DIFFUSION DUE TO A SINGLE WAVE IN A MAGNETIZED PL SMA

Gary R. Smith and Allan N. Kaufman

Department of Physics and Lawrence Berkele Laboratory

University of California, Berkeley, Califor ia 94720

The nature of charged-particle motion in the presence of a spectrum of waves usually depends on the width of the spectrum. In a narrow spectrum (modeled as a single wave), particles may be trapped in the potential wells of the wave and thereby have a limited acceleration. In a broad spectrum, resonant particles diffuse in velocity space, and thereby undergo a more extensive (stochastic) acceleration. In contrast to these well-known results we find[1] that a single wave in a magnetized plasma may cause particle diffusion.

In a magnetized plasma, a wave propagating at an oblique angle $\theta = \tan^{-1}(k_\perp/k_z)$ to a uniform magnetostatic field $B_0\hat{z}$ has a set of resonant parallel velocities

$$V_\ell = (\omega + \ell\Omega)/k_z \ , \ \ell = 0, \pm 1, \pm 2, \cdots ,$$

where the gyrofrequency $\Omega = eB_0/mc$. Near each V_ℓ there is a trapping layer $(V_\ell \pm w_\ell)$, with a half-width

$$w_\ell = 2 \left| e\Phi_0 J_\ell(k_\perp \rho)/m \right|^{\frac{1}{2}} ,$$

for an electrostatic wave of amplitude Φ_0 . For small Φ_0 the oscillations of a particle's parallel velocity are limited to the trapping width $2w_\ell$. For Φ_0 large enough that the trapping layers overlap, i.e., roughly when

$$2w_\ell \gtrsim |v_{\ell+1} - v_\ell| = \Omega/k_z \ , \tag{1}$$

a particle can move from one resonance region to another, changing its parallel velocity much more than $2w_\ell$. In the latter case we call the particle motion stochastic.

We use Hamiltonian methods to study the motion of a particle in the field $B_o\hat{z}$ and a single electrostatic wave. We work in the wave frame, which moves with velocity $(\omega/k_z)\hat{z}$ with respect to the plasma. (We consider only waves with $\omega \ll k_z c$.) The Hamiltonian is then

$$H = (\underline{p} + m\Omega y\hat{x})^2/2m + e\Phi_o \sin(k_z z + k_\perp y) \ .$$

Instead of the Cartesian variables (x,y,p_x,p_y) we use a more convenient set of canonical variables. The gyrophase, $\phi = \tan^{-1}(v_x/v_y)$, is conjugate to the angular momentum $p_\phi = \frac{1}{2} m\Omega\rho^2 = mv_\perp^2/2\Omega = (mc/e)\mu$, where ρ is the gyroradius and μ is the magnetic moment. The y-component of the guiding center, $Y = y + \rho \sin\phi$, is conjugate to $m\Omega X = m\Omega(x - \rho\cos\phi)$. In terms of these variables

$$H = p_z^2/2m + p_\phi\Omega + e\Phi_o \sin(k_z z + k_\perp Y - k_\perp\rho \sin\phi) \ .$$

We redefine the origin of z by performing a canonical transformation to the new variables

$$z' = z + k_\perp Y/k_z \ , \ p_z' = p_z$$

$$Y' = Y \ , \ X' = X - k_\perp p_z/k_z m\Omega \ .$$

Since Y' and X' do not appear in the transformed Hamiltonian, they are each constants of the motion. Dropping the primes on p_z' and z' , we write

$$H = p_z^2/2m + p_\phi\Omega + e\Phi_o \sin(k_z z - k_\perp\rho \sin\phi) \ . \tag{2}$$

If the wave amplitude is small, perturbation methods can be used to study the particle motion. There exists an additional constant of the motion I ; Taylor and Laing[2] have found the

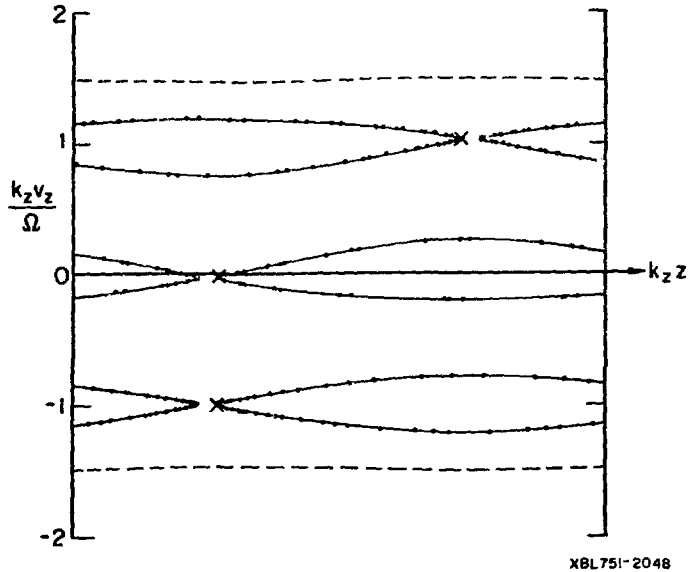

Fig. 1. Particle trajectories, calculated from Hamiltonian (2), represented by dots at the values of z and v_z whenever $\Phi = (2N + 1)\pi$, $N = 1,2,\cdots$. The three initial conditions, indicated by crosses, were chosen near the separatrices. The parameters are the same for all three trajectories: $k_\perp \rho_E \equiv k_\perp (2E/m)^{\frac{1}{2}}/\Omega = 1.48$, $(\tilde{\omega}/\Omega)^2 \equiv k_z^{\ 2}|e\Phi_o|/m\Omega^2 = 0.025$, $\theta = 45^\circ$.

expression

$$I = \cos\left(\frac{\pi k_z v_z}{\Omega}\right) - \pi\left(\frac{k_z^{\ 2} e\Phi_o}{m\Omega^2}\right)\sin\left(\frac{\pi k_z v_z}{\Omega}\right)\sum_\ell J_\ell \frac{\sin(k_z z - \ell\Phi)}{(k_z v_z/\Omega) - \ell} \quad ,$$

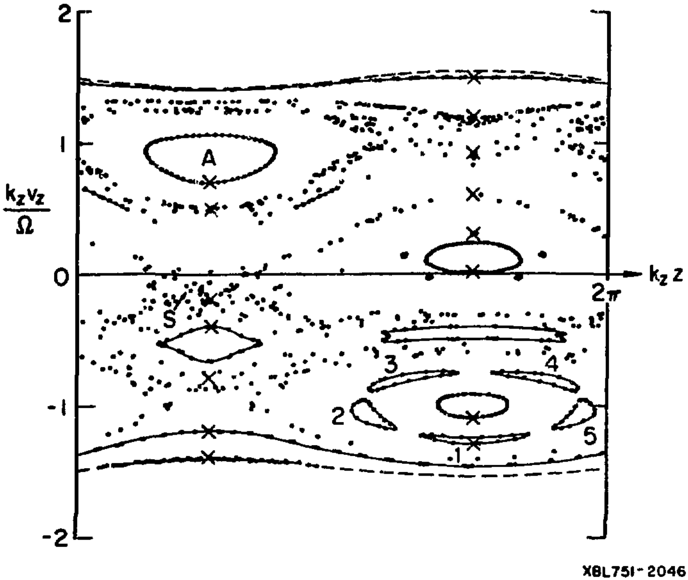

Fig. 2. Similar to Fig. 1 but for a larger wave amplitude, $(\tilde{\omega}/\Omega)^2 = 0.1$. The plane is divided into adiabatic regions (like A) and stochastic regions (like S) . A chain of five islands is indicated by the numbers 1-5.

accurate to first order in Φ_0 . The existence of the constant I is verified by the nature of the particle trajectories shown in Fig. 1. The dots representing a trajectory lie on a smooth curve only if a constant I exists.

We study the particle motion by numerically integrating the equations of motion derivable from (2) . A transition to stochastic motion occurs when the wave amplitude satisfies the threshold condition (1) . Near the threshold, the motion has the interesting character illustrated in Fig. 2. This type of

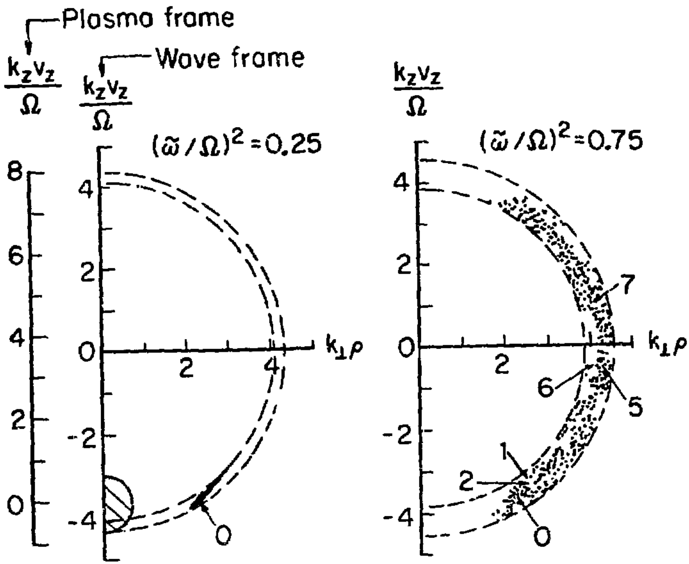

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Fig. 3. Particle trajectories represented by v_z and $\rho = v_\perp/\Omega$ whenever $2N\pi = \phi - \pi \approx \Omega t$. The numbers 0,1,2,5,6,7 are the values of N for a trajectory initiated with a particular $z(t = 0)$. The wave amplitude is given by the parameter $(\tilde{\omega}/\Omega)^2$; $\theta = 45^\circ$ and $k_z v/\Omega = 4.24$, where v is the initial speed in the wave frame.

motion has been observed for several other problems.[3-7]

Fig. 3 shows the particle motion in velocity space. In the small wave amplitude case $[(\tilde{\omega}/\Omega)^2 = 0.25]$, particles with initial conditions at the point marked 0 are accelerated from the initial parallel and perpendicular velocities by only small amounts. In the large wave amplitude case $[(\tilde{\omega}/\Omega)^2 = 0.75]$, the particle acceleration is substantial. The particle remains

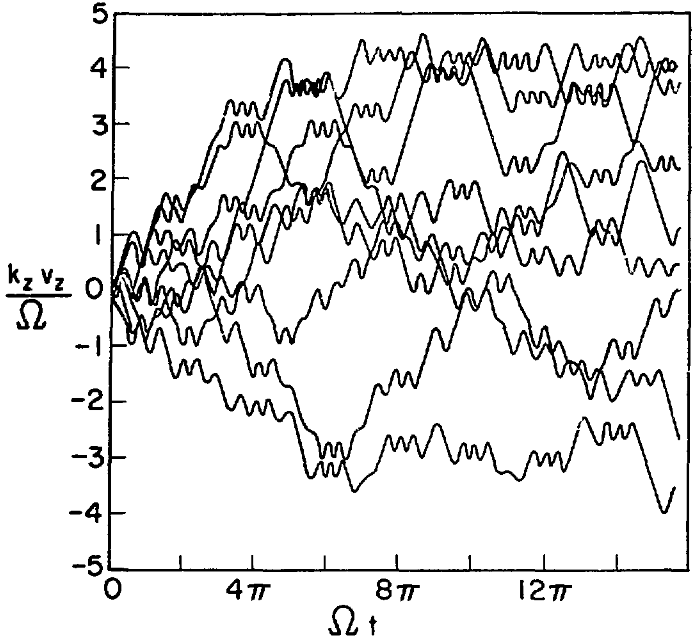

Fig. 4. Particle trajectories represented by v_z vs. time. $(\tilde{\omega}/\Omega)^2 = 0.75$, $\theta = 45°$, $k_z v/\Omega = 5$ and $v_z(t = 0) = 0$.

near a curve of constant speed in the wave frame (the dashed semicircles in Fig. 3), because the Hamiltonian (2) is a constant of the motion. In the plasma frame the particle's parallel and perpendicular energies can increase substantially.

The stochastic particle motion has the nature of a diffusion process. We establish this fact by numerically calculating a set of 50 to 200 trajectories. The initial velocity components (v_z and $v_\perp$) are the same for the whole set, but the initial phases are distributed throughout $0 \leq k_z z$, $\phi < 2\pi$. A subset

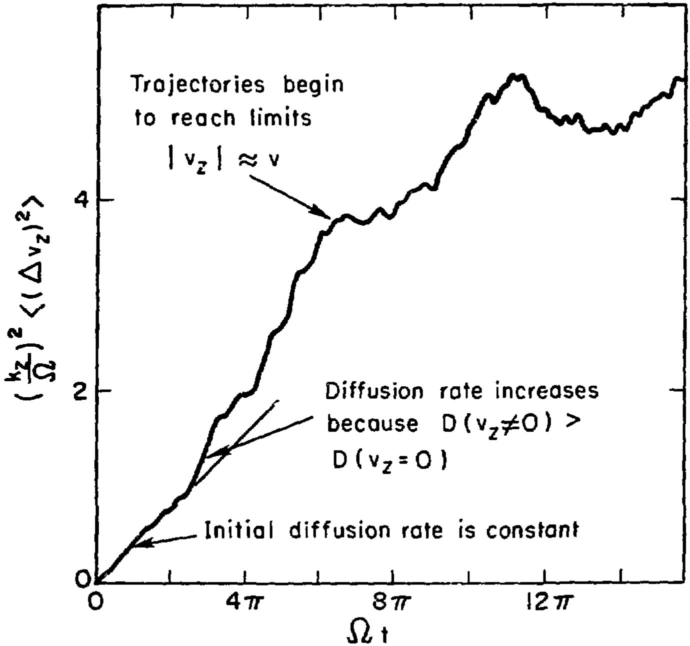

Fig. 5. Mean square deviation in parallel velocity vs. time for the same parameters as in Fig. 4.

of the trajectories is shown in Fig. 4. From the set of trajectories we calculate the mean square deviation in parallel velocity, $\langle [\Delta v_z(t)]^2 \rangle$, and find that it increases linearly with time, as shown in Fig. 5.

To explain analytically the observed diffusion rates we use methods developed for the general theory of stochastic instability of nonlinear oscillations.[8] Using (2) and a Bessel function identity, we write the potential due to the wave as

$$\Phi(z,\phi,\rho) = \Phi_o \sum_{\ell} J_\ell(k_\perp\rho)\sin(k_z z - \ell\phi) .$$

We replace J_ℓ by an average, $\bar{J}_\ell$, over ℓ , with resonant values of ℓ (those near $k_z v_z/\Omega$) weighted most heavily. We then use another identity to write

$$\Phi(t) \approx \Phi_o \bar{J}_\ell \sin[k_z z(t)]T \sum_n \delta(t - nT + \phi_o/\Omega) , \qquad (3)$$

where $T \equiv 2\pi/\Omega$ is the gyroperiod. The parallel velocity undergoes a sudden jump once each gyroperiod in this approximation. (Some trajectories in Fig. 4 show rapid changes in v_z each gyroperiod.) The jump in v_z is

$$\Delta v_z = - \frac{e}{m}\int_0^T dt \frac{\partial\Phi}{\partial z} . \qquad (4)$$

The diffusivity D is defined by

$$D \equiv \langle[\Delta v_z(t)]^2\rangle/2t , \qquad (5)$$

where the average is over the initial phases z and ϕ . Combining (3) - (5) and choosing the typical value 0.2 for $\bar{J}_\ell$, we find

$$D \sim (0.1)(k_z e\Phi_o/m)^2/\Omega .$$

The particle motion has a diffusive nature only when (1) is satisfied, which implies roughly

$$k_\perp\rho \gtrsim 1$$

$$|e\Phi_o| \gtrsim m(\Omega/k_z)^2 .$$

A more sophisticated theory is necessary to explain the dependence of the diffusivity D on v_z and ρ . A rigorous theory for problems involving stochastic instability does not yet exist.[8] As a model to describe our observations we take

$$D \approx \frac{1}{2}(k_z e\Phi_o/m)^2 \sum_\ell J_\ell^2(k_\perp\rho)R_\ell(k_z v_z - \ell\Omega) . \qquad (6)$$

A quasilinear theory would yield a resonance function

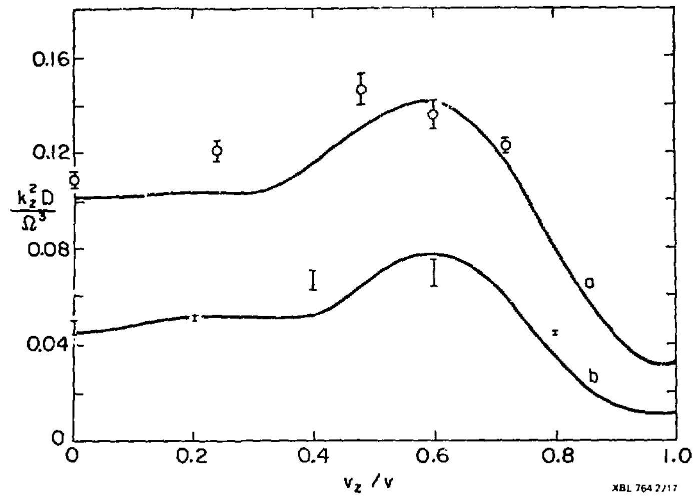

Fig. 6. Agreement between model (6) for the diffusivity D and the numerically observed diffusion, for several values of $v_z(t = 0)$. $\theta = 45^\circ$.

(a) $k_z v/\Omega = 5$, $(\tilde{\omega}/\Omega)^2 = 1.00$.

(b) $k_z v/\Omega = 6$, $(\tilde{\omega}/\Omega)^2 = 0.75$.

The measurement of the observed points and the error bars is somewhat subjective.

$$R_\ell(\omega) = \int_0^\infty d\tau \cos\omega\tau = \pi\,\delta(\omega) .$$

Instead we allow the resonance to be broadened:

$$R_\ell(\omega) = \int_0^\infty d\tau \cos\omega\tau \exp(-\nu_\ell\tau)$$
$$= \nu_\ell(\omega^2 + \nu_\ell^2)^{-1} .$$

The amount of broadening is determined by the trapping half-width:

$$\nu_\ell = k_z w_\ell .$$

(Conventional resonance broadening theories[9] do not adequately explain our observations.) Shown in Fig. 6 is the agreement between this model and the numerically observed diffusivity.

A specific wave which would lead to heating of an ion distribution is an ion-acoustic wave with frequency $\omega \approx kc_s > \Omega_i$ and oblique propagation angle θ (e.g., 45^o). The cross-hatched region in Fig. 3 shows the extent of the thermal ions for $\omega = 3.6\ \Omega_i$ and $T_e = 16\ T_i$. Ions in the tail of an initially Maxwellian distribution would be stochastically accelerated by a wave with $\delta n/n \gtrsim 0.1$. A Langmuir wave with analogous properties would lead to heating of the tail of the electron distribution.

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