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PENETRATION OF AN ORDINARY WAVE INTO A WEAKLY INHOMOGENEOUS MAGNETOPLASMA AT OBLIQUE INCIDENCE

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ABSTRACT

A theory is given describing the propagation of high-frequency electromagnetic waves in a plane-stratified weakly inhomogeneous plasma. The density gradient is supposed to be perpendicular to the external magnetic field and the wave vector is expected not to be generally parallel to the plane given by both the preceding vectors. The analysis points out that the ordinary wave can penetrate through the plasma resonance region if the direction of vacuum wave vector is chosen appropriately. Analytical expressions for the reflexion and transmission coefficients are obtained and their dependence on the direction cosines of the wave vector of the incident wave is studied. The paper further shows in outline that, after transmission through the plasma resonance, the ordinary wave is transformed into an extraordinary wave and the latter is reflected back to the region of the hybrid resonance. In this region the extraordinary wave is fully transformed into the Bernstein modes.

I. INTRODUCTION

When we want to heat a plasma effectively by high-frequency waves we encounter with great obstacles at the transfer of high-frequency energy from vacuum to a plasma. To solve this problem we may utilise a well known effect of the linear transformation of waves. In this case an electromagnetic wave penetrating into a plasma must get over an evanescent layer before it reaches a place in which it is transformed into an electrostatic wave. Owing to a damping incurred in tunnelling such a layer the transformation coefficient can be of the order of unity only in case that a wave-length of an incident wave is comparable with the width of the evanescent layer. In a number of experimental devices the characteristic parameters, such as a plasma density or an intensity of the magnetic field, reach, however, high values. It is therefore necessary for the heating of electrons to apply waves with wave-lengths much shorter than the dimensions of devices. Such a situation is typical for example for the Tokamak-devices. It turns out, however, that the appropriately polarized waves obliquely incident on a plasma get over only a narrow evanescent layer before they penetrate in the region of the linear transformation.

The linear transformation of waves as well as the wave propagation in an inhomogeneous plasma have already been treated in many papers, most of them quoted in [2]. Among many problems studied so far little attention was paid until lately to the penetration and transformation of waves obliquely incident on an inhomogeneous magnetized plasma. The authors of paper [1] dealt with the special case of oblique incidence and supposed that the plane of incidence is parallel to the plane given by the density gradient and by the external magnetic field. In that paper it was shown that the ordinary wave having the appropriate angle of incidence can be transmitted through region of the plasma resonance. A survey of the wave propagation in an inhomogeneous plasma at oblique incidence was given there and the original references were mentioned,too. The oblique incidence of a long wave length extraordinary wave on a plasma with steep density gradient has been discussed in [5]. It has been shown there that the transformation coefficient of this wave into the Bernstein mode is small in this case.

In our paper we study the propagation of high-frequency waves of a small amplitude obliquely incident on a collisionless magnetized plasma. The density gradient in supposed to be perpendicular to the magnetic field. We confine ourselves to the waves with frequencies ω greated than the electron cyclotron frequency ω_c and we assume that the vacuum wave--length is essentially smaller than the characteristic length of the density inhomogeneity $\varkappa^{-1} (\varkappa = d \ln n / dy)$. According to the first assumption we neglect the ion motion, according to the second one we solve the problem using the WKB method. The penetration of the ordinary wave into a dense plasma is treated in the cold plasma approximation. The transformation of the ordinary wave to an extraordinary one and the transformation of this wave to the Bernstein mode can be investigated in a similar way as it was done in [1]. Thus, we discuss this problems only briefly at the end of our paper.

II. FUNDAMENTAL EQUATIONS

In this chapter we collect the equations and the boundary conditions governing the electric field of waves in a plane-stratified magnetised plasma. We assume the plasma to be inhomogeneous along the y-direction and the uniform static magnetic field to be directed along the z-axis. Let the electromagnetic wave of the form

$$(1)\quad \vec{E} = \vec{\mathcal{E}}_0 \exp\left(-i\omega t + ik_x x + ik_z z + i(k_v^2 - k_x^2 - k_z^2)^{1/2} y\right)$$

be incident from vacuum on the plasma; k_x and k_z are the x and y component of the wave vector, respectively; $k_v = \frac{\omega}{c}$. $\vec{\mathcal{E}}_0$ is to be chosen in accordance with the condition $\operatorname{div} \vec{E} = 0$. The electric field within the plasma can then be supposed to have the form

$$(2)\quad \vec{E} = \vec{E}_{k_x, k_z, \omega}(y) \exp\left(-i\omega t + ik_x x + ik_z z\right).$$

Using the Maxwell equations and the linearized equations of cold magnetohydrodynamics we obtain for the vector $\vec{E}_{k_x, k_z, \omega}(y)$

$$(3)\quad \begin{aligned} &\frac{d^2 E_x}{d\bar{y}^2} + (\varepsilon_1 - N_z^2)E_x - iN_x \frac{dE_y}{d\bar{y}} + igE_y + N_x N_z E_z = 0, \\ &iN_x \frac{dE_x}{d\bar{y}} + igE_x - (\varepsilon_1 - N_x^2 - N_z^2)E_y + iN_z \frac{dE_z}{d\bar{y}} = 0, \end{aligned}$$

$$iN_xN_zE_x - iN_z\frac{dE_y}{d\bar{y}} + \frac{d^2E_z}{d\bar{y}^2} + (\varepsilon_\parallel - N_x^2)E_z = 0 \ ,$$

where $\varepsilon_\parallel = 1 - \frac{\omega_0^2(y)}{\omega^2}$, $\varepsilon_\perp = 1 - \frac{\omega_0^2(y)}{\omega^2 - \omega_c^2}$,

$$g = \frac{\omega_0^2(y)\,\omega_c}{\omega(\omega^2 - \omega_c^2)} \ ,$$

ω_0 is the plasma frequency, $\bar{y} = k_y y$, $N_x = k_x/k_y$ and $N_z = k_z/k_y$. To make notation simpler, we have omitted the indices k_x , k_z and ω in the components of the electric field.

The conservation law of the energy flow can be derived from the system of eq. (3) and its complex conjugate

$$E_x^*\frac{dE_x}{d\bar{y}} - E_x\frac{dE_x^*}{d\bar{y}} + E_z^*\frac{dE_z}{d\bar{y}} - E_z\frac{dE_z^*}{d\bar{y}} -$$

(4)

$$- iN_x(E_x^*E_y + E_xE_y^*) - iN_z(E_y^*E_z + E_yE_z^*) = const$$

The expression on the left-hand side of (4) is proportional to the component of the energy flow which is parallel to the density gradient.

III. THE WKB SOLUTIONS AND THE PROPAGATION OF WAVES

The derivation of the solution of the system (3) is very complicated in case of the general dependence of the density on y. The problem is considerably simplified if the vacuum wave-length of incident waves is small compared with the characteristic length of inhomogeneity i.e. $\varkappa/k_y \ll 1$. Then the WKB approximation can be used and a solution can be found in the form

$$\vec{E} = (\vec{E}_{(0)}(y) + \vec{E}_{(1)}(y))\, e^{i\int^{y} k_y(y')dy'} \qquad (5)$$

$\vec{E}_{(0)}$ and k_y are the zero order quantities in the parameter $\varkappa/k_y$ and $\vec{E}_{(1)}$ is the first order quantity in the same parameter. All the three quantities are assumed to be slowly varying functions of y. Using (5) we obtain from (3) the equations for the zero and first order WKB approximations (see also [3])

$$a_{\kappa q}\, E_{(0)q} = 0, \quad \kappa, q = 1,2,3, \qquad (6)$$

$$a_{\kappa q}\, E_{(1)q} = -G_\kappa, \quad 1 \leftrightarrow x,\ 2 \leftrightarrow y,\ 3 \leftrightarrow z, \qquad (7)$$

where the coefficients $a_{\kappa q}$ have the form

$$a_{11} = -N_y^2 - N_z^2 + \varepsilon_\perp \ ; \quad a_{12} = N_x N_y + ig \ ; \quad a_{13} = a_{31} = N_x N_z \ ;$$

$$a_{21} = -N_x N_y + ig \ ; \quad a_{22} = N_x^2 + N_z^2 - \varepsilon_\perp \ ; \quad a_{23} = -a_{32} = -N_y N_z \ ;$$

$$a_{33} = \varepsilon_{\parallel} - N_x^2 - N_y^2 \ ; \quad N_y = \frac{k_y(y)}{k_0} .$$

The absolute terms G_κ are given by

$$G_1 = 2iN_y \frac{d}{d\bar{y}} E_{(0)x} + i\frac{dN_y}{d\bar{y}} E_{(0)x} - iN_x \frac{dE_{(0)y}}{d\bar{y}} ,$$

$$G_2 = iN_x \frac{dE_{(0)x}}{d\bar{y}} + iN_z \frac{dE_{(0)z}}{d\bar{y}} ,$$

$$G_3 = 2iN_y \frac{dE_{(0)z}}{d\bar{y}} + i\,\frac{dN_y}{d\bar{y}}\,E_{(0)z} - iN_z \frac{dE_{(0)y}}{d\bar{y}} \ .$$

The system (6) has a solution different from the trivial solution if its determinant A

$$A \equiv |a_{kq}| = 0 \tag{8}$$

From (8) we can obtain for the dimensionless wave vector

$$N^2_{y1,2} = \frac{1}{2\varepsilon_\perp}\Big\{(\varepsilon_\perp - N_z^2)(\varepsilon_\parallel + \varepsilon_\perp) - g^2 - 2N_x^2\varepsilon_\perp \pm$$

$$\pm\Big[\Big((\varepsilon_\perp - N_z^2)(\varepsilon_\perp + \varepsilon_\parallel) - g^2 - 2N_x^2\varepsilon_\perp\Big)^2 - 4\varepsilon_\perp\Big(\varepsilon_\parallel\big((\varepsilon_\perp - N_z^2)^2 - g^2\big) - N_x^2\big((\varepsilon_\perp + \varepsilon_\parallel)(\varepsilon_\perp - N_z^2) - g^2 - N_x^2\varepsilon_\perp\big)\Big)\Big]^{1/2}\Big\}. \tag{9}$$

Introducing dimensionless quantities $p = \omega_0^2/\omega_c^2$ and $\alpha = \omega/\omega_c$ the expression (9) may be rewritten as

$$N^2_{y1,2} = -N_x^2 + \frac{1}{2\alpha^2(\alpha^2 - 1 - p)}\Big\{2p^2 - p\big[4\alpha^2 - 1 - N_z^2(2\alpha^2 - 1)\big]$$
$$+ 2\alpha^2(\alpha^2 - 1)(1 - N_z^2) \pm p\big[(1 - N_z^2)^2 + 4N_z^2\,(\alpha^2 - p)\big]^{1/2}\Big\}. \tag{10}$$

In (9) or (10) the indeces 1 and 2 correspond to the ordinary and extraordinary wave, respectively. The second term on the right-hand side of (10) representing a square of the y-component of dimensioneless wave vector at $N_x = 0$ has been studied in [1]. Thus, to plot the dependence of N_y^2 on the dimensionless density p , it is sufficient to carry out a translation of the point 0 on the ordinate axis by an amount N_x^2 in figure 1 of that paper.

If the dispersion equation (8) is fulfilled we can express the y-and z-components of the electric field as functions of $E_{(0)x}$

(11) $$E_{(0)y} = \frac{-M_{32}}{M_{31}} E_{(0)x} , \qquad E_{(0)z} = \frac{M_{23}}{M_{31}} E_{(0)x} ,$$

where M_{kq} are minors of the system determinant A . The dependence of $E_{(0)x}$ on y can be determined from the solvability condition of the system (7)

(12) $$G_1 M_{13} - G_2 M_{23} + G_3 M_{33} = 0$$

Using (11) we obtain from (12) a linear first order differential equation for $E_{(0)x}$. The solution of this equation is given in the appendix.

Making use of (A5) we can write the general WKB solution of the system (3) in the form

(13) $$E_x = \sum_{k=1,2} \left(\frac{M_{11}}{N_{yk}\,\varepsilon_1 (N_{y1}^2 - N_{y2}^2)}\right)^{1/2} \cdot \left(C_{+,k}\, e^{-\frac{iN_x}{2}\mathcal{F}_k + i\int^y k_{yk}\,dy'} + C_{-,k}\, e^{+\frac{iN_x}{2}\mathcal{F}_k - i\int^y k_{yk}\,dy'} \right).$$

Let us notice one interesting feature of the solution (13). We can see that the phase of wave in addition to the usual quickly varying term $\int k_y\,dy'$ contains the slowly varying term $\frac{N_x}{2}\mathcal{F}$, too. Such a term is just typical of the general oblique incidence and is absent in the majority of cases solved till now. This is also clear from the deep-rooted term for $E_{(0)x}$, namely the amplitude. It may be shown that the slowly-varying phase appears in the WKB solutions only when the elipse of polarisation is twisting in its own plane as a result of the wave translation along its ray in an inhomogeneous plasma.

The WKB solutions cannot be used in the neighbourhood of the points where $N_{yi} = 0$, $N_{yi} = \infty$ or $N_{y1} = N_{y2}$. Considering that the reflexion or transformation of waves can occur only in these points we must specify their location in the dependence on the plasma density. It is only the wave vector of the extraordinary wave that is infinite, viz., in the hybride resonance

$$p = \alpha^2 - 1 \tag{14}$$

The extraordinary and ordinary wave branches intersect in the points $p = 0$ and

$$p = \frac{(1-N_z^2)^2 + 4\alpha^2 N_z^2}{4N_z^2} \tag{15}$$

Nothing but the location of the points where $N_{yi} = 0$ depends on N_x . From (9) we obtain for these points .

$$\varepsilon_{\parallel}\left((\varepsilon_{\perp}-N_z^2)^2-g^2\right)=N_x^2\left((\varepsilon_{\perp}+\varepsilon_{\parallel})(\varepsilon_{\parallel}-N_z^2)-g^2 N_x^2\varepsilon_{\perp}\right) \tag{16}$$

The equation (16) is an algebraic equation of degree 3 in p and it may be solved only numerically. The analysis of this equation and some solutions in special cases can be found in [4]. For our purpose it is, however, sufficient to determine the roots of (16) for small values of N_x^2 ($N_x^2 \ll 1$). Then, the smallest root corresponding to the extraordinary wave is given approximately by

$$(17) \qquad p_1 = \alpha(\alpha-1)(1-N_z^2) - \frac{N_x^2}{2}\alpha(\alpha-1)\left(1+\frac{\alpha N_z^2}{1+(\alpha-1)N_z^2}\right).$$

On the assumption that $\left|N_z^2-\frac{1}{\alpha+1}\right| \ll 1$ and $N_x^2 \ll 1$ two remaining roots of (16) correspond to the ordinary wave and are located in the vicinity of the plasma resonance

$$(18) \quad p_{2,3} = \alpha^2 + \frac{\alpha^2}{2}\left\{\frac{\alpha+1}{\alpha}\left(\frac{1}{\alpha+1}-N_z^2\right) \mp \left[\left(\frac{\alpha+1}{\alpha}\right)^2\left(\frac{1}{\alpha+1}-N_z^2\right)^2 + \frac{2N_x^2}{\alpha}\right]^{1/2}\right\}.$$

The evanescent layer situated between the points p_2 and p_3 prevents partly the penetration of the ordinary wave into a dense plasma.

If $\left|N_z^2-\frac{1}{\alpha+1}\right| \sim 1$, we could derive the expressions similar to (17) for $p_{2,3}$. The thickness of the evanescent layer would be, however, comparable with $\varkappa^{-1}$ in this case and the ordinary wave would be reflected here practically completely. The same situation could be found also for $N_x^2 \sim 1$. Deriving the transmission coefficient of the ordinary wave through this layer in the next chapter we shall, therefore, suppose that $N_x^2 \ll 1$, $\left|N_z^2-\frac{1}{\alpha+1}\right| \ll 1$.

We shall now briefly discuss the situation arising if the wave (1) is incident on the plasma-vacuum boundary. Supposing the density and its first derivative to be continuous in this region we can neglect the reflexion of wave there. On entering the plasma the incident wave is split into an ordinary and an extraordinary wave propagating independently afterwards. The amplitudes and phases of these waves may be expressed as functions of $\vec{\mathcal{E}}_o$. The expressions for these quantities are very complicated but supposing that $N_x^2 \ll 1$, they turn out practically the same as those in [1]. The extraordinary wave is mostly reflected in the point (17) back to the boundary of a plasma.

IV. REFLEXION AND TRANSMISSION COEFFICIENTS OF THE ORDINARY WAVE

Provided that $N_x^2 \ll 1$ and $\left| N_z^2 - \frac{1}{\alpha + 1} \right| \ll 1$ two points p_2 and p_3, where $k_{y1} = 0$, are located in the vicinity of the plasma resonance. The ordinary wave is evanescent between these points and the validity of the WKB approximation is violated in this region. To obtain the transmission coefficient of the wave through the evanescent layer we must continue the solutions (13) to the y-complex plane. To this purpose we deduce the approximate form of (13) for this region and make it clear where it may be used. We suppose that the plasma density near the plasma resonance has a linear profile

$$p = \alpha^2 \left(1 + \varkappa y \right) , \quad \varkappa > 0 . \tag{19}$$

The wave vector of the ordinary wave can be then rewritten as

$$N_{y1}^2 = 2 \alpha \varkappa^2 (y - y_2)(y - y_3) \tag{20}$$

where $\quad y_{2,3} = \frac{1}{\varkappa} \left(\frac{p_{2,3}}{\alpha^2} - 1 \right)$

Both the WKB approximation and the expansion (19) are valid simultaneously if the variable y fulfils the inequality

(21) $$\left(\frac{\varkappa}{k_y}\right)^{1/2} \ll |\varkappa y| \ll 1$$

On the basis of the previous assumptions we can see that the points y_2 and y_3 are close together and thus $\varkappa(y_3 - y_2) \ll 1$.

In the region of the plasma resonance the expression for slowly varying phase (A5) may be put into a simple form

$$\mathcal{F}_1(y) = \frac{-1}{\varkappa(2\alpha)^{1/2}} \int^{y} \frac{dy}{(y+y_0)\left[(y-y_2)(y-y_3)\right]^{1/2}} =$$

(22)
$$= \frac{i}{\varkappa(2\alpha u)^{1/2}} \lg \frac{i\left(2y_2y_3 + y_0(y_2+y_3) - y(2y_0+y_2+y_3)\right) + 2\left[u(y-y_2)(y-y_3)\right]^{1/2}}{(y_3-y_2)(y+y_0)},$$

where $y_0 = \dfrac{N_x^2(\alpha-1)}{\alpha\varkappa}$, $u = -(y_2+y_0)(y_3+y_0) \doteq \dfrac{N_x^2}{2\alpha\varkappa^2}$

Deriving (22) we have made use of the approximate formulae for M_{11}, F and $\varepsilon_\perp(N_{y1}^2 - N_{y2}^2)$

(23) $$M_{11} \doteq -\frac{\alpha\varkappa}{\alpha^2-1}(y+y_0), \quad F \doteq \varepsilon_\perp(N_{y1}^2 - N_{y2}^2) \doteq \frac{\alpha}{(\alpha+1)(\alpha^2-1)}.$$

The root $-y_0$ of M_{11} is situated between the points y_2 and y_3. If the conditions (21) are fulfilled we get the expression for the electric field of the ordinary wave

(24) $$E_x^{or} = C_{\pm,1} A_\pm^{1/2}(y)\, e^{\pm i\int^{y} k_{y1}\, dy}$$

where

$$A_\pm = \pm\frac{\alpha+1}{(2\alpha)^{1/2}} \frac{i\left(2y_2y_3 + y_0(y_2+y_3) - y(2y_0+y_2+y_3)\right) \pm 2\left[u(y-y_2)(y-y_3)\right]^{1/2}}{(y_2-y_3)\left[(y-y_2)(y-y_3)\right]^{1/2}}.$$

Now we shall proceed to deduce the transmission and reflexion coefficients of the ordinary wave. To this purpose we look for a solution representing, in the region $y > y_3$, a wave transferring energy to the region of a denser plasma, i.e. a wave given by

$$(25)\quad E_x^{o} = H A_-^{1/2}(y)\, e^{-i\int_{y_3}^{y} k_{y1}\,dy}, \qquad y > y_3 .$$

Continuing the solution (25) into the lower half plane and going around the turning points y_2 and y_3 we get a combination of an incident and a reflected ordinary wave for $y < y_2$ (see [6])

$$(26)\quad E_x^{o} = H e^{\int_{y_2}^{y_3} |k_{y1}|\,dy} \left(A_+^{1/2} e^{+i\int_{y_2}^{y} k_{y1}\,dy} + R A_-^{1/2} e^{-i\int_{y_2}^{y} k_{y1}\,dy} \right).$$

R is the Stokes constant corresponding to the Stokes line r from figure 1. The absolute value of this constant can be determined by the conservation law of the energy flow (4). From (26) it is clear that $|R|$ represents also the absolute value of the reflexion coefficient. Our method does not make it possible to find the phase of the Stokes constant; therefore the phase of the reflected wave cannot be determined either. Using (11), (25) and (26) we get the relation for $|R|$ from (4)

$$(27)\quad 1 - |R|^2 = \exp\left\{-2\int_{y_2}^{y_3} |k_{y1}|\,dy\right\}$$

On substituting for k_{y1} from (20) we obtain for $|R|$

$$(28)\quad |R| = \left\{ 1 - \exp\left[-\frac{\pi}{4}(2\alpha)^{1/2} k_y\, \varkappa (y_3 - y_2)^2\right]\right\}^{1/2}$$

By means of the relations (18) the former expression can be written in form

$$(29)\quad |R| = \left\{1 - \exp\left[\frac{\pi}{4}(2\alpha)^{1/2}\frac{k_v}{\varkappa}\left(\left(\frac{\alpha+1}{\alpha}\right)^2\left(\frac{1}{\alpha+1} - N_z^2\right)^2 + \frac{2N_x^2}{\alpha}\right)\right]\right\}^{1/2}$$

The amplitude of the transmission coefficient T is

$$(30)\quad |T| = \exp\left\{-\frac{\pi}{8}(2\alpha)^{1/2}\frac{k_v}{\varkappa}\left(\left(\frac{\alpha+1}{\alpha}\right)^2\left(\frac{1}{\alpha+1} - N_z^2\right)^2 + \frac{2N_x^2}{\alpha}\right)\right\}$$

When matching (26) to (13) and (1) we could get the relation between the constant H and the vector $\vec{\mathcal{E}}_0$. Assuming $N_x^2 \ll 1$ the expression for $H\exp\left\{\int_{y2}^{y3} |k_{y1}|\, dy\right\}$ is, however, approximately the same as the expression (25) for H in [1]. It is thus clear that in this case it is only the coefficients of transmission and reflexion that depend on N_x^2 strongly.

Having passed through the evanescent layer, the ordinary wave propagates further into a dense plasma up to the point (15) where $k_{y1} = k_{y2}$. In an analogous way as in [1] it might be demonstrated that in this place the backward ordinary wave is fully transformed into a forward extraordinary wave. This wave travels then back to lower densities up to the region of the hybride resonance.

Owing to the singularity of k_{y2} in the hybride resonance both the WKB approximation and the approximation of cold magnetohydrodynamics break down in this region. To obtain an adequate description of the electric field of the extraordinary wave in the vicinity of the resonance we should include the effect of a finite temperature of electrons in the system(3). Like in [1] we could then derive a sixth-order differential equation for E_x applicable for waves with $\omega \in (\omega_c, 3\omega_c)$. From the analysis of this equation it would follow that the full transformation of the extraordinary waves into the Bernstein modes takes place in the hybride resonance also if $N_x \neq 0$. These short-wavelength Bernstein modes are practically electrostatic and they propagate back to a denser plasma.

V. CONCLUSION

In the foregoing chapters we have investigated the propagation of electromagnetic waves in an inhomogeneous magnetized plasma at oblique incidence. Main attention have been paid to the propagation of the ordinary wave in a weakly inhomogeneous plasma and to the transmission of this wave through the region of the plasma resonance. We have discussed the general case of oblique incidence and obtained also the new form for the WKB solutions of this problem. Problems connected with the transformation of the ordinary wave to an extraordinary one and with the transformation of this wave to the Bernstein mode have been treated in outline as the results of [1] may be used to solve them.

Now, analysing the transmission coefficient T, we shall clear up somewhat the conditions at which the ordinary wave can penetrate through the region of the plasma resonance. If we consider that N_x and N_z represent the direction cosines of an incident ray $(N_x = \cos\gamma_x, N_z = \cos\gamma_z)$ we may determine the vertex angle and the axis direction of the cone consisting of those rays that can penetrate into a denser plasma. The wave is fully transmitted if $\gamma_{x,0} = \frac{\pi}{2}$ and $\gamma_{z,0} = \arccos \frac{\pm 1}{(\alpha+1)^{1/2}}$ because $T = 1$ for this ray. It is seen from (30) that T is greater than $1/e$

when γ_x and γ_z fullfil the inequality

$$(31) \quad (\gamma_x - \gamma_{x,0})^2 + 2(\gamma_z - \gamma_{z,0})^2 \leqq 2 \frac{(2\alpha)^{1/2}}{\pi} \frac{\varkappa}{k_y} .$$

Supposing $\varkappa/k_y$ not to be too small we may deduce from (31) that the waves having γ_x and γ_z fairly different from $\gamma_{x,0}$, $\gamma_{z,0}$ can be transmitted through the region of the plasma resonance. For instance if $\alpha = 1,5$ and $\varkappa/k_y = 10^{-1}$ the maximum differences of γ_x and γ_z are $2\Delta\gamma_x = 38°$ and $2\Delta\gamma_z = 27°$; if $\varkappa/k_y = 10^{-2}$ then $2\Delta\gamma_x = 12°$ and $2\Delta\gamma_z = 8°$; if $\varkappa/k_y = 10^{-3}$ then $2\Delta\gamma_x = 4°$ and $2\Delta\gamma_z = 2°30'$.

On the whole, we conclude that electromagnetic waves having their angles of incidence in a fairly wide range can penetrate into a denser plasma and eventually be transformed into the Bernstein modes. Owing to the fact that these modes have a small group velocity and are damped much strongly they can effectively heat a plasma.

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APPENDIX

From (12) we get for $E_{(0)x}$

$$\left\{ 2N_y \left(1 + \frac{M_{22}}{M_{11}}\right) + N_x \left(\frac{M_{21}}{M_{31}} - \frac{M_{21}}{M_{13}}\right) + N_z \frac{M_{31} - M_{13}}{M_{11}} \right\} \frac{dE_{(0)x}}{d\bar{y}} +$$

$$\text{(A1)}\quad +\left\{\frac{dN_y}{d\bar y}\left(1+\frac{M_{33}}{M_{11}}\right)+2N_y\frac{M_{33}}{M_{11}}\frac{d}{d\bar y}\left(\frac{M_{13}}{M_{31}}\right)+N_z\frac{d}{d\bar y}\left(\frac{M_{32}}{M_{31}}\right)+\right.$$

$$\left.+N_z\left[\frac{M_{33}}{M_{13}}\frac{d}{d\bar y}\left(\frac{M_{32}}{M_{31}}\right)-\frac{M_{23}}{M_{31}}\frac{d}{d\bar y}\left(\frac{M_{33}}{M_{31}}\right)\right]\right\}E_{(0)x}=0\ .$$

Deriving this equation we have used the relation between the minors $M_{11}M_{33}=M_{31}M_{13}$. Making use of dispersion relation (9) and the expressions for M_{pq} we can rewrite (A1) in a form

$$\text{(A2)}\quad \frac{2N_y\varepsilon_1(N_{y1}^2-N_{y2}^2)}{M_{11}}\frac{dE_{(0)x}}{d\bar y}+\left[\frac{d}{d\bar y}\left(\frac{N_y\varepsilon_1(N_{y1}^2-N_{y2}^2)}{M_{11}}\right)+\frac{iN_xF}{M_{11}^2}\frac{dg}{d\bar y}\right]E_{(0)x}=0,$$

in which F is given by

$$\frac{iN_xF}{M_{11}}\frac{dg}{d\bar y}=N_yM_{33}\left(\frac{1}{M_{13}}\frac{dM_{13}}{d\bar y}-\frac{1}{M_{31}}\frac{dM_{31}}{d\bar y}\right)+$$

$$\text{(A3)}\quad +\frac{N_z}{2}M_{11}\frac{d}{d\bar y}\left(\frac{M_{32}}{M_{31}}+\frac{M_{23}}{M_{13}}\right)+\frac{N_z}{2}\left[\frac{d}{d\bar y}(M_{32}+M_{23})-\frac{M_{32}+M_{23}}{M_{33}}\frac{dM_{33}}{d\bar y}+\right.$$

$$\left.+(M_{32}-M_{23})\left(\frac{1}{M_{13}}\frac{dM_{13}}{d\bar y}-\frac{1}{M_{31}}\frac{dM_{31}}{d\bar y}\right)\right].$$

After some algebra we can show that F has relatively simple form

$$F=(N_x^2+N_y^2-\varepsilon_{11})^2(N_x^2-1)+N_z^2\left[(N_x^2-1)(\varepsilon_1-N_x^2-N_z^2)+\right.$$

$$\text{(A4)}\quad \left.+(\varepsilon_{11}-N_x^2)^2+(\varepsilon_{11}-N_x^2)(1+\varepsilon_1-2N_x^2-2N_y^2-2N_z^2)\right].$$

To obtain (A4) we had to consider that $g\frac{d\varepsilon_{11}}{d\bar y}=(\varepsilon_{11}-1)\frac{dg}{d\bar y}$, $\varepsilon_1\,dg/d\bar y-g\,d\varepsilon_1/d\bar y=dg/d\bar y$, and to remove the term $dN_y/d\bar y$ from the right-hand side of (A3) by using the dispersion relation. The solution of the equation (A2) may then be written as

(A5) $$E_{(0)x} = C\left[\frac{M_{11}}{N_y \varepsilon_\perp (N_{y1}^2 - N_{y2}^2)}\right]^{1/2} e^{-\frac{iN_x}{2}\mathcal{F}} ,$$

where C is a constant of integration, the minor $M_{11} = N_y^2(\varepsilon_\perp - N_x^2) - (\varepsilon_\perp - N_x^2 - N_z^2)(\varepsilon_\parallel - N_x^2)$ and

(A6) $$\mathcal{F} = \int^y dy \, \frac{\frac{dg}{dy} F}{M_{11} N_y \varepsilon_\perp (N_{y1}^2 - N_{y2}^2)} .$$

To clarify a little the origin and structure of the slowly-varying term $\mathcal{F}N_x/2$ in the phase of wave we shall derive the expressions (A6) in another way in a simple case that $N_z = 0$. For the wave having the electric vector parallel to the plane of incidence we obtain from (3) this equation for

$$\varepsilon_\perp(\varepsilon_\perp - N_x^2)\frac{d^2E_x}{dy^2} - N_x^2 \frac{d\varepsilon_\perp}{dy}\frac{dE_x}{dy} +$$

(A7) $$+ \left\{(\varepsilon_\perp - N_x^2)\left[(\varepsilon_\perp - N_x^2)\varepsilon_\perp - g^2\right] + N_x(1 - N_x^2)\frac{dg}{dy}\right\}E_x = 0.$$

The WKB solution of this equation has a form

(A8) $$E_x = C\left(\frac{\varepsilon_\perp - N_x^2}{N_y \varepsilon_\perp}\right)^{1/2} \exp\left\{\frac{iN_x(1-N_x^2)}{2}\int^y \frac{dg}{dy}\frac{dy}{N_y\varepsilon_\perp(\varepsilon_\perp - N_x^2)} + i\int^y k_y\, dy\right\},$$

where $N_y^2 = k_y^2/k_0^2 = \varepsilon_\perp - N_x^2 - g^2/\varepsilon_\perp$. It is thus clear that the slowly-varying phase is connected with the term $N_x(1-N_x)\,dg/dy$ in the coefficient at E_x in (A7). In general case when $N_z \neq 0$ it is also possible to deduce from the system (3) the differential equation of the fourth order for E_x and similar terms of the first order in the parameter $\varkappa/k_0$ may be found out in the coefficients at even derivatives of E_x.

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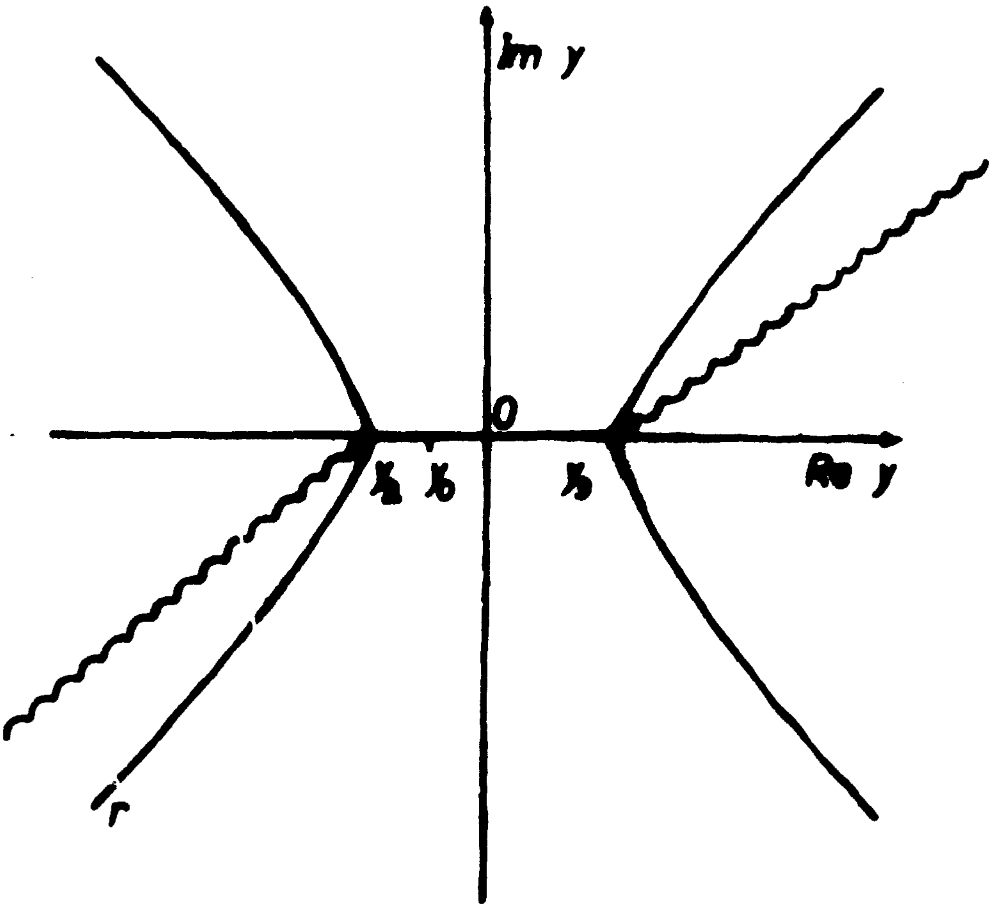

Fig. 1 The y-complex plane with cuts (wavy line) and Stokes lines (smooth) corresponding to the expression (24) for E_x^{or} .

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