also the frequency ω vary during the interaction. The intensity of the leading wave appears in ω already in the zeroth approximation.

The analogy with the nonlinear oscillator in the field of periodical δ-pulses requires that the effect of the spectrum on the particles can really be approximated in such form (apparently, the requirement is not necessary for the existence of stochasticity; however, the model was solved only for this case).

As it is shown in (Zaslavskij G.M., 1970) (and as it follows simply from the properties of the Fourier series) the spectrum of N discrete lines in the distance Ω , with the constant amplitude and in the limit $N \to \infty$ in the time picture is equivalent to repeated δ-pulses with the period $T = \frac{2\pi}{\Omega}$. For finite N the width of particular pulses is $\Delta t \sim (N\Omega)^{-1}$. In our case the inequality

$$N\Omega \gg \omega_{HK}$$

must hold, where ω_{HK} is the circular frequency of the nonlinear oscillator. As we have seen, the frequency ω_{HK} of the unperturbed system (particle and leading wave) is, in the coordinate system of the wave, $\omega = \frac{\pi q}{\tau_0 F}$. Let us give several typical parameters: For $\omega_1 = 9{,}4{,}10^9\,\mathrm{sec}^{-1}$, $\frac{2\pi}{k_1} = 2{,}2.10^{-2}\,\mathrm{m}$ and $N_T \sim 10\,eV$ with $E_1 \sim 10^4\,\mathrm{V/m}$ the corresponding ω is $\omega = 9{,}3.10^9 \sec^{-1}$; for $E_1 \sim 10^5$ N/m we get $\omega = 9{,}1.10^9 \sec^{-1}$. In our range of parameters we thus require approximately

$$2N\Omega >> \omega_1 \ .$$

This is, of course, rather strong requirement; because $N\Omega \sim \Delta\omega_{eff}$ ($\Delta\omega_{eff}$ is the effective width of the spectrum) we then necessarily require $2\Delta\omega_{eff} >> \omega_1$. The requirement will be fulfilled more easily for $k_1 v_T \geq \frac{1}{2}\omega_1$. However, here already the effect of absorption of one wave (Dawson J.M., Shanny R., 1968) plays a role being more intensive than the beat effect.

A case when the first or second wave appears in very short regular pulses is nearer to the requirement. Nevertheless it seems that more attainable is the case when the velocity of particle is modulated; this happens e.g. for the motion of particles in the magnetic mirror system (from another point of view discussed in (Jaeger F. et al., 1972)).

Let us suppose we can in similar way approach the δ-form of interaction (the process can be at least quasiergodic). The sequence of δ-pulses exactly corresponds to the equation (6) with $\frac{1}{\omega\tau_0^2}\ \frac{4\pi}{F}\ \frac{\overset{*}{q}^2}{1+\overset{*}{q}^4}$ = const., $\psi_i = \psi$ = const. The first condition could be satisfied in a narrow spectrum band; some difficulties arrow due to the second condition. Because ψ_i are rather accidental (if one does not know the generation law of spectrum) and result rather in some modulation of the sequence of δ-pulses, some simulation of this effect is necessary. We took the extreme case with the form $\sim(-1)^m . \delta(t-mT)$. Because we need for the determination of $\mathcal{K}$ only maximal change of ΔJ, causes this change no difficulties. The periodicity only changes in the following the expression $\langle\Delta J^2\rangle$ in the diffusion coefficient.

Then we can very easily determine the change ΔJ

$$\Delta J = \frac{1}{\omega\tau_0^2}\cdot\frac{4\pi}{F}\cdot\frac{\overset{*}{q}^2}{1+\overset{*}{q}^4}\cdot\frac{\pi}{\Omega}\cdot\varepsilon \ .$$

From a simple calculation it follows that the change of ω is given by

$$\frac{d\omega}{dJ} = \frac{\pi^2}{4}\cdot\frac{E}{F^3}\cdot\frac{q^2}{q^2-1}\cdot\frac{1}{\tau_0} \ .$$

Then, according definition of $\mathcal{K}$ we have

$$\mathcal{K} = \frac{d\omega}{dJ}\,\frac{\Delta J}{\Omega} = 4\pi^4\,\frac{E}{F^4}\cdot\frac{q^2\cdot\overset{*}{q}^2}{\overset{*}{q}^4-1}\cdot\frac{1}{\omega\tau_0^3}\cdot\frac{\varepsilon}{\Omega^2}\cdot\frac{1}{q^2-1} \ .$$

In this way also the basic parameter of the mixing process - the time of correlation decay - is determined

$$\tau_r = 2.\,(\mathcal{K}.\Omega.\lg\mathcal{K})^{-1}; \qquad \mathcal{K} >> 1 \ .$$

Now, consider the spectrum of the type b , defined as follows. Let this spectrum has a rectangle envelope with constant amplitude of all lines and let the distance between neighbouring spectral lines is Ω . For the solution we use the second perturbation model - motion of a particle within two waves of the same amplitude.

Still before determination of τ_{κ} let us mention the interaction mechanism of this model. We use the analogy with the mixing process at the interaction of three waves (Zaslavskij G.M., 1970). Let the Hamiltonian of individual waves be expressed in the action - angle coordinates and let their frequency be dependent on amplitude. Suppose further

$$\left(\frac{d\omega_k}{dJ_k}\frac{\Delta J_k}{\Omega}\right)^{\frac{1}{2}} \gg 1 \; ; \qquad N\Omega \gg \omega_k \; .$$

ω_k is the frequency of the k -th nonlinear oscillator, N is the number of simultaneously interacting groups of waves and ΔJ_k is the maximum variation of action of the k -th wave due to the resonant action of all waves from the spectrum satisfying the resonant condition $\omega_k = \omega_m + \omega_n$; $k_k = k_m + k_n$. We can formally describe the interaction by the Hamiltonian

$$H = \sum \omega_k \left(J_\ell\right) J_k + V_{int} . \qquad (8)$$

V_{int} gives the energy of interaction between modes. Then, the system of three waves interaction forms mixing system and the distribution function of waves satisfy the Fokker--Planck equation and has thus diffusion character.

We need to express the total Hamiltonian of particles and waves as a system of bounded nonlinear oscillators in the form (8). Finding analogous resonant conditions, the model of mixing for three - wave process can be used for interaction of particles and waves for spectrum b .

We shall in Chapt. 7. discuss development of the whole wave-particle system; now we content with only giving a condition when the particle motion in a given system of waves is stochastic. Therefore, we need to follow only the change of velocity of particles.

As it is known, just the nonlinear Landau damping is often considered as a three--wave process (with a virtual third wave resonantly interacting with particles). Thus, let us describe the particle in the coordinates action-angle in accordance with Chapt. 4. with the Hamiltonian in the coordinate system of one waves and normalised to the amplitude of the spectrum (according (5))

$$H = \frac{1}{2m} k_1^2 J^2 - \sum_i \sum_k \left[e\varphi_{i0} \cos w_{1i} - e\varphi_{k0} \cos\left(\varkappa_k w_k + \alpha_k t + \varphi_k\right)\right] \qquad (9)$$

and with the frequency substantially nonlinear

$$\omega = \frac{1}{m} k_1^2 J \; .$$

Let our spectrum is discrete in such a way that its adjoining lines satisfy

$$v_T = \frac{\omega_2-\omega_1}{k_2-k_1}$$

where v_T is the thermal velocity of particles. In a given spectrum let us look for the maximum change of ΔJ of the particle with resonant thermal velocity. As long as

$$N\Omega >> \omega_{osc}\,; \qquad \frac{d\omega}{dJ}\,\frac{\Delta J}{\Omega} >> 1$$

where Ω is the corresponding discreteness and N the number of simultaneously resonant interacting pairs of waves with a given particle, the particle motion can be considered to be stochastic.

The following equality generaly holds

$$\Omega_i = 2\omega\left(x_{i+1}-x_i\right)+k_{i+1}\,\alpha_{i+1}-k_i\alpha_i \quad .$$

In our case, the angular frequency of a nonlinear oscillator can be expressed as

$$\omega_{osc} \rightarrow v_T\left(k_2-k_1\right)$$

and the inequality

$$N\left[2\omega\left(x_{i+1}-x_i\right)+k_{i+1}\,\alpha_{i+1}-k_i\alpha_i\right] >> v_T\left(k_{i+1}-k_i\right)$$

must be satisfied. This is fulfilled in the system with particles with sufficiently small v_T and with sufficiently wide spectrum. Satisfying the above conditions, ΔJ could be simply determined. Using Hamiltonian (8), considering beat resonances and their harmonics and supposing

$$x_i \sim 1\,; \quad \frac{\omega_i}{k_i} \sim \frac{\omega_1}{k_1} >> v_T\,; \quad \varphi_{io} = \varphi_{ko}\,; \quad \beta = \frac{1}{m}k_i^2\,\frac{e\varphi_{io}}{\omega_1^2}$$

we get

$$\frac{dJ}{dt} \sim -e\,\varphi_{10}\,J_1(\beta)\sum_n\sum_m \sin\left(n\Omega+\psi_{nm}\right).$$

Here the first sum describes the fact that particles see besides the beat resonance for basic distance of two lines Ω further sequence of harmonics $n\Omega$. The second sum expresses that in the basic beat resonance could act not only one couple of waves but a large amount of corresponding resonant combinations (approximately also N).
We can take rather uniform distribution of phases ψ_{nm_i}. Using again the same consideration as for the spectrum a , expessing

$$\psi_{nm} = \psi_n + \psi_m\,; \quad \sum_{n=0,\,\psi_n=0}^{n=\frac{N}{2},\,\psi_N=\frac{\pi}{2}} \sin\psi_n \sim \frac{N}{\pi}$$

and considering $J_1(\beta_i)\,x_i \sim J_1(\beta)\,x$ we have

$$\frac{dJ}{dt} \sim -2e\varphi_{10}\,J_1(\beta)\,\frac{N}{\pi}\sum(-1)^n\delta\left(t-n\,\Omega^{-1}\right)\cdot\frac{\pi}{\Omega}$$

$$\left|\Delta J\right|_{max} \sim 2e\,\varphi_{10}\,J_1(\beta)\,\frac{N}{\Omega}\ .$$

The corresponding value of $\mathcal{K}$ is thus

$$\mathcal{K} \doteq \frac{\omega_2^{4}}{\omega_1^{2}\Omega^{2}} N$$

and τ_r^{-1}

$$\tau_r^{-1} = \frac{1}{2} N \Omega \lg \mathcal{K} ; \qquad \mathcal{K} >> 1 .$$

The distinction from the model of spectrum a is not therefore only formal. In the spectrum a the basic line forms a strong nonlinearity; the side spectrum lines contribute with their resonances. The conditions of the model require rather wide spectrum; however, for the present one cannot say that this is the necessary condition for existence of stochasticity.

In the model b the unperturbed velocity $v^{(0)}$ is taken as a motion with possible nonlinear features; by increasing velocity the particle can go over to another resonant beat. From the point of view of realisation, the model b is, of course, more accetable. Of course, for higher accuracy of approximation both models approach each other.

6. ESTIMATION OF ABSORPTION RATE

In Chapt. 5 we have derived the time of decay of the correlation for two kinds of discrete spectrum. Let us now have a look at changes of energy of particles. In this Chapt. we suppose the spectrum to be stationary with constant energy. Thus, also from the energetical point of view the problem is not self-consistent. In Chapt. 7 we discuss consequencies of this approximation and some more general aspects of this form of interaction.

As follows from the analysis of mixing systems, the probability of appearance of a particle with velocity v (or the distribution function $f(v,t)$) can be described by the diffusion equation (Zaslavskij G.M., 1970)

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left[D(v)\frac{\partial f}{\partial v}\right] .$$

For simplicity we denote the parameters subject to diffusion by v ; in general this can be any coordinate of the phase space. The diffusion coefficient is defined (as in the case of stochastic acceleration)

$$D(v) = \frac{1}{\tau} \left| (\Delta v)^2 \right|$$

where Δv is the increasing of v during two subsequent "collisions" and τ is the mean "collision" period. In our case we take $\tau = \frac{2\pi}{\Omega}$ (here also $\tau_r \ll \tau$) and Δv as the increase of v during the δ -pulse. As long as the spectrum can be considered to be dense so that particles practically in every state can finde a corresponding resonant interaction, the resonant diffusion coefficient can be used (Krlín L., 1969)

$$D(v) = \frac{1}{2}\,\Delta v_{max}^2 \left(\frac{2\pi}{\Omega}\right)^{-1}$$

where Δv_{max} is the maximum change of v in the time $\delta(t - mT)$.

In the previous discussion we have studied the dynamics of a particle with the action variable J ; therefore we need the inverse transformation $J \rightarrow v$.
In sec. 5 we derived ΔJ for passing a particle through one δ -pulse hence, the increase in time $T = \frac{2\pi}{\Omega}$. Denoting a corresponding change of the coordinate ΔJ we have for D_J

$$D_J = \frac{1}{2}\,\Delta J^2\,\frac{\Omega}{2\pi} \quad .$$

Since for the spectrum of the type a we have

$$\Delta v^2 = \frac{\pi^2}{8}\cdot\frac{\Delta J^2}{E^2 k_1^2}\,\omega_B^2$$

and for the type b

$$\Delta v^2 = v_\phi^{-2}\,\Delta J^2 \frac{\omega_1^2}{m^2}\,; \qquad v_\phi \doteq \frac{\omega_1}{k_1}$$

considering the results of Chapt 6 we have diffusion coefficient for the spectrum a

$$D_v^{(a)} = \frac{\pi^3}{16}\,\frac{1}{E^2F^2}\,\frac{1}{\Omega}\,e^2\,\frac{q^4}{(1+q)^2}\cdot\frac{\omega_B^6}{k_1^2\omega_1^2}\,; \qquad \omega \rightarrow \omega_1 \;.$$

Similarly for the spectrum of the type b

$$D_v^{(b)} = \sim \frac{1}{4\pi}\,\frac{N^2}{k_1^2}\,\frac{\omega_B^8}{\omega_1^4\Omega}\,. \qquad \omega_i \rightarrow \omega_1\,, \quad k_i \rightarrow k_1\,, \quad x_i \rightarrow 1.$$

Because of the diffusion form of the equation the change of the longitudinal energy $\frac{dQ}{dt}$ is given by

$$\frac{dQ}{dt} = \int \frac{1}{2} m v^2 \frac{\partial}{\partial v}\, D_v \frac{\partial f}{\partial v}\, dv \,.$$

If we consider D_v not depending on energy (this, of course, is approximation for D_v both ω and q are functions of some average velocity) we have, under usual convergence assumptions

$$\lim_{v\rightarrow\infty} v^2 \frac{\partial f}{\partial v} \rightarrow 0\,; \qquad \lim_{v\rightarrow\infty} v f \rightarrow 0$$

for the velocity of absorption the following simple relation

$$\frac{dQ}{dt} = D_v\, n.m$$

where n is the particle density. The total increase in the same approximation is

$$\Delta Q = D_v\, n\, m\, \Delta t$$

(validity of D = const for all v requires an infinitely wide spectrum; approximately it is sufficient if $D \sim const$ will cover the assumed mean increase of velocities).

For the case of narrow spectrum band and greater thermal velocity one has consider that the nonlinear resonances cover a limited velocity interval of distribution function; inside this interval has the motion of particles diffusion character. According (Zaslavskij G.M., 1970) the particles are in this interval[1] rather closed (this supposition consists in the fundamental difference of the character of the particle motion inside and outside of stochastic layer). If further $D(v)$ is constant in this interval the consequence of diffusion is the plateau-like form of the distribution function. The limit amount of absorbed energy (for stationary spectrum) is then

$$\Delta Q = \int \frac{1}{2} m v^2 \left[f(v, t_0) - f(v, t \to \infty) \right] dv \sim m v_f \left. \frac{\partial f}{\partial v} \right|_{v = v_f} \cdot \Delta v^3 .$$

As a concrete case of the above mentioned analysis we now consider a possibility of absorption of the discrete spectrum of waves in the range of parameters close to experiment (Piffl V. et al., 1971). A direct comparison of results with experiments could be done, nevertheless, only carefully; as we have already mentioned, we have not sufficient experimental picture of the spectrum; parameters of our models are therefore chosen rather ad hoc - especially Ω/ω_1. Further difficulty arises from our not-selfconsistent scheme. We are therefore unable to judge to what degree the usually considered integral properties of nonlinear Landau damping are changed by means of discussed mechanism; some possible changes are mentioned in Chapt. 7. Thus, we must therefore consider that particle heating could be really covered from the spectrum energy due to these effects. If not so, we can without fail only state the change of the behaviour of resonant particles due to discussed mechanism; our numerical results are then only valid as an estimation of the rate at the beginning of the interaction.

As we have seen from the discussion of the stochasticity limit, from the point of view of calculation effort it is more advantageous to use the spectrum b ; here, for sufficiently small v_T the stochasticity condition is satisfied still for not too wide spectrum. We take therefore the diffusion coefficient $D(v)^{(4)}$ and $\mathcal{K}_b$ in the form

$$D_v^{(4)} \sim \frac{1}{4\pi} \frac{N^2}{k_1^2} \frac{\omega_2^8}{\omega_1^4 \Omega} \quad ; \qquad \mathcal{K}_v^{(4)} = \mathcal{K} = \frac{\omega_2^4}{\omega_1^2 \Omega^2} N$$

Let the original velocity distribution of plasma particles be δ-functional and let the spectrum be sufficiently broad so that for supposed mean thermal spread due to absorption the diffusion coefficient can be taken as a constant. Then, the increase of particles energy in time Δt is simply given by

$$\Delta Q = D_v^{(4)} n . m . \Delta t .$$

1) (interval of stochastic instability)

Now, let us choose, according (Piffl et al. 1971; Jungwirth and Šunka 1973) following parameters: $n \sim 2.10^{16}$ particles/m^3, $\omega_1 = 9{,}6.10^9$ sec^{-1}, $k_1' = 2{,}8.10^2$ m^{-1}; choose $N = 10^2$. In the table the results of calculation of temperature increasing ΔT are given for several groups of parameters $E, N, \frac{\Omega}{\omega_1}$. W_{tot} is equal to the total energy density of the wave spectrum. W_{acc} is an accessible energy density and is defined as $W_{acc} = W_{tot} - W_{k=1}$; $W_{k=1}$ corresponds to the stochastic instability limit $\mathcal{K} = 1$. Δt is here defined 1) as a time Δt_1 of absorption of the energy W_{acc} if D could be considered as a constant, and 2) as a time Δt_2 of absorption of the half of the accessible energy, if waves and resonant particles form conservative system. Both $\Delta t_{1,2}$ give, of course, only rough estimation of healing rate. ΔT is accessible increasing of temperature for Δt_1 .

E V/m	N	Ω/ω_1	W_{tot} J/m^3	K	W_{acc} J/m^3	Δt_1/μsec	Δt_2/μsec	ΔT eV
1.10^4	10^2	10^{-3}	5.10^{-2}	$1{,}3.10^4$	$\sim W_{tot}$	$1{,}5.10^{-9}$	$2{,}3.10^{-9}$	15
$6{,}6.10^3$	10^2	10^{-3}	2.10^{-2}	$5{,}1.10^3$	$\sim W_{tot}$	$8{,}5.10^{-9}$	$1{,}3.10^{-8}$	6
$3{,}3.10^3$	10^2	10^{-3}	6.10^{-3}	$1{,}4.10^3$	$\sim W_{tot}$	$1{,}3.10^{-7}$	$1{,}9.10^{-7}$	1,8
1.10^3	10^2	10^{-3}	5.10^{-4}	$1{,}3.10^2$	$\sim W_{tot}$	$1{,}5.10^{-5}$	$2{,}3.10^{-5}$	0,15

In mentioned experiments the interaction time is 10^{-6} sec. Initial temperature of plasma was 6 eV and estimated increasing of temperature in the range of several eV. The maximum amplitude of single wave was cca 10^4 V/m and the width of the spectrum after a breaking of single wave $\frac{\Delta f}{f}$ up to 10%. The spectrum covers with beat resonances the range of several tens of eV of plasma thermal velocity. The estimation of spectrum energy is difficult. Let us suppose that the spectrum energy approximately equals to the maximum energy of the single wave; then this corresponds to the density of energy $W \sim 5.10^{-4}$ J/m^3. We see that for this total energy the interaction is unsufficiently slow and must be in mentioned experiment as a negligible one covered by some stronger mechanism. As was already mentioned, imperfection of our model consists in rather artificial choice of N, Ω and can be overcome by better knowledge of spectrum genealogy. Our formulae can give only following more general conclusion. If we take as experimental values W_{tot} and spectrum width $\Delta\omega \sim N\Omega$, then for given W_{tot} and $\Delta\omega$ more subtile structure of spectrum ℓ gives greater K and D.

7. CONSEQUENCES OF STOCHASTIC INSTABILITIES FOR ENERGY TRANSFER IN THE REGIME OF NONLINEAR LANDAU DAMPING

Till now we have discussed a case of stationary spectrum. Such not selfconsistent model can generally describe only the character of particle motion, but not changes in energy. It is well known (e.g. Ott and Dum 1971) that for two waves and particles with resonant interaction velocity $(\omega_2 - \omega_1) = v(k_2 - k_1)$ integral exists (U_i is wave energy)

$$\frac{U_1}{\omega_1} + \frac{U_2}{\omega_2} = const.$$

For $|\omega_2 - \omega_1| << \omega_1$ the main effect of interaction consists therefore in energy transfer between two waves; an analogy is valid for spectrum (Tsytovitch 1967).

There are some possibilities of overcoming this severe restriction. The first consists in non-conservativity of the system. This can happen if the waves with some amplitude level are still feeded by means of external driver (e.g. in beam-plasma interaction, if the waves are nonlineary limited and where - in a final volume - is the energy still supplied by the beam).

The second possibility appears in disturbation of the resonant condition (if the particle interacts with more than two waves). This could be caused by collisions and in our case by τ_{κ}^{-1} . This effective collisions frequency corresponds to the decay of particle-phase correlation, caused by nonlineary effects, and is a complicated function of the total spectrum region. It is therefore not possible to discuss the dynamics of two waves, forming beat, in such isolated form, as, e.g. in (Ott and Dum, 1971) description. Such complicated nonlinear problem extends farther the aim of our paper. Notwithstanding, closely related is the next approach, giving some general informations.

The further - and interesting - possibility exists, if we can the whole system (waves and resonant particles) regard as a system with mixing and if individual waves from the spectrum could be formally represented by means of nonlinear oscillators. Let us express the total Hamiltonian in action-angles in form

$$H_{tot} = \sum \omega_{Fi}(J_{Fk}) J_{Fi} + \sum H_p(J_p) + V_{int\,F}(w_F, J_F, w_p, J_p) + V_{int\,p}(w_F, J_p, w_p, J_p)$$

where two first sums at the right side represent energy of waves, resp. particles and two last terms give the interaction energy between waves and between waves and particles. If

$$\frac{\partial \omega_F}{\partial J_F} \frac{\Delta J_F}{\Omega} >> 1 ; \qquad \frac{\partial \omega_p}{\partial J_p} \frac{\Delta J_p}{\Omega} >> 1$$

is valid, where ΔJ_F , resp. ΔJ_p is the maximum change of J_F , resp. J_p due to the resonant interaction, the whole system is possible regare as system with mixing; such system can overgo into the Rayleigh-Jeans distribution in energies.[x] One third of wave energy can then overgo into particles.

We believe that a combination of these effects is nearest to the experiment (probably the first case is most important); then we can consider our numerical estimation as a good one at least for the beginning of the interaction.

x

see also Kaufman A., Phys. Rev. Lett. 27, 376 (1971)

8. CONCLUSION

We have discussed the effect of beat resonances on the absorption of wave energy in particles (the mechanism of nonlinear Landau damping) in the stochastic instability regime. We have shown that in suitable chosen spectrum the motion of particles has - due to the stochastic instability - diffusion character. Supposing stationary spectrum this results in monotonous increasing of particle energy. The energetic considerations in the selfconsistent case are more difficult; we have discussed the effect of effective collisions and of mixing of the whole system (wave and resonant particles) on the energetic balance. As an application we have estimated the absorption by means of discussed mechanism for beam-plasma experiment. Our estimation for parameters from experiment give lower heating rate then it results from measurement, nevertheless it premises for larger amplitude spectrum more intensive absorption. In any case the mechanism under consideration requires - - due to used models and approximations - more detailed study and the generalization for the self - consistent case.

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Státní tiskárna, n. p., závod 6, Praha 1