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BOUNDARY CONDITIONS FOR CYLINDRICAL GEOMETRY IN NEUTRON TRANSPORT THEORY

I.

CIRCULAR VACUUM - CAVITY - BOUNDARY IN CYLINDRICAL SYMMETRY

ŠKODA WORKS
Nuclear Power Construction Department, Information Centre
PLZEŇ - CZECHOSLOVAKIA

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ABSTRACT

In this paper we give the a n a l y t i c a l f o r m u l a t i o n of the boundary condition for a circular vacuum-cavity-boundary in c y l i n d r i c a l s y m m e t r y. It is linked to the report /2 / presenting the results of the experimental research of neutron penetration through a cavity of a radius R and a length H, made by the research workers of the ŠKODA WORKS, PLZEŇ, CZECHOSLOVAKIA. The n u m e r i c a l a n a l y s i s of the problem convenient for practical calculations is given in the following paper (to be published):

O. Veverka, V. Valenta, V. Krýsl

B O U N D A R Y C O N D I T I O N S F O R
C Y L I N D R I C A L G E O M E T R Y I N
N E U T R O N T R A N S P O R T T H E O R Y

II.

C I R C U L A R V A C U U M - C A V I T Y -
B O U N D A R Y I N C Y L I N D R I C A L
S Y M M E T R Y

The mathematical verification of the mentioned experiment is given in the paper (to be published):

V. Krýsl, V. Valenta, O. Veverka

C A L C U L A T I O N O F N E U T R O N P E N E T R A T I O N
T H R O U G H A S T R A I G H T C I R C U L A R V A C U U M
C H A N N E L I N (r, z) - G E O M E T R Y

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Introduction

The task of this report and of the following set of papers is to verify the above mentioned experimental results and to develop an effective method, simple enough, for calculations of neutron fields in the vicinity of a straight circular cavity using the P_1-approximation in the form of an equivalent effective diffusion model or in the simple diffusion approximation in the two-dimensional (r, z)-geometry and both the isotropical diffusion model with a scalar diffusion coefficient D and the anisotropical diffusion model with a tensor diffusion coefficient (D_r, D_z).

The applications of these formulations consist, for instance, in the analysis of the neutron distribution in a reactor core cell including an empty technological channel or in an effective cell of the axial neutron shield penetrated with technological channels.

Both the P_1-approximation in the form of an equivalent effective diffusion model and the simple diffusion model here are written in the isotropical diffusion approximation with the scalar diffusion coefficient D. The generalization for an anisotropical diffusion approximation may be easily written writing the tensor component D_r together with the differential operator according to r and with the tensor component D_z together with the differential operator according to z .

Note: In the formulae for the group constants (2.41) to (2.45) the index i defines the individual isotopes and the index j defines the reactions, which are:

Σ^{j} .... capture, fission, elastic scattering, inelastic scattering, (n, 2n)-reaction, ((n, 3n)-reaction);

Σ_{i}^{j} .... the same;

$\Sigma_{\nu}^{h\to j}$......... elastic scattering ($\nu = 1$), inelastic scattering ($\nu = 1$), (n, 2n)-reaction ($\nu = 2$), ((n, 3n)-reaction ($\nu = 3$));

$\mu_{\nu}^{h\to j}\Sigma_{\nu}^{h\to j}$... the same;

$\chi^{j}\nu_{j}^{h}\Sigma_{j}^{h}$ ... fission (ν is a function of the energy of the primary neutron and of the fission isotope).

The formulae for the group constants (2.41) to (2.45) include the integration according to $\vec{r}$ across any homogeneous subregion. These formulae are of course too complicated and for practical calculations there are used some of the well-known simplifications.

There is given the boundary condition together with the symmetry conditions in the following approximations:

i) Infinite cavity:

General formulation: (1.1), (1.2), (1.3).

P_N-approximation: (1.22).

P_1-approximation: (2.69), (2.70), (2.71) or (2.72), (2.73).

Diffusion approximation: (2.74), (2.75).

ii) Finite cavity:

General formulation: (2.1), (2.2).

P_N-approximation: (2.13) to (2.27).

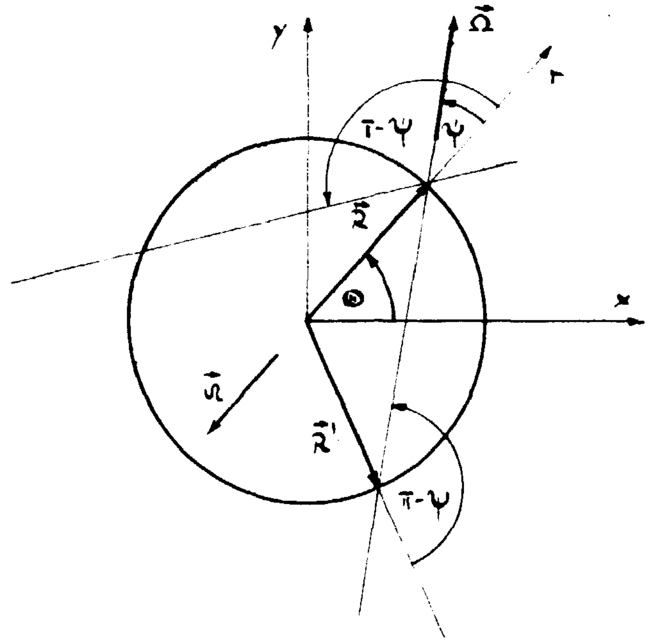
P_1-approximation: (2.35), (2.36), (2.37), (2.38), (2.32), (2.33), (2.34) or (2.53), (2.50), (2.51), (2.52), (2.57) to (2.62), (2.32), (2.33), (2.34).

Diffusion approximation: (2.63) to (2.68), (2.32), (2.33), (2.34).

1. Infinite cavity

We consider an infinite circular vacuum channel irradiated with a cylindrically symmetrical neutron flux (for simplicity we do not write the energy-dependence of the neutron flux). First the general boundary condition together with the symmetry conditions are given:

Boundary condition:



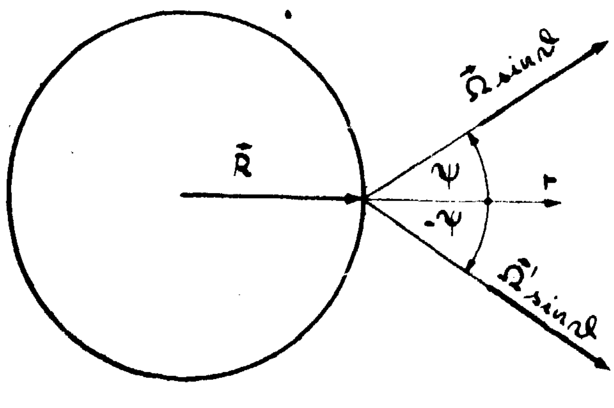
$$\varphi(\vec{R}, \vartheta, \psi) = \varphi(\vec{R}', \vartheta, \pi - \psi) \quad \text{for} \quad 0 \le \vartheta \le \pi, \quad -\frac{\pi}{2} \le \psi \le \frac{\pi}{2},$$

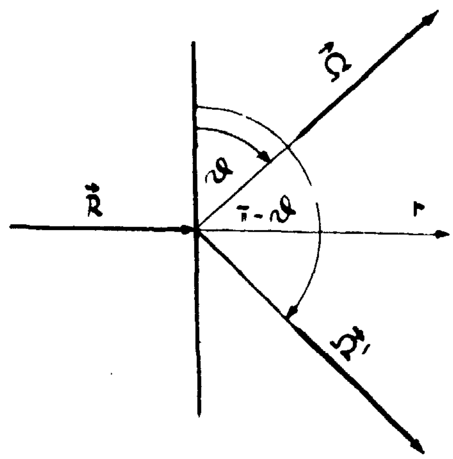
from cylindrical symmetry follows

$$\varphi(\vec{R}', \vartheta, \pi - \psi) = \varphi(\vec{R}, \vartheta, \pi - \psi) \quad \text{for} \quad 0 \le \vartheta \le \pi, \quad 0 \le \psi < 2\pi,$$

i.e. $$\varphi(R, \vartheta, \psi) = \varphi(R, \vartheta, \pi - \psi) \quad \text{for} \quad 0 \le \vartheta \le \pi, \quad -\frac{\pi}{2} \le \psi \le \frac{\pi}{2}. \qquad (1.1)$$

Symmetry conditions:





$$\varphi(R, \vartheta, \psi) = \varphi(R, \vartheta, -\psi) \quad \text{for} \quad 0 \le \vartheta \le \pi, \quad 0 \le \psi < 2\pi, \qquad (1.2)$$

$$\varphi(R, \vartheta, \psi) = \varphi(R, \pi - \vartheta, \psi) \quad \text{for} \quad 0 \le \vartheta \le \pi, \quad 0 \le \psi < 2\pi. \qquad (1.3)$$

Using the Fourier series

$$\varphi(R,\vartheta,\psi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{2n+1}{4\pi}\,\varphi_{nm}(R)\,P_n^m(\vec\Omega)\,, \qquad (1.4)$$

$$\varphi(R,\vartheta,\pi-\psi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{2n+1}{4\pi}\,\varphi_{n,-m}(R)\,P_n^m(\vec\Omega)\,, \qquad (1.5)$$

$$\varphi(R,\vartheta,-\psi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (-1)^m\,\frac{2n+1}{4\pi}\,\varphi_{n,-m}(R)\,P_n^m(\vec\Omega)\,, \qquad (1.6)$$

$$\varphi(R,\pi-\vartheta,\psi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (-1)^{n-m}\,\frac{2n+1}{4\pi}\,\varphi_{nm}(R)\,P_n^m(\vec\Omega)\,, \qquad (1.7)$$

we get from (1.2) $\quad \varphi_{nm}(R) = (-1)^m\,\varphi_{n,-m}(R)$ for all n, m, $\qquad$ (1.8)

and from (1.3) $\quad \varphi_{nm}(R) = 0$ for odd values of n + m. $\qquad$ (1.9)

Note: To get the F. series the following was used:

$$P_n^m(\vartheta,\psi) = (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,e^{im\psi}\,P_n^m(\cos\vartheta) = P_n^m(\vec\Omega)\,,$$

$$P_n^m(\vartheta,\pi-\psi) = (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,e^{im(\pi-\psi)}\,P_n^m(\cos\vartheta) = (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,(-1)^m e^{-im\psi}\,P_n^m(\cos\vartheta) =$$
$$= (-1)^m\,P_n^{m*}(\vec\Omega) = P_n^{-m}(\vec\Omega)\,,$$

$$P_n^m(\vartheta,-\psi) = (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,e^{-im\psi}\,P_n^m(\cos\vartheta) = P_n^{m*}(\vec\Omega) = (-1)^m\,P_n^{-m}(\vec\Omega)\,,$$

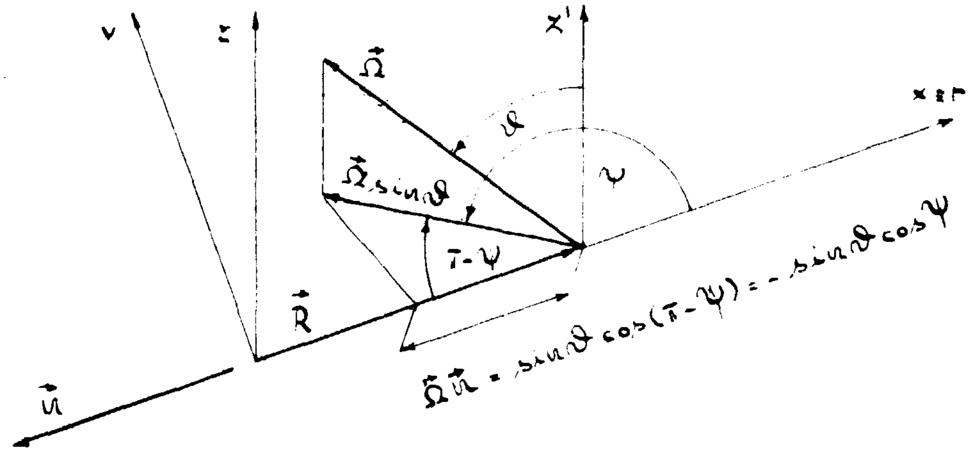
$$P_n^m(\pi-\vartheta,\psi) = (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,e^{im\psi}\,P_n^m[\cos(\pi-\vartheta)] = (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,e^{im\psi}\,P_n^m(-\cos\vartheta) =$$
$$= (-1)^m\sqrt{\frac{(n-m)!}{(n+m)!}}\,e^{im\psi}(-1)^{n-m}\,P_n^m(\cos\vartheta) = (-1)^{n-m}\,P_n^m(\vec\Omega)\,,$$

where $\vec\Omega = [\sin\vartheta\cos\psi,\ \sin\vartheta\sin\psi,\ \cos\vartheta]$ for $0 \le \vartheta \le \pi,\ 0 \le \psi < 2\pi$.

Further the boundary condition (1.1) will be discussed. According to Wladimirow [illegible] the following optimum boundary condition for cylindrical geometry is valid:

$$\int\limits_{\vec\Omega\vec n<0} d\vec\Omega\,\vec\Omega\vec n\,\varphi(R,\vartheta,\psi)\,P_{2l}^k(\vec\Omega) = \int\limits_{\vec\Omega\vec n<0} d\vec\Omega\,\vec\Omega\vec n\,\varphi(R,\vartheta,\pi-\psi)\,P_{2l}^k(\vec\Omega)\,, \qquad (1.10)$$

where $d\vec\Omega = \sin\vartheta\,d\vartheta\,d\psi$, $\vec\Omega\vec n = -\sin\vartheta\cos\psi$:



Using (1.4), (1.5), (1.8), (1.9), and (1.10), we get in the P_N-approximation the following system of boundary conditions

$$\sum_{n=0}^{N}\sum_{m=-n}^{n} C_{nmlk}\,\varphi_{nm}(R) = 0 \qquad (1.11)$$

$$\left[\begin{array}{l} \varphi_{n,-m}(R) = (-1)^m \varphi_{nm}(R) \text{ for all } n,\ m \\ \varphi_{nm}(R) = 0 \text{ for odd } n+m\,. \end{array}\right]$$

$$C_{nmlk} = \left[1-(-1)^m\right]\frac{2n+1}{4\pi}\int_{-\pi/2}^{\pi/2} d\psi \int_0^{\pi} d\vartheta\,\sin^2\vartheta\cos\psi\, P_n^m(\vec{\Omega})\,P_{2l}^k(\vec{\Omega}) =$$

$$= (-1)^{m+k}\left[1-(-1)^m\right]\frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}}\int_{-\pi/2}^{\pi/2} e^{i(m+k)\psi}\cos\psi\, d\psi\int_0^{\pi} d\vartheta\,\sin^2\vartheta\, P_n^m(\cos\vartheta)\,P_{2l}^k(\cos\vartheta);$$

using $\cos\psi = \frac{1}{2}\left[e^{i\psi} + e^{-i\psi}\right]$ there is $C_{nmlk} = C^1_{nmlk} + C^2_{nmlk}$, where

$$C^1_{nmlk} = (-1)^{m+k}\left[1-(-1)^m\right]\frac{2n+1}{8\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}}\int_{-\pi/2}^{\pi/2} e^{i(m+k+1)\psi}\, d\psi\int_0^{\pi} d\vartheta\,\sin^2\vartheta\, P_n^m(\cos\vartheta)\,P_{2l}^k(\cos\vartheta),$$

$$C^2_{nmlk} = (-1)^{m+k}\left[1-(-1)^m\right]\frac{2n+1}{8\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}}\int_{-\pi/2}^{\pi/2} e^{i(m+k-1)\psi}\, d\psi\int_0^{\pi} d\vartheta\,\sin^2\vartheta\, P_n^m(\cos\vartheta)\,P_{2l}^k(\cos\vartheta).$$

As $C^i_{nmlk} = 0$ for even values of m, the m must have an odd value; as $\varphi_{nm} = 0$ for odd values of $n+m$, the $n+m$ must have an even value; thence both the values of n, m must be odd, and N in the (1.11) is odd, too.

Therefore the number of equations (1.11) is defined by the following values of ℓ and k : $\ell = 0, 1, 2, \dots, \frac{N-1}{2}$, $k = -2\ell, \dots, 2\ell$. The integrals according to ψ are:

$$\int_{-\pi/2}^{\pi/2} e^{i(m+k+1)\psi} d\psi = \begin{cases} \pi & \text{for } m+k+1=0, \\ \frac{2}{m+k+1}\sin\frac{m+k+1}{2}\pi & \text{for } m+k+1 \neq 0; \end{cases}$$

$$\int_{-\pi/2}^{\pi/2} e^{i(m+k-1)\psi} d\psi = \begin{cases} \pi & \text{for } m+k-1=0, \\ \frac{2}{m+k-1}\sin\frac{m+k-1}{2}\pi & \text{for } m+k-1 \neq 0, \end{cases}$$

i.e.

$$\int_{-\pi/2}^{\pi/2} e^{i(m+k+1)\psi} d\psi = \begin{cases} \pi & \text{for } m+k+1=0, \\ 0 & \text{for } m+k+1 \neq 0, \text{ even}, \\ 1 & \text{for } m+k+1 \neq 0, \text{ odd}; \ m+k+1 = 4j+1, \\ -1 & \text{for } m+k+1 \neq 0, \text{ odd}; \ m+k+1 = 4j+3; \end{cases}$$

$$\int_{-\pi/2}^{\pi/2} e^{i(m+k-1)\psi} d\psi = \begin{cases} \pi & \text{for } m+k-1=0, \\ 0 & \text{for } m+k-1 \neq 0, \text{ even}, \\ -1 & \text{for } m+k-1 \neq 0, \text{ odd}; \ m+k-1 = 4j-1, \\ 1 & \text{for } m+k-1 \neq 0, \text{ odd}; \ m+k-1 = 4j-1, \end{cases}$$

where $j = 0, \pm 1, \pm 2, \dots$; for $m+k+1=0$ there is $m+k-1 \neq 0$ even and $J_+ = \pi$, $J_- = 0$; for $m+k-1=0$ there is $m+k+1 \neq 0$ even and $J_+ = 0$, $J_- = \pi$; for odd values of $m+k+1$ the $m+k-1$ is odd as well, and either $J_+ = 1$, $J_- = -1$ or $J_+ = -1$, $J_- = 1$; here J_+ is the first integral, J_- is the second integral; hence

$$\int_{-\pi/2}^{\pi/2} e^{i(m+k)\psi} \cos\psi \, d\psi = \begin{cases} \frac{\pi}{2} & \text{for either } m+k+1=0 \text{ or } m+k-1=0, \\ 0 & \text{in the contrary.} \end{cases}$$

For $m+k \pm 1 = 0$ and odd values of m the values of k are always even. Thence, using $\sin\vartheta \cdot P_1^1(\cos\vartheta) = -2P_1^{-1}(\cos\vartheta)$, $\mu = \cos\vartheta$, $P_\alpha^{-\beta}(\mu) = (-1)^\beta \frac{(\alpha-\beta)!}{(\alpha+\beta)!} P_\alpha^\beta(\mu)$, we get

$$C_{nm\ell k} = \begin{cases} C^1_{nm\ell k} = -\frac{2n+1}{4}\sqrt{\frac{(n-m)!(2\ell+m+1)!}{(n+m)!(2\ell-m-1)!}} \int_{-1}^{1} d\mu \, P_n^m(\mu) P_1^1(\mu) P_{2\ell}^{-m-1}(\mu) = \\ = -\frac{2n+1}{4}\sqrt{\frac{(n-m)!(2\ell-m-1)!}{(n+m)!(2\ell+m+1)!}} \int_{-1}^{1} d\mu \, P_n^m(\mu) P_1^1(\mu) P_{2\ell}^{m+1}(\mu) \\ \qquad \text{for } m+k+1=0, \end{cases}$$

$$C^2_{nmlk} = \frac{2n+1}{2}\sqrt{\frac{(n-m)!\,(2l+m-1)!}{(n+m)!\,(2l-m+1)!}}\int_{-1}^{1} d\mu\, P_n^m(\mu)\, P_1^{-1}(\mu)\, P_{2l}^{-m+1}(\mu) =$$

$$= \frac{2n+1}{2}\sqrt{\frac{(n-m)!\,(2l-m+1)!}{(n+m)!\,(2l+m-1)!}}\int_{-1}^{1} d\mu\, P_n^m(\mu)\, P_1^{-1}(\mu)\, P_{2l}^{m-1}(\mu)$$

for $m + k - 1 = 0$,

0 for $m - k \pm 1 \neq 0$.

Using the following formula according to [1], (1.16) :

$$\int_{-1}^{1} d\mu\, P_n^m(\mu)\, P_s^t(\mu)\, P_l^{m+t}(\mu) =$$

$$= \frac{2}{2l+1}\sqrt{\frac{(n+m)!\,(s+t)!\,(l+m+t)!}{(n-m)!\,(s-t)!\,(l-m-t)!}}\, C(n,s,l;m,t,m+t)\, C(n,s,l;0,0,0) \,, \qquad (1.12)$$

we get

$$\int_{-1}^{1} d\mu\, P_n^m(\mu)\, P_1^1(\mu)\, P_{2l}^{m+1}(\mu) =$$

$$= \frac{2}{4l+1}\sqrt{\frac{(n+m)!\, 2!\, (2l+m+1)!}{(n-m)!\, 0!\, (2l-m-1)!}}\, C(n,1,2l;m,1,m+1)\, C(n,1,2l;0,0,0) \,,$$

$$\int_{-1}^{1} d\mu\, P_n^m(\mu)\, P_1^{-1}(\mu)\, P_{2l}^{m-1}(\mu) =$$

$$= \frac{2}{4l+1}\sqrt{\frac{(n+m)!\, 0!\, (2l+m-1)!}{(n-m)!\, 2!\, (2l-m+1)!}}\, C(n,1,2l;m,-1,m-1)\, C(n,1,2l;0,0,0) \,,$$

whence

$$C^1_{nmlk} = -\frac{1}{\sqrt{2}}\,\frac{2n+1}{4l+1}\, C(n,1,2l;m,1,m+1)\, C(n,1,2l;0,0,0)\, \delta_{k,-m-1} \,,$$

$$C^2_{nmlk} = \frac{1}{\sqrt{2}}\,\frac{2n+1}{4l+1}\, C(n,1,2l;m,-1,m-1)\, C(n,1,2l;0,0,0)\, \delta_{k,-m+1} \,.$$

Here $C(j_1, j_2, j_3; m_1, m_2, m_3)$, $C(j_1, j_2, j_3; 0, 0, 0)$ are the Clebsch-Gordan coefficients defined as

$$C(j_1, j_2, j_3, 0, 0, 0) =$$

$$= (-1)^{g-j_3}\sqrt{\frac{(2j_3+1)(j_1-j_2+j_3)!\,(j_1+j_2-j_3)!\,(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}}\;\frac{g!}{(g-j_1)!\,(g-j_2)!\,(g-j_3)!}\;\delta_{j_1+j_2+j_3,\,2g} \,, \qquad (1.13)$$

$$C(j_1,j_2,j_3;m_1,m_2,m_1+m_2)=$$

$$=(-1)^{j_1-j_3+m_2}\sqrt{\frac{(2j_3+1)(j_3+m_1+m_2)!(j_3-m_1-m_2)!(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}\times$$

$$\times\sum_{i=\max(0,-j_1+j_2-m_1-m_2)}^{\min(-j_1+j_2+j_3,\,j_3-m_1-m_2)}\frac{(-1)^i(j_1+m_1+i)!(j_2+j_3-m_1-i)!}{i!(-j_1+j_2+j_3-i)!(j_1-j_2+m_1+m_2+i)!(j_3-m_1-m_2-i)!}\,, \qquad (1,14)$$

where the factorial is supposed to be defined for non-negative integer values including zero; therefore $j_1+j_2-j_3 \geq 0$, $j_1-j_2+j_3 \geq 0$, and $-j_1+j_2+j_3 \geq 0$; choosing for instance $j_1 = 0, 1, 2, \ldots$, and $j_2 = 0, 1, 2, \ldots$, the values of j_3 are limited: from the first relation follows $j_3 \leq j_1+j_2$, from the other two relations follows $|j_1-j_2| \leq j_3$; further $|m_1| \leq j_1$, $|m_2| \leq j_2$, $|m_3| \leq j_3$; for $m_3 \neq m_1+m_2$ there is $C(j_1,j_2,j_3;m_1,m_2,m_3)=0$, and for odd values of $j_1+j_2+j_3$ there is $C(j_1,j_2,j_3;0,0,0)=0$.

Finally the coefficients C_{nmlk} may be written as

$$C_{nmlk}=\frac{1}{\sqrt{2}}\frac{2n+1}{4l+1}\Big[C(n,1,2l;m,-1,m-1)\,\delta_{k,m-1}-C(n,1,2l;m,1,m+1)\,\delta_{k,-m-1}\Big]\,C(n,1,2l;0,0,0). \qquad (1,15)$$

Using the following formula in accordance with /1/, (3.15), (3.19)

$$C(j_1,j_2,j_3;m_1,m_2,m_1+m_2)=(-1)^{j_3-j_1-j_2}\,C(j_1,j_2,j_3;-m_1,-m_2,-m_1-m_2)\,, \qquad (1,16)$$

we get easily

$$C_{n,-mlk}=-\frac{1}{\sqrt{2}}\frac{2n+1}{4l+1}\Big[C(n,1,2l;m,-1,m-1)\,\delta_{k,m-1}-C(n,1,2l;m,1,m+1)\,\delta_{k,m+1}\Big]\,C(n,1,2l;0,0,0), \qquad (1,17)$$

and

$$C_{nml,-k}=-C_{n,-mlk}\,;\quad C_{n,-ml,-k}=-C_{nmlk}\,;\quad C_{nml,-k}-C_{n,-ml,-k}=-\big[C_{nmlk}-C_{n,-mlk}\big]\,. \qquad (1,18)$$

Now the condition (1,11) may be written as

$$\sum_{n=1}^{N}\sum_{m=1}^{n}\big[C_{nmlk}-C_{n,-mlk}\big]\,\varphi_{nm}(R)=0\,, \qquad (1,19)$$

$$\left[\begin{array}{l} n+m \text{ even};\ N,\ n,\ m \text{ odd};\\ l=0,1,2,\ldots,\frac{N-1}{2};\quad l\leq\frac{n-1}{2},\ \frac{m-1}{2};\\ k=0,2,\ldots,2l \text{ even.}\end{array}\right]$$

In the P_1-approximation ($N=1$) there is $n=1$; $m=1$; $l=0$; $k=0$, and, using (1,18) for $k=0$, the condition is

$$C_{1100}\,\varphi_{11}(R)=0\,.$$

In the P_3-approximation ($N=3$) there is $n=1$; $m=1$; $l=0$; $k=0$, and $l = 1$; $k = 0, 2$, and $n = 3$; $m = 1, 3$; $l = 1$; $k = 0, 2$, and, using (1.18) for $k = 0$, the conditions are

$$c_{1100}\,\varphi_{11}(R) = 0 \; ,$$

$$c_{1110}\,\varphi_{11}(R) + c_{3110}\,\varphi_{31}(R) = 0 \; ,$$

$$c_{1,-1,12}\,\varphi_{11}(R) + c_{3,-1,12}\,\varphi_{31}(R) + c_{3,-3,12}\,\varphi_{33}(R) = 0 \; .$$

We have written only the non-zero coefficients, the values of which may be found, for instance, by using the reference [illegible], where the designation is $(j_1\, j_2\, m_1\, m_2 \mid j_1\, j_2\, j\, m) = C(j_1, j_2, j;\, m_1, m_2, m)$. However, we do not need these values in this case, as we can easily see that the conditions for both the P_1- and the P_3- approximations, respectively, are

$$P_1 : \qquad \varphi_{11}(R) = 0 \; , \qquad (1.20)$$

$$P_3 : \qquad \varphi_{11}(R) = 0 \; , \quad \varphi_{31}(R) = 0 \; , \quad \varphi_{33}(R) = 0 \; . \qquad (1.21)$$

This result may be obtained in general using, for instance, the principle of mathematical induction. Hence the conditions (1.19) may be replaced by more simple relations

$$\varphi_{nm}(R) = 0 \qquad \text{for} \quad n = 1, 3, \ldots, N; \quad m = 1, 3, \ldots, n; \qquad (1.22)$$

$n + m$ even; N, n, m odd.

The components of the vector current $\vec{J}(\vec{r})$ in the cylindrical symmetry are

$$J_r = \frac{1}{\sqrt{2}}\left[\varphi_{1,-1} - \varphi_{11}\right] = -\frac{2}{\sqrt{2}}\,\varphi_{11} \; ,$$

$$J_\psi = \frac{i}{r\sqrt{2}}\left[\varphi_{1,-1} + \varphi_{11}\right] = 0 \; , \qquad (1.23)$$

$$J_z = \varphi_{10} = 0 \; .$$

Hence the boundary condition (1.20) for the P_1-approximation is

$$J_r(R) = 0 \; . \qquad (1.24)$$

The complete formulations of the problem in both the P_1-approximation and the diffusion approximation are given at the end of the chapter [illegible].

2. Finite cavity

In this chapter the more general case of a finite straight circular cavity of the radius R and the length H is discussed. For $R \ll H$ the cavity is supposed to be infinite and this has been already discussed in the previous chapter. Further we suppose $R \sim H$. Assuming the boundary of the cavity to be not irradiated from outside the boundary condition is defined as

$$\varphi(R,x,\vartheta,\psi) = \begin{cases} 0 & \text{for} \quad 0 \le \vartheta < \vartheta_1,\ \vartheta_2 < \vartheta \le \pi,\ -\frac{\pi}{2} \le \psi \le \frac{\pi}{2}, \\ \varphi(R,x',\vartheta,\pi-\psi) & \text{for} \quad \vartheta_1 \le \vartheta \le \vartheta_2,\ -\frac{\pi}{2} \le \psi \le \frac{\pi}{2}. \end{cases} \tag{2.1}$$

The following symmetry condition is true:

$$\varphi(R,x,\vartheta,\psi) = \varphi(R,x,\vartheta,-\psi) \qquad \text{for} \quad 0 \le \vartheta \le \pi,\ 0 \le \psi < 2\pi, \tag{2.2}$$

the equivalence of which is

$$\varphi_{nm}(R,x) = (-1)^m \varphi_{n,-m}(R,x) \qquad \text{for} \quad n = 0,1,2,\dots;\ m = -n,\dots,n. \tag{2.3}$$

Now the boundary condition (2.1) will be discussed. According to the picture on the next page, there is

$tg\,\vartheta_1 = \frac{\varrho}{x}$, $\vartheta_1 = arctg\frac{\varrho}{x}$; $tg(\pi-\vartheta_2) = \frac{\varrho}{H-x} = -tg\,\vartheta_2$, $\vartheta_2 = \pi - arctg\frac{\varrho}{H-x} = arctg\frac{\varrho}{x-H}$; $tg\,\vartheta = \frac{\varrho}{x-x'} \gtrless 0$, $\vartheta = arctg\frac{\varrho}{x-x'}$, $\vartheta = \frac{\pi}{2}$ for $x' = x$; further $x' = x - \frac{\varrho}{tg\,\vartheta} = x - \varrho\,cotg\,\vartheta$, and $\varrho^2 = 2R^2[1-\cos(\pi-2\psi)] = 4R^2\cos^2\psi$, $\varrho = 2R\cos\psi$ for $\psi \in \langle -\frac{\pi}{2}, \frac{\pi}{2}\rangle$; hence

$$\vartheta_1 = arctg\frac{2R\cos\psi}{x}, \tag{2.4}$$

$$\vartheta_2 = arctg\frac{2R\cos\psi}{x-H} = \pi - arctg\frac{2R\cos\psi}{H-x}, \tag{2.5}$$

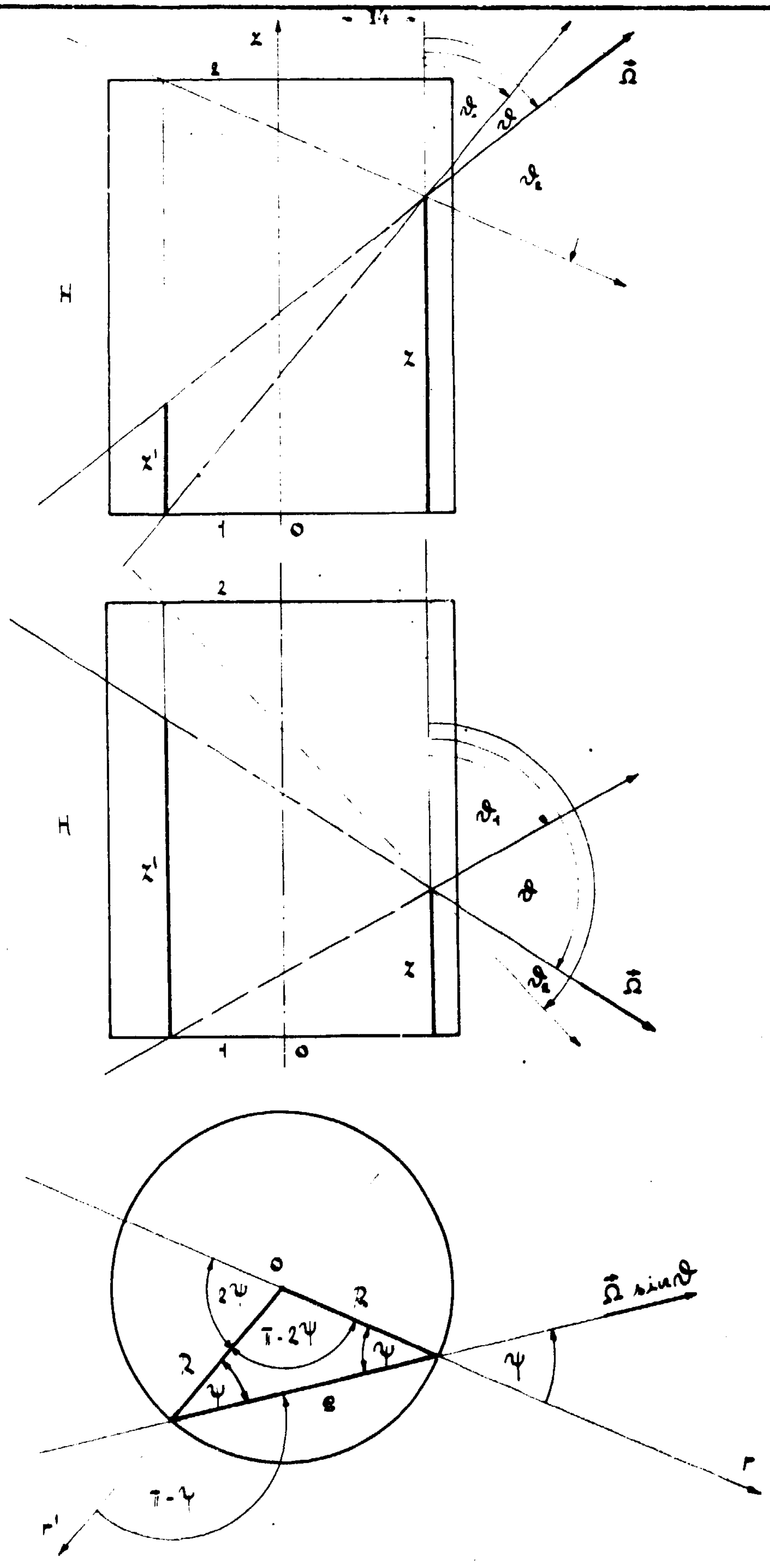
$$\vartheta = arctg\frac{2R\cos\psi}{x-x'}, \tag{2.6}$$

$$x' = x - \frac{2R\cos\psi}{tg\,\vartheta} = x - 2R\cos\psi\,cotg\,\vartheta. \tag{2.7}$$

Using the series (1.1), (1.7), multiplying the condition by the spherical harmonic $\vec{\Omega}\vec{n}P_{2l}^{k}(\vec{\Omega})$, and integrating it over $\vec{\Omega}\vec{n} < 0$, we get the Wladimirow boundary condition in the P_N-approximation

$$\sum_{n=0}^{N}\sum_{m=-n}^{n}\frac{2n+1}{4\pi}\int\limits_{\vec{\Omega}\vec{n}<0} d\vec{\Omega}\ \vec{\Omega}\vec{n}\ P_n^m(\vec{\Omega})P_{2l}^k(\vec{\Omega})\varphi_{nm}(R,x) =$$

$$= \sum_{n=0}^{N}\sum_{m=-n}^{n}(-1)^m\frac{2n+1}{4\pi}\int\limits_{(\vartheta_1\le\vartheta\le\vartheta_2,\ -\frac{\pi}{2}\le\psi\le\frac{\pi}{2})} d\vec{\Omega}\ \vec{\Omega}\vec{n}\ P_n^m(\vec{\Omega})P_{2l}^k(\vec{\Omega})\varphi_{nm}(R,x'). \tag{2.8}$$


H
z
z'
1
0
2
R
2ψ
π - 2ψ
ψ
π - ψ
r
r'

The coefficient on the left of (2,3) is

$$K_{nmlk} = (-1)^{m+k}\,\frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}}\int_{-\pi/2}^{\pi/2} e^{i(m+k)\psi}\cos\psi\, d\psi \int_0^{\pi} d\vartheta\, \sin^2\vartheta\, P_n^m(\cos\vartheta)\, P_{2l}^k(\cos\vartheta).$$

Here $K_{0000} = \frac{1}{4}$ as $P_0^0(\cos\vartheta) = 1$; further, analogicaly[1] to the calculation of the C_{nmlk} in chapter 1, and to (1,17), (1,18), we get

$$K_{nmlk} = \frac{1}{2\sqrt{2}}\,\frac{2n+1}{4l+1}\left[C(n,1,2l;m,-1,m-1)\,\delta_{k,-m+1} - C(n,1,2l;m,1,m+1)\,\delta_{k,-m-1}\right] C(n,1,2l;0,0,0),$$

$$K_{n,-mlk} = -\frac{1}{2\sqrt{2}}\,\frac{2n+1}{4l+1}\left[C(n,1,2l;m,-1,m-1)\,\delta_{k,m-1} - C(n,1,2l;m,1,m+1)\,\delta_{k,m+1}\right] C(n,1,2l;0,0,0),$$

$$K_{nml,-k} = -K_{n,-mlk}, \quad K_{n,-ml,-k} = -K_{nmlk}.$$

According to the definition of the Clebsh-Gordan coefficients there is $K_{nmlk} = 0$ for even values of $n > 0$. Further $K_{00lk} = 0$ for $l \neq 0$, which also follows from $\int_{-1}^{1} d\mu\, P_l^k(\mu)\, P_n^k(\mu) = \frac{(n+k)!}{(n-k)!}\,\frac{2}{2n+1}\,\delta_{nl}$: using the relations on p. 9 we can write

$$K_{00lk} = (-1)^k\,\frac{1}{4\pi}\sqrt{\frac{(2l-k)!}{(2l+k)!}}\int_{-\pi/2}^{\pi/2} e^{ik\psi}\cos\psi\, d\psi \int_0^{\pi} d\vartheta\, \sin^2\vartheta\, P_{2l}^k(\cos\vartheta);$$

$$\int_{-\pi/2}^{\pi/2} e^{ik\psi}\cos\psi\, d\psi = \begin{cases} \frac{\pi}{2} & \text{for either } k+1=0 \text{ or } k-1=0, \\ 0 & \text{in the contrary;} \end{cases}$$

$$k = 1: \quad \int_0^{\pi} d\vartheta\, \sin^2\vartheta\, P_{2l}^k(\cos\vartheta) = \int_{-1}^{1} d\mu\, P_1^1(\mu)\, P_{2l}^1(\mu) = 0,$$

$$k = -1: \quad \int_0^{\pi} d\vartheta\, \sin^2\vartheta\, P_{2l}^k(\cos\vartheta) = \int_{-1}^{1} d\mu\, P_1^1(\mu)\, P_{2l}^{-1}(\mu) = -\frac{(2l-1)!}{(2l+1)!}\int_{-1}^{1} d\mu\, P_1^1(\mu)\, P_{2l}^1(\mu) = 0;$$

Hence $K_{00lk} = 0$ for $l > 0$.

For $n > 0$, $l = 0$ there is

$$K_{nm00} = (-1)^m\,\frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!}{(n+m)!}}\int_{-\pi/2}^{\pi/2} e^{im\psi}\cos\psi\, d\psi \int_0^{\pi} d\vartheta\, \sin^2\vartheta\, P_n^m(\cos\vartheta);$$

$$\int_{-\pi/2}^{\pi/2} e^{im\psi} \cos\psi \, d\psi = \begin{cases} \frac{\pi}{2} & \text{for either } m+1=0 \text{ or } m-1=0, \\ 0 & \text{in the contrary;} \end{cases}$$

$$m = 1: \quad \int_0^{\pi} d\vartheta \sin^2\vartheta \, P_n^m(\cos\vartheta) = \int_{-1}^{1} d\mu \, P_1^1(\mu) P_n^1(\mu) = \frac{4}{3}\delta_{n1} \, ,$$

$$m = -1: \quad \int_0^{\pi} d\vartheta \sin^2\vartheta \, P_n^m(\cos\vartheta) = \int_{-1}^{1} d\mu \, P_1^1(\mu) P_n^{-1}(\mu) = -\frac{(n-1)!}{(n+1)!}\int_{-1}^{1} d\mu \, P_1^1(\mu) P_n^1(\mu) = -\frac{2}{3}\delta_{n1} \, ;$$

hence $K_{nmoo} = 0$ for $n \neq 1$, $K_{11oo} = -\frac{1}{2\sqrt{2}}$, $K_{1,-1,oo} = \frac{1}{2\sqrt{2}}$.

The values of the Clebsh-Gordan coefficients $C(n,\ 1,\ 2l;\ m,\ -1,\ m-1)$, $C(n,\ 1,\ 2l;\ m,\ 1,\ m+1)$, and $C(n,\ 1,\ 2l;\ 0,\ 0,\ 0)$ may be found, for instance, using the tables /[illegible]/, or they may be calculated using the following table, which may be found, for instance, in /[illegible]/, p. 5, or in /[illegible]/, (3.28):

$C(n,\ 1,\ 2l;\ m,\ t,\ m+t)$ for $n > 0$ odd, $m = -n, \ldots, n$:

$2l$ \ t	-1	0	1
$n+1$	$\sqrt{\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+2)}}$	$\sqrt{\frac{(n-m+1)(n+m+1)}{(n+1)(2n+1)}}$	$\sqrt{\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+2)}}$
$n-1$	$\sqrt{\frac{(n+m-1)(n+m)}{2n(2n+1)}}$	$-\sqrt{\frac{(n+m)(n-m)}{n(2n+1)}}$	$\sqrt{\frac{(n-m-1)(n-m)}{2n(2n+1)}}$

Example: $l = 0$, $n = 1$; $m = -1$ (second line, third column),
$m = 0$ (second line, second column),
$m = 1$ (second line, first column);

we get the same result as above: $K_{11oo} = -\frac{1}{2\sqrt{2}}$, $K_{1,-1,oo} = \frac{1}{2\sqrt{2}}$,
where the formula for k_{nmlk} was used.

Now the integral on the right of (2.8) will be discussed. From $\operatorname{tg}\vartheta = \frac{2R\cos\psi}{x-x'}$ follows $d\operatorname{tg}\vartheta = \frac{d\vartheta}{\cos^2\vartheta} = \frac{2R\cos\psi}{(x-x')^2}dx'$; further for $0 \leq \vartheta \leq \pi/2$ there is

$$\cos\vartheta = \frac{1}{\sqrt{1+\operatorname{tg}^2\vartheta}} = \frac{|x-x'|}{\sqrt{(x-x')^2+4R^2\cos^2\psi}} \ ; \quad \sin\vartheta = \frac{\operatorname{tg}\vartheta}{\sqrt{1+\operatorname{tg}^2\vartheta}} =$$

$$= \frac{|x-x'|}{x-x'} \frac{2R\cos\psi}{\sqrt{(x-x')^2+4R^2\cos^2\psi}}$$

and for $\pi/2 \leq \vartheta \leq \pi$ there is

$$\cos\vartheta = -\frac{1}{\sqrt{1+\operatorname{tg}^2\vartheta}} = -\frac{|x-x'|}{\sqrt{(x-x')^2+4R^2\cos^2\psi}} \ , \quad \sin\vartheta = -\frac{\operatorname{tg}\vartheta}{\sqrt{1+\operatorname{tg}^2\vartheta}} \ .$$

$$= -\frac{|x-x'|}{x-x'}\,\frac{2R\cos\Psi}{\sqrt{(x-x')^2+4R^2\cos^2\Psi}}\,;$$ hence $$d\vartheta = \cos^2\vartheta\,\frac{2R\cos\Psi}{(x-x')^2}\,dx' =$$

$$= \frac{2R\cos\Psi}{(x-x')^2+4R^2\cos^2\Psi}\,dx'\,;$$ further $0 \le \vartheta_1(x) \le \pi/2$, $\pi/2 \le \vartheta_2(x) \le \pi$, where

$\vartheta_1 = \operatorname{arctg}\dfrac{2R\cos\Psi}{x}$, $\vartheta_2 = \operatorname{arctg}\dfrac{2R\cos\Psi}{x-H}$; the following formulae will be used:

$m \ge 0$:
$$P_n^m(\cos\vartheta) = \sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]} \frac{(-1)^\alpha(2n-2\alpha)!}{2^n\,\alpha!\,(n-\alpha)!\,(n-m-2\alpha)!}\sin^m\vartheta\,\cos^{n-m-2\alpha}\vartheta\,, \qquad (2.10)$$

$k \ge 0$:
$$P_{2l}^k(\cos\vartheta) = \sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]} \frac{(-1)^\beta(4l-2\beta)!}{2^{2l}\,\beta!\,(2l-\beta)!\,(2l-k-2\beta)!}\sin^k\vartheta\,\cos^{2l-k-2\beta}\vartheta\,. \qquad (2.11)$$

For $m < 0$, $k < 0$ there is $P_n^{-m}(\cos\vartheta) = (-1)^m\dfrac{(n-m)!}{(n+m)!}P_n^m(\cos\vartheta)$,

$P_{2l}^{-k}(\cos\vartheta) = (-1)^k\dfrac{(2l-k)!}{(2l+k)!}P_{2l}^k(\cos\vartheta)$. Now for $m \ge 0$, $k \ge 0$ there may be written

$$(-1)^m\frac{2n+1}{4\pi}\int\limits_{\left[\substack{\vartheta_1\le\vartheta\le\vartheta_2\\ -\frac{\pi}{2}\le\Psi\le\frac{\pi}{2}}\right]} d\vec\Omega\;\vec\Omega\vec n\;P_n^m(\vec\Omega)\,P_{2l}^k(\vec\Omega)\,\varphi_{nm}(R,x') =$$

$$= (-1)^k\frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}}\int_{-\pi/2}^{\pi/2} e^{i(m+k)\Psi}\cos\Psi\int_{\vartheta_1(x)}^{\vartheta_2(x)}\sin^2\vartheta\,P_n^m(\cos\vartheta)P_{2l}^k(\cos\vartheta)\,\varphi_{nm}(R,x')\,d\Psi\,d\vartheta =$$

$$= \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}}\sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]}\sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]}\frac{(-1)^{k+\alpha+\beta}(2n-2\alpha)!\,(4l-2\beta)!}{2^{n+2l}\,\alpha!\,\beta!\,(n-\alpha)!\,(2l-\beta)!\,(n-m-2\alpha)!\,(2l-k-2\beta)!}\times$$

$$\times\int_{-\pi/2}^{\pi/2} e^{i(m+k)\Psi}\cos\Psi\int_{\vartheta_1(x)}^{\vartheta_2(x)}\sin^{m+k+2}\vartheta\,\cos^{n+2l-m-k-2\alpha-2\beta}\vartheta\,\varphi_{nm}(R,x')\,d\Psi\,d\vartheta =$$

$$= \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}}\sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]}\sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]}\frac{(-1)^{k+\alpha+\beta}(2n-2\alpha)!\,(4l-2\beta)!}{2^{n+2l-m-k-4}\,\alpha!\,\beta!\,(n-\alpha)!\,(2l-\beta)!\,(n-m-2\alpha)!\,(2l-k-2\beta)!}\times$$

$$\times R^{m+k+3}\int_0^H (x-x')^{n+2l-m-k-2\alpha-2\beta}\int_0^{\pi/2}\frac{\left[\cos(m+k)\Psi + i\sin(m+k)\Psi\right]\cos^{m+k+4}\Psi}{\left[(x-x')^2+4R^2\cos^2\Psi\right]^{\frac{n}{2}+l-\alpha-\beta+2}}\,d\Psi\;\varphi_{nm}(R,x')\,dx'\,.$$

$$= \int_0^H K_{nmlk}(x,x')\,\varphi_{nm}(R,x')\,dx' + i\int_0^H K^*_{nmlk}(x,x')\,\varphi_{nm}(R,x')\,dx' .$$

The integral including the complex unit ' i ' is equal to zero, as the boundary condition is real. Hence

$$\int_0^H K_{nmlk}(x,x')\,\varphi_{nm}(R,x')\,dx' =$$

$$= (-1)^k \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}} \int_{-\pi/2}^{\pi/2} \cos(m+k)\psi \cos\psi \int_{\vartheta_1(x)}^{\vartheta_2(x)} \sin^2\vartheta\, P_n^m(\cos\vartheta)\, P_{2l}^k(\cos\vartheta)\, \varphi_{nm}(R,x')\, d\psi\, d\vartheta ,$$

$$\int_0^H K_{n,-mlk}(x,x')\,\varphi_{n,-m}(R,x')\,dx' = \int_0^H K_{n,-mlk}(x,x')(-1)^m \varphi_{nm}(R,x')\,dx' =$$

$$= (-1)^{m+k} \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}} \int_{-\pi/2}^{\pi/2} \cos(m-k)\psi \cos\psi \int_{\vartheta_1(x)}^{\vartheta_2(x)} \sin^2\vartheta\, P_n^m(\cos\vartheta)\, P_{2l}^k(\cos\vartheta)(-1)^m \varphi_{nm}(R,x')\, d\psi\, d\vartheta ,$$

$$\int_0^H K_{nml,-k}(x,x')\,\varphi_{nm}(R,x')\,dx' =$$

$$= (-1)^{2k} \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}} \int_{-\pi/2}^{\pi/2} \cos(m-k)\psi \cos\psi \int_{\vartheta_1(x)}^{\vartheta_2(x)} \sin^2\vartheta\, P_n^m(\cos\vartheta)\, P_{2l}^k(\cos\vartheta)\, \varphi_{nm}(R,x')\, d\psi\, d\vartheta ,$$

$$\int_0^H K_{n,-ml,-k}(x,x')\,\varphi_{n,-m}(R,x')\,dx' = \int_0^H K_{n,-ml,-k}(x,x')(-1)^m \varphi_{nm}(R,x')\,dx' =$$

$$= (-1)^{m+2k} \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}} \int_{-\pi/2}^{\pi/2} \cos(m+k)\psi \cos\psi \int_{\vartheta_1(x)}^{\vartheta_2(x)} \sin^2\vartheta\, P_n^m(\cos\vartheta)\, P_{2l}^k(\cos\vartheta)(-1)^m \varphi_{nm}(R,x')\, d\psi\, d\vartheta ,$$

where

$$K_{nmlk}(x,x') =$$

$$= \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}} \sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]} \sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]} \frac{(-1)^{k+\alpha+\beta}(2n-2\alpha)!\,(4l-2\beta)!}{2^{n+2l-m-k-4}\,\alpha!\,\beta!\,(n-\alpha)!\,(2l-\beta)!\,(n-m-2\alpha)!\,(2l-k-2\beta)!} \times$$

$$\times R^{m+k+3}(x-x')^{n+2l-m-k-2\alpha-2\beta} \int_0^{\pi/2} \frac{\cos(m+k)\psi \cos^{m+k+4}\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{\frac{n}{2}+l-\alpha-\beta+2}}\, d\psi ,$$

$K_{n,-ml k}(x,x') =$

$$= (-1)^{m}\frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!(2l-k)!}{(n+m)!(2l+k)!}}\sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]}\sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]}\frac{(-1)^{k+\alpha+\beta}(2n-2\alpha)!(4l-2\beta)!}{2^{n+2l-m-k-4}\alpha!\beta!(n-\alpha)!(2l-\beta)!(n-m-2\alpha)!(2l-k-2\beta)!}\times$$

$$\times R^{m+k+3}(x-x')^{n+2l-m-k-2\alpha-2\beta}\int_0^{\pi/2}\frac{\cos(m-k)\psi\,\cos^{m+k+4}\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{\frac{n}{2}+l-\alpha-\beta+2}}d\psi\,,$$

$K_{nml,-k}(x,x') =$

$$= (-1)^{k}\frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!(2l-k)!}{(n+m)!(2l+k)!}}\sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]}\sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]}\frac{(-1)^{k+\alpha+\beta}(2n-2\alpha)!(4l-2\beta)!}{2^{n+2l-m-k-4}\alpha!\beta!(n-\alpha)!(2l-\beta)!(n-m-2\alpha)!(2l-k-2\beta)!}\times$$

$$\times R^{m+k+3}(x-x')^{n+2l-m-k-2\alpha-2\beta}\int_0^{\pi/2}\frac{\cos(m-k)\psi\,\cos^{m+k+4}\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{\frac{n}{2}+l-\alpha-\beta+2}}d\psi\,,$$

$K_{n,-ml,-k}(x,x') =$

$$= (-1)^{m+k}\frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!(2l-k)!}{(n+m)!(2l+k)!}}\sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]}\sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]}\frac{(-1)^{k+\alpha+\beta}(2n-2\alpha)!(4l-2\beta)!}{2^{n+2l-m-k-4}\alpha!\beta!(n-\alpha)!(2l-\beta)!(n-m-2\alpha)!(2l-k-2\beta)!}\times$$

$$\times R^{m+k+3}(x-x')^{n+2l-m-k-2\alpha-2\beta}\int_0^{\pi/2}\frac{\cos(m+k)\psi\,\cos^{m+k+4}\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{\frac{n}{2}+l-\alpha-\beta+2}}d\psi\,,$$

$$K_{nml,-k}(x,x') = (-1)^{m+k}K_{n,-ml k}(x,x')\,,$$

$$K_{n,-ml,-k}(x,x') = (-1)^{m+k}K_{nml k}(x,x')\,.$$

According to the definition of the Clebsh-Gordan coefficients the values of l are in relation with the values of n: $l = \frac{n-1}{2}, \frac{n+1}{2}$. Hence the boundary condition may be written as a system of integral equations

$$\sum_{n=0}^{N}\sum_{m=-n}^{n}K_{nml k}\varphi_{nm}(R,x) = \sum_{n=0}^{N}\sum_{m=-n}^{n}\int_0^{H}K_{nml k}(x,x')\varphi_{nm}(R,x')\,dx'\,, \qquad (2.12)$$

$\Big[\varphi_{n,-m}(R,x) = (-1)^{m}\varphi_{nm}(R,x)\,;$
N is supposed to be odd;
$l = 0, 1, 2, \ldots, \frac{N-1}{2}$; $k = 0, \pm 1, \pm 2, \ldots, \pm 2l$;
$\left[\,n = 0;\right.$ and $n > 0$ odd, $l = \frac{n-1}{2}, \frac{n+1}{2}$ for $K_{nml k}$.$\left.\right]\Big]$

which may be rewritten into a more convenient form :

$$K_{00\ell\kappa}\varphi_{00}(R,x) + \sum_{n=1}^{N} K_{n0\ell\kappa}\varphi_{n0}(R,x) + \sum_{n=1}^{N}\sum_{m=1}^{n}\left[K_{nm\ell\kappa} + (-1)^{m} K_{n,-m\ell\kappa}\right]\varphi_{nm}(R,x) =$$

$$= \int_0^H K_{00\ell\kappa}(x,x')\varphi_{00}(R,x')\,dx' + \sum_{n=1}^{N}\int_0^H K_{n0\ell\kappa}(x,x')\varphi_{n0}(R,x')\,dx' +$$

$$+ \sum_{n=1}^{N}\sum_{m=1}^{n}\int_0^H\left[K_{nm\ell\kappa}(R,x') + (-1)^{m} K_{n,-m\ell\kappa}(x,x')\right]\varphi_{nm}(R,x')\,dx' , \tag{2.13}$$

$$\left[\begin{array}{l} N \text{ odd: } \ell = 0,\ 1,\ 2,\ \dots,\ \frac{N-1}{2},\quad k = 0,\ \pm 1,\ \pm 2,\ \dots,\ \pm 2\ell; \\ \left[\ell = \frac{n-1}{2},\ \frac{n+1}{2} \text{ for } K_{nm\ell k}\right] \end{array}\right]$$

with the coefficients

$$K_{00\ell\kappa} = \frac{1}{4}\,\delta_{\ell 0}\,\delta_{\kappa 0} , \tag{2.14}$$

$$K_{nm00} = \frac{4}{3}\,\delta_{n1}\,\delta_{m1} , \quad \text{and} \quad K_{nm00} = -\frac{2}{3}\,\delta_{n1}\,\delta_{m,-1} , \tag{2.15}$$

$$K_{nm\ell\kappa} = \frac{1}{2\sqrt{2}}\,\frac{2n+1}{4\ell+1}\left[C(n,1,2\ell;m,-1,m-1)\,\delta_{\kappa,m-1} - C(n,1,2\ell;m,1,m+1)\,\delta_{\kappa,m+1}\right] C(n,1,2\ell;0,0,0), \tag{2.16}$$

$$K_{n,-m\ell\kappa} = -\frac{1}{2\sqrt{2}}\,\frac{2n+1}{4\ell+1}\left[C(n,1,2\ell;m,-1,m-1)\,\delta_{\kappa,m-1} - C(n,1,2\ell;m,1,m+1)\,\delta_{\kappa,m+1}\right] C(n,1,2\ell;0,0,0), \tag{2.17}$$

$$K_{nm\ell,-\kappa} = -K_{n,-m\ell\kappa} , \tag{2.18}$$

$$K_{n,-m\ell,-\kappa} = -K_{nm\ell\kappa} , \tag{2.19}$$

$$K_{nm\ell,-\kappa} + (-1)^{m} K_{n,-m\ell,-\kappa} = -(-1)^{m}\left[K_{nm\ell\kappa} + (-1)^{m} K_{n,-m\ell\kappa}\right] \tag{2.20}$$

on the left, and with the integral kernels

$$K_{nm\ell\kappa}(x,x') =$$

$$= \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2\ell-\kappa)!}{(n+m)!\,(2\ell+\kappa)!}}\sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]}\sum_{\beta=0}^{\left[\frac{2\ell-\kappa}{2}\right]}\frac{(-1)^{n+\alpha+\beta}\,(2n-2\alpha)!\,(4\ell-2\beta)!}{2^{n+2\ell-m-\kappa-4}\,\alpha!\,\beta!\,(n-\alpha)!\,(2\ell-\beta)!\,(n-m-2\alpha)!\,(2\ell-\kappa-2\beta)!}\times$$

$$\times R^{m+\kappa+3}\,(x-x')^{n+2\ell-m-\kappa-2\alpha-2\beta}\int_0^{\pi/2}\frac{\cos(m+\kappa)\psi\,\cos^{m+\kappa+4}\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{\frac{n}{2}+\ell-\alpha-\beta+2}}\,d\psi , \tag{2.21}$$

$$K_{n,-mlk}(x,x') =$$

$$= (-1)^m \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}} \sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]} \sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]} \frac{(-1)^{k+\alpha+\beta}\,(2n-2\alpha)!\,(4l-2\beta)!}{2^{n+2l-m-k-4}\,\alpha!\,\beta!\,(n-\alpha)!\,(2l-\beta)!\,(n-m-2\alpha)!\,(2l-k-2\beta)!} \times$$

$$\times R^{m+k+3}\,(x-x')^{n+2l-m-k-2\alpha-2\beta} \int_0^{\pi/2} \frac{\cos(m-k)\psi\,\cos^{m+k+4}\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{\frac{n}{2}+l-\alpha-\beta+2}}\,d\psi\,, \qquad (2.22)$$

$$K_{nml,-k}(x,x') =$$

$$= (-1)^k \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}} \sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]} \sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]} \frac{(-1)^{k+\alpha+\beta}\,(2n-2\alpha)!\,(4l-2\beta)!}{2^{n+2l-m-k-4}\,\alpha!\,\beta!\,(n-\alpha)!\,(2l-\beta)!\,(n-m-2\alpha)!\,(2l-k-2\beta)!} \times$$

$$\times R^{m+k+3}\,(x-x')^{n+2l-m-k-2\alpha-2\beta} \int_0^{\pi/2} \frac{\cos(m-k)\psi\,\cos^{m+k+4}\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{\frac{n}{2}+l-\alpha-\beta+2}}\,d\psi\,, \qquad (2.23)$$

$$K_{n,-ml,-k}(x,x') =$$

$$= (-1)^{m+k} \frac{2n+1}{4\pi}\sqrt{\frac{(n-m)!\,(2l-k)!}{(n+m)!\,(2l+k)!}} \sum_{\alpha=0}^{\left[\frac{n-m}{2}\right]} \sum_{\beta=0}^{\left[\frac{2l-k}{2}\right]} \frac{(-1)^{k+\alpha+\beta}\,(2n-2\alpha)!\,(4l-2\beta)!}{2^{n+2l-m-k-4}\,\alpha!\,\beta!\,(n-\alpha)!\,(2l-\beta)!\,(n-m-2\alpha)!\,(2l-k-2\beta)!} \times$$

$$\times R^{m+k+3}\,(x-x')^{n+2l-m-k-2\alpha-2\beta} \int_0^{\pi/2} \frac{\cos(m+k)\psi\,\cos^{m+k+4}\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{\frac{n}{2}+l-\alpha-\beta+2}}\,d\psi\,, \qquad (2.24)$$

$$K_{nml,-k}(x,x') = (-1)^{m+k} K_{n,-mlk}(x,x')\,, \qquad (2.25)$$

$$K_{n,-ml,-k}(x,x') = (-1)^{m+k} K_{nmlk}(x,x')\,, \qquad (2.26)$$

$$K_{nml,-k}(x,x') + (-1)^m K_{n,-ml,-k}(x,x') = (-1)^k\left[K_{nmlk}(x,x') + (-1)^m K_{n,-mlk}(x,x')\right] \qquad (2.27)$$

on the right. In the P_1-approximation $(N = 1)$ we get

$$K_{0000}\,\varphi_{00}(R,x) + K_{1000}\,\varphi_{10}(R,x) + \left[K_{1100} - K_{1,-100}\right]\varphi_{11}(R,x) =$$

$$= \int_0^H K_{0000}(x,x')\,\varphi_{00}(R,x')\,dx' + \int_0^H K_{1000}(x,x')\,\varphi_{10}(R,x')\,dx' +$$

$$+ \int_0^H \left[K_{1100}(x,x') - K_{1,-100}(x,x')\right]\varphi_{11}(R,x')\,dx'\,, \qquad (2.28)$$

where $K_{0000} = \frac{1}{4}$ according to (2.14), $K_{1000} = 0$ according to (2.15), and

$k_{11oo} = -\frac{1}{2\sqrt{2}}$, $k_{1,-1,oo} = \frac{1}{2\sqrt{2}}$ in accordance with the page 16 ; further

$$\Phi(R,z) = \varphi_{oo}(R,z) \qquad (2,29)$$

is the scalar flux, and

$$J_r = \frac{1}{\sqrt{2}}[\varphi_{1,-1} - \varphi_{11}] = -\frac{2}{\sqrt{2}}\varphi_{11} ,$$

$$J_\psi = \frac{i}{R\sqrt{2}}[\varphi_{1,-1} + \varphi_{11}] = 0 , \qquad (2,30)$$

$$J_z = \varphi_{10}$$

are components of the vector current in the point (R, z); hence we may write the boundary condition as an integral equation

$$\Phi(R,z) + 2J_r(R,z) = \int_0^H K_1(z,z')\Phi(R,z')dz' + \int_0^H K_2(z,z')J_r(R,z')dz' + \int_0^H K_3(z,z')J_z(R,z')dz' , \qquad (2,31)$$

$$K_1(z,z') = \frac{16R^3}{\pi}\int_0^{[\pi/2]} \frac{\cos^4\psi}{[(z-z')^2 + 4R^2\cos^2\psi]^2} d\psi , \qquad (2,32)$$

$$K_2(z,z') = -\frac{96R^4}{\pi}\int_0^{[\pi/2]} \frac{\cos^6\psi}{[(z-z')^2 + 4R^2\cos^2\psi]^{5/2}} d\psi , \qquad (2,33)$$

$$K_3(z,z') = \frac{48R^3}{\pi}(z-z')\int_0^{[\pi/2]} \frac{\cos^4\psi}{[(z-z')^2 + 4R^2\cos^2\psi]^{5/2}} d\psi , \qquad (2,34)$$

where the entire [1/2] is equal to zero. Now we may write the problem for the neutron distribution in the multi-group P_1-approximation

$$\left[\frac{\partial}{\partial r} + \frac{1}{r}\right]J_r^g + \frac{\partial}{\partial z}J_z^g + \Sigma_{rem}^g\Phi^g = \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \Sigma_s^{h\to g}\Phi^h + \sum_{h=1}^{M} \mathcal{F}^g \nu_f^h \Sigma_f^h \Phi^h + S^g , \qquad (2,35)$$

$$\frac{1}{3}\frac{\partial}{\partial r}\Phi^g + \Sigma_{tr}^g J_r^g = \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \mu_s^{h\to g}\Sigma_s^{h\to g} J_r^h + S_r^g , \qquad (2,36)$$

$$\frac{1}{3}\frac{\partial}{\partial z}\Phi^g + \Sigma_{tr}^g J_z^g = \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \mu_s^{h\to g}\Sigma_s^{h\to g} J_z^h + S_z^g , \qquad (2,37)$$

$$\phi^{g}(R,x)+2\,J_{r}^{g}(R,x)=\int_{0}^{H}K_{1}(x,x')\,\phi^{g}(R,x')\,dx'+\int_{0}^{H}K_{2}(x,x')\,J_{r}^{g}(R,x')\,dx'+\int_{0}^{H}K_{3}(x,x')\,J_{x}^{g}(R,x')\,dx', \tag{2.33}$$

where

$$\Sigma_{rem}^{g}=\Sigma^{g}-\Sigma_{p}^{g\to g}, \tag{2.39}$$

$$\Sigma_{tr}^{g}=\Sigma_{1}^{g}-\mu_{p}^{g\to g}\,\Sigma_{p}^{g\to g}; \tag{2.40}$$

the group constants defined by the balance theory are given by the following formulae

$$\Sigma^{g}=\frac{\int_{G}d\vec{r}\int_{\Delta E_{g}}dE\sum_{i}N_{i}\,\sigma_{i,g}(E)\,\phi(\vec{r},E)}{\int_{G}d\vec{r}\int_{\Delta E_{g}}dE\,\phi(\vec{r},E)}, \tag{2.41}$$

$$\Sigma_{1}^{g}=\frac{\int_{G}d\vec{r}\int_{\Delta E_{g}}dE\sum_{i}N_{i}\,\sigma_{i,g}(E)\,J(\vec{r},E)}{\int_{G}d\vec{r}\int_{\Delta E_{g}}dE\,J(\vec{r},E)}, \tag{2.42}$$

$$\Sigma_{p}^{h\to g}=\frac{\int_{G}d\vec{r}\int_{\Delta E_{g}}dE\int_{\Delta E_{h}}dE'\sum_{i}N_{i}\,\nu_{i,g}(E')\,\sigma_{i,g}(E'\to E)\,\phi(\vec{r},E')}{\int_{G}d\vec{r}\int_{\Delta E_{h}}dE'\,\phi(\vec{r},E')}, \tag{2.43}$$

$$\mu_{p}^{h\to g}\,\Sigma_{p}^{h\to g}=\frac{\int_{G}d\vec{r}\int_{\Delta E_{g}}dE\int_{\Delta E_{h}}dE'\sum_{i}N_{i}\,\nu_{i,g}(E')\,\mu_{i,g}(E'\to E)\,\sigma_{i,g}(E'\to E)\,J(\vec{r},E')}{\int_{G}d\vec{r}\int_{\Delta E_{h}}dE'\,J(\vec{r},E')}, \tag{2.44}$$

$$\mathcal{F}^g \nu_f^h \Sigma_f^h = \frac{\int\limits_G d\vec{r} \int\limits_{\Delta E_h} dE' \sum_\lambda N_\lambda \mathcal{F}_\lambda^g \nu_{\lambda f}(E') \sigma_{\lambda f}(E') \phi(\vec{r}, E')}{\int\limits_G d\vec{r} \int\limits_{\Delta E_h} dE' \phi(\vec{r}, E')} \qquad (2.45)$$

The equations (2.35), (2.36), and (2.37) are a special case of the P_1-approximation equations for general geometry

$$\operatorname{div} \vec{J}^g(\vec{r}) + \Sigma_{rem}^g \phi^g(\vec{r}) = \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} \Sigma_\lambda^{h\to g} \phi^h(\vec{r}) + \sum_{h=1}^{M} \mathcal{F}^g \nu_f^h \Sigma_f^h \phi^h(\vec{r}) + S^g(\vec{r}), \qquad (2.46)$$

$$\frac{1}{3} \operatorname{grad} \phi^g(\vec{r}) + \Sigma_{tr}^g \vec{J}^g(\vec{r}) = \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} \mu_\lambda^{h\to g} \Sigma_\lambda^{h\to g} \vec{J}^h(\vec{r}) + \vec{S}^g(\vec{r}). \qquad (2.47)$$

In general the group parameters in (2.46) and (2.47) are functions of $\vec{r}$. But in any homogeneous subregion the group parameters are constants, and the system (2.46), (2.47) may be easily rewritten into the form of an effective diffusion equation, which is convenient for programming. Beneath we are going to find out this equivalent diffusion equation.

Defining the diffusion coefficient as

$$D^g = \frac{1}{3 \Sigma_{tr}^g}, \qquad (2.48)$$

the (2.47) may be written as

$$\vec{J}^g = -D^g \operatorname{grad} \phi^g + 3 D^g \vec{S}^g + 3 D^g \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} \mu_\lambda^{h\to g} \Sigma_\lambda^{h\to g} \vec{J}^h,$$

from which

$$\operatorname{div} \vec{J}^g = -D^g \Delta \phi^g + 3 D^g \operatorname{div} \vec{S}^g + 3 D^g \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} \mu_\lambda^{h\to g} \Sigma_\lambda^{h\to g} \operatorname{div} \vec{J}^h;$$

according to (2.46), writing h instead of g, there is

$$\operatorname{div} \vec{J}^h = -\Sigma_{rem}^h \phi^h + \sum_{\substack{\beta=1 \\ h \neq 1}}^{h-1} \Sigma_\lambda^{\beta\to h} \phi^\beta + \sum_{\beta=1}^{M} \mathcal{F}^h \nu_f^\beta \Sigma_f^\beta \phi^\beta + S^h;$$

therefore there is

$$\mathrm{div}\,\vec{J}^g = -D^g \Delta \Phi^g + 3D^g \mathrm{div}\,\vec{S}^g - 3D^g \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \mu_p^{h\to g} \Sigma_p^{h\to g} \Sigma_{rem}^h \Phi^h +$$

$$+ 3D^g \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \sum_{\substack{\beta=1\\ h\neq 1}}^{h-1} \mu_p^{h\to g} \Sigma_p^{h\to g} \Sigma_p^{\beta\to h} \Phi^\beta +$$

$$+ 3D^g \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \sum_{\beta=1}^{M} \mu_p^{h\to g} \Sigma_p^{h\to g} \mathcal{F}^h \nu_f^\beta \Sigma_f^\beta \Phi^\beta +$$

$$+ 3D^g \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \mu_p^{h\to g} \Sigma_p^{h\to g} S^h ;$$

expanding the series on both the left and the right side the following two relations will be prooved:

$$\sum_{\substack{h=1\\ g\neq 1}}^{g-1} \sum_{\substack{\beta=1\\ h\neq 1}}^{h-1} \mu_p^{h\to g} \Sigma_p^{h\to g} \Sigma_p^{\beta\to h} \Phi^\beta = \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \sum_{\substack{\beta=h+1\\ h\neq g-1}}^{g-1} \mu_p^{\beta\to g} \Sigma_p^{\beta\to g} \Sigma_p^{h\to \beta} \Phi^h ,$$

$$\sum_{\substack{h=1\\ g\neq 1}}^{g-1} \sum_{\beta=1}^{M} \mu_p^{h\to g} \Sigma_p^{h\to g} \mathcal{F}^h \nu_f^\beta \Sigma_f^\beta \Phi^\beta = \sum_{\substack{\beta=1\\ g\neq 1}}^{g-1} \sum_{h=1}^{M} \mu_p^{\beta\to g} \Sigma_p^{\beta\to g} \mathcal{F}^\beta \nu_f^h \Sigma_f^h \Phi^h ;$$

therefore we can write

$$\mathrm{div}\,\vec{J}^g = -D^g \Delta \Phi^g + 3D^g \mathrm{div}\,\vec{S}^g + 3D^g \sum_{\substack{\beta=1\\ g\neq 1}}^{g-1} \sum_{h=1}^{M} \mu_p^{\beta\to g} \Sigma_p^{\beta\to g} \mathcal{F}^\beta \nu_f^h \Sigma_f^h \Phi^h -$$

$$- 3D^g \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \Big[\mu_p^{h\to g} \Sigma_p^{h\to g} \Sigma_{rem}^h - \sum_{\substack{\beta=h+1\\ h\neq g-1}}^{g-1} \mu_p^{\beta\to g} \Sigma_p^{\beta\to g} \Sigma_p^{h\to \beta} \Big] \Phi^h +$$

$$+ 3D^g \sum_{\substack{h=1\\ g\neq 1}}^{g-1} \mu_p^{h\to g} \Sigma_p^{h\to g} S^h ,$$

substituting this into the equation 2.10) we get the searched effective diffusion equation together with its effective groupconstants and with the effective source functions;

$$-D^g \Delta \phi^g + \Sigma^g_{rem} \phi^g = \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} [\Sigma_p^{h\to g}]_{ef} \phi^h + \sum_{h=1}^{M} [\mathcal{F}^g \nu_f^h \Sigma_f^h]_{ef} \phi^h + S^g_{ef} , \qquad (2.49)$$

$$[\Sigma_p^{h\to g}]_{ef} = \Sigma_p^{h\to g} + 3D^g \mu_p^{h\to g} \Sigma_p^{h\to g} \Sigma^h_{rem} - 3D^g \sum_{\substack{\beta=h+1 \\ h \neq g-1}}^{g-1} \mu_p^{\beta\to g} \Sigma_p^{\beta\to g} \Sigma_p^{h\to\beta} , \qquad (2.50)$$

$$[\mathcal{F}^g \nu_f^h \Sigma_f^h]_{ef} = \mathcal{F}^g \nu_f^h \Sigma_f^h - 3D^g \sum_{\substack{\beta=1 \\ g \neq 1}}^{g-1} \mu_p^{\beta\to g} \Sigma_p^{\beta\to g} \mathcal{F}^\beta \nu_f^h \Sigma_f^h , \qquad (2.51)$$

$$S^g_{ef} = S^g - 3D^g \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} \mu_p^{h\to g} \Sigma_p^{h\to g} S^h - 3D^g \operatorname{div} \vec{S}^g ; \qquad (2.52)$$

for a finite cylinder with cylindrical symmetry the diffusion equation (2.49) has the following form

$$-D^g \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right] \phi^g + \Sigma^g_{rem} \phi^g = \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} [\Sigma_p^{h\to g}]_{ef} \phi^h + \sum_{h=1}^{M} [\mathcal{F}^g \nu_f^h \Sigma_f^h]_{ef} \phi^h + S^g_{ef} . \qquad (2.53)$$

Now we have to eliminate the J_r, J_z from the boundary condition (2.48). Writing the equations (2.36), (2.37)

$$J^g_r - 3D^g \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} \mu_p^{h\to g} \Sigma_p^{h\to g} J^h_r = -D^g \frac{\partial}{\partial r} \phi^g + 3D^g S^g_r ,$$

$$J^g_z - 3D^g \sum_{\substack{h=1 \\ g \neq 1}}^{g-1} \mu_p^{h\to g} \Sigma_p^{h\to g} J^h_z = -D^g \frac{\partial}{\partial z} \phi^g + 3D^g S^g_z ,$$

and using the following designation

$$c^{h\to g} = -3D^g \mu_p^{h\to g} \Sigma_p^{h\to g} ,$$

$$\Psi^g_r = -D^g \frac{\partial}{\partial r} \phi^g + 3D^g S^g_r ,$$

$$\Psi^g_z = -D^g \frac{\partial}{\partial z} \phi^g + 3D^g S^g_z ,$$

we may write the following algebraic equations

$$J_r^g + \sum_{\substack{h=1 \\ g+1}}^{g-1} c^{h\to g} J_r^h = \Psi_r^g ,$$

$$J_x^g + \sum_{\substack{h=1 \\ g+1}}^{g-1} c^{h\to g} J_x^h = \Psi_x^g$$

for $g = 1, 2, \ldots, N$ with the solutions

$$J_r^g = \sum_{h=1}^{g} k^{h\to g} \Psi_r^h ,$$

$$J_x^g = \sum_{h=1}^{g} k^{h\to g} \Psi_x^h ,$$

where $k^{h\to g}$ are terms of the inverse matrix; therefore the components of the vector current $\vec{J}$ are

$$J_r^g(r,x) = -\sum_{h=1}^{g} D^h k^{h\to g} \left[\frac{\partial}{\partial r} \phi^h(r,x) + 3\, S_r^h(r,x) \right] , \tag{2.54}$$

$$J_x^g(r,x) = -\sum_{h=1}^{g} D^h k^{h\to g} \left[\frac{\partial}{\partial x} \phi^h(r,x) + 3\, S_x^h(r,x) \right] ; \tag{2.55}$$

using the method of elimination and defining $k^{g\to g} = 1$ we get easily

$$\left. \begin{aligned} & k^{g\to g} = 1 , \\ & k^{g-i\to g} = -c^{g-i\to g} - \sum_{j=1}^{i-1} c^{g-i\to g-j}\, k^{g-j\to g} , \\ & i = 1, 2, \ldots, g-1 ; \end{aligned} \right| \tag{2.56}$$

substituting the (2.54), (2.55) into the boundary condition (2.[illegible]) we get the final form of the boundary condition :

$$\Phi^{g}(R,x) + \sum_{h=1}^{g} K^{h\to g}\left[\frac{\partial}{\partial r}\Phi^{h}(r,x)\right]_R = \int_0^H K_1(x,x')\Phi^{g}(R,x')\,dx' +$$

$$+\sum_{h=1}^{g}\int_0^H K_2^{h\to g}(x,x')\left[\frac{\partial}{\partial r}\Phi^{h}(r,x')\right]_R dx' + \sum_{h=1}^{g}\int_0^H K_3^{h\to g}(x,x')\left[\frac{\partial}{\partial x'}\Phi^{h}(R,x')\right]dx' +$$

$$+F^{g}(R,x), \tag{2.57}$$

$$K^{h\to g} = -2D^{h}k^{h\to g}, \tag{2.58}$$

$$K_2^{h\to g}(x,x') = -D^{h}k^{h\to g}K_2(x,x'), \tag{2.59}$$

$$K_3^{h\to g}(x,x') = -D^{h}k^{h\to g}K_3(x,x'), \tag{2.60}$$

$$F^{g}(R,x) = -3\sum_{h=1}^{g}K^{h\to g}S^{h}(R,x) + K^{g}(R,x), \tag{2.61}$$

$$K^{g}(R,x) = 3\sum_{h=1}^{g}\left[\int_0^H K_2^{h\to g}(x,x')S_r^{h}(R,x')\,dx' + \int_0^H K_3^{h\to g}(x,x')S_x^{h}(R,x')\,dx'\right]. \tag{2.62}$$

Hence for the P_1-approximation we have got the equations (2.35), (2.36), (2.37), (2.38), or (2.53), (2.50), (2.51), (2.72), (2.57), (2.58), (2.59), (2.60), (2.61), (2.62).

Now we are going to write the formulation of the problem in the diffusion approximation: here the terms including the cosine $\mu(h\to g)$ for $h \neq g$ are equal to zero and the source function is isotropic, whence from the second formulation of the P_1-approximation we get immediately

$$-D^{g}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right]\Phi^{g} + \Sigma_{rem}^{g}\Phi^{g} = \sum_{\substack{h=1\\g\neq 1}}^{g-1}\Sigma_{s}^{h\to g}\Phi^{h} + \sum_{h=1}^{M}\mathcal{F}^{g}\nu_f^{h}\Sigma_f^{h}\Phi^{h} + S^{g}, \tag{2.63}$$

$$\Phi^{g}(R,x) + K^{g\to g}\left[\frac{\partial}{\partial r}\Phi^{g}(r,x)\right]_R = \int_0^H K_1(x,x')\Phi^{g}(R,x')\,dx' +$$

$$+\int_0^H K_2^{g\to g}(x,x')\left[\frac{\partial}{\partial r}\Phi^g(r,x')\right]_R dx' + \int_0^H K_3^{g\to g}(x,x')\left[\frac{\partial}{\partial x'}\Phi^g(R,x')\right]dx' + F^g(R,x), \qquad (2.64)$$

$$K^{g\to g} = -2D^g, \qquad (2.65)$$

$$K_2^{g\to g}(x,x') = -D^g K_2(x,x'), \qquad (2.66)$$

$$K_3^{g\to g}(x,x') = -D^g K_3(x,x'), \qquad (2.67)$$

$$F^g(R,x) = -3K^{g\to g}S^g(R,x). \qquad (2.68)$$

Beneath the formulations of the problem for an infinite cavity with cylindrical symmetry in both the classic and the effective diffusion P_1-approximations and in the classic diffusion approximation are given:

$$\left[\frac{\partial}{\partial r}+\frac{1}{r}\right]J_r^g + \Sigma_{rem}^g\Phi^g = \sum_{\substack{h=1\\ g\neq 1}}^{g-1}\Sigma_s^{h\to g}\Phi^h + \sum_{h=1}^{M}\mathcal{F}^g\nu_f^h\Sigma_f^h\Phi^h + S^g, \qquad (2.69)$$

$$\frac{1}{3}\frac{\partial}{\partial r}\Phi^g + \Sigma_{tr}^g J_r^g = \sum_{\substack{h=1\\ g\neq 1}}^{g-1}\mu_s^{h\to g}\Sigma_s^{h\to g}J_r^h + S_r^g, \qquad (2.70)$$

$$J_r^g(R) = 0. \qquad (2.71)$$

$$-D^g\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\Phi^g + \Sigma_{rem}^g\Phi^g = \sum_{\substack{h=1\\ g\neq 1}}^{g-1}\left[\Sigma_s^{h\to g}\right]_{ef}\Phi^h + \sum_{h=1}^{M}\left[\mathcal{F}^g\nu_f^h\Sigma_f^h\right]_{ef}\Phi^h + S_{ef}^g, \qquad (2.72)$$

$$\sum_{h=1}^{g} k^{h\to g}\left[\left[\frac{\partial}{\partial r}\Phi^h(r)\right]_R + 3S_r^h(R)\right] = 0. \qquad (2.73)$$

$$-D^g\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\Phi^g + \Sigma_{rem}^g\Phi^g = \sum_{\substack{h=1\\ g\neq 1}}^{g-1}\Sigma_s^{h\to g}\Phi^h + \sum_{h=1}^{M}\mathcal{F}^g\nu_f^h\Sigma_f^h\Phi^h + S^g, \qquad (2.74)$$

$$\left[\frac{\partial}{\partial r}\Phi^g(r)\right]_R = 0. \qquad (2.75)$$

N o t e : The formula for the effective source function (2.32) includes the term div $\vec{S}^g$. For a finite cavity there is

$$\operatorname{div} \vec{S}^g = \left[\frac{\partial}{\partial r} + \frac{1}{r} \right] S_r^g + \frac{\partial}{\partial x} S_x^g \; ;$$

for an infinite cavity there is

$$\operatorname{div} \vec{S}^g = \left[\frac{\partial}{\partial r} + \frac{1}{r} \right] S_r^g \; .$$

3. Appendix

Here we are going to show the formulae (1.2?). The vector current $\vec{J}(\vec{r})$ is defined as

$$\vec{J}(\vec{r}) = \int_{4\pi} d\vec{\Omega}\, \vec{\Omega}\, \Psi(\vec{r},\vec{\Omega}), \quad d\vec{\Omega} = \sin\vartheta\, d\vartheta\, d\psi .$$

For the cylindrical geometry there may be written

$$\vec{J}(\vec{r}) = \int_0^{2\pi} d\psi \int_0^{\pi} \sin\vartheta\, d\vartheta\, [\Omega_r \vec{e}_r + \Omega_\varphi \vec{e}_\varphi + \Omega_z \vec{e}_z]\, \Psi(\vec{r},\vartheta,\psi).$$

The components of the vector $\vec{\Omega} = [\sin\vartheta\cos\psi, \sin\vartheta\sin\psi, \cos\vartheta]$, $0 \le \vartheta \le \pi$, $0 \le \psi < 2\pi$ may be found using the definition of the spherical harmonics

$$P_n^m(\vec{\Omega}) = (-1)^m \sqrt{\frac{(n-m)!}{(n+m)!}}\, e^{im\psi} P_n^m(\cos\vartheta),$$

$$P_n^{m*}(\vec{\Omega}) = (-1)^m \sqrt{\frac{(n-m)!}{(n+m)!}}\, e^{-im\psi} P_n^m(\cos\vartheta),$$

from which follows

$$P_1^{-1}(\vec{\Omega}) - P_1^1(\vec{\Omega}) = \sqrt{2}\sin\vartheta\cos\psi ,$$

$$P_1^{-1}(\vec{\Omega}) + P_1^1(\vec{\Omega}) = -i\sqrt{2}\sin\vartheta\sin\psi ,$$

$$P_1^0(\vec{\Omega}) = \cos\vartheta ,$$

$$P_1^{-1*}(\vec{\Omega}) - P_1^{1*}(\vec{\Omega}) = \sqrt{2}\sin\vartheta\cos\psi ,$$

$$P_1^{-1*}(\vec{\Omega}) + P_1^{1*}(\vec{\Omega}) = i\sqrt{2}\sin\vartheta\sin\psi ,$$

$$P_1^{0*}(\vec{\Omega}) = \cos\vartheta ;$$

hence there is

$$\Omega_r = \sin\vartheta\cos\psi = \frac{1}{\sqrt{2}}[P_1^{-1*}(\vec{\Omega}) - P_1^{1*}(\vec{\Omega})],$$

$$\Omega_\varphi = -\frac{1}{r}\sin\vartheta\sin\psi = \frac{i}{r\sqrt{2}}[P_1^{-1*}(\vec{\Omega}) + P_1^{1*}(\vec{\Omega})],$$

$$\Omega_z = \cos\vartheta = P_1^{0*}(\vec{\Omega}) .$$

Therefore we may write

$$\vec{J}(\vec{r}) = \frac{1}{\sqrt{2}}\int_{4\pi} d\vec{\Omega}\,[P_1^{-1*}(\vec{\Omega}) - P_1^{1*}(\vec{\Omega})]\,\vec{e}_r\,\varphi(\vec{r},\vec{\Omega}) +$$

$$+\frac{i}{r\sqrt{2}}\int_{4\pi} d\vec{\Omega}\,[P_1^{-1*}(\vec{\Omega}) + P_1^{1*}(\vec{\Omega})]\,\vec{e}_\varphi\,\varphi(\vec{r},\vec{\Omega}) +$$

$$+\int_{4\pi} d\vec{\Omega}\,P_1^{0*}(\vec{\Omega})\,\vec{e}_z\,\varphi(\vec{r},\vec{\Omega}) \ .$$

Using the Fourier series (1.4) and the orthogonality relation for spherical harmonics

$$\int_{4\pi} d\vec{\Omega}\,P_n^{m*}(\vec{\Omega})P_l^{k*}(\vec{\Omega}) = \frac{4\pi}{2n+1}\delta_{nl}\,\delta_{mk} \ ,$$

we get

$$J_r = \frac{1}{\sqrt{2}}[\varphi_{1,-1} - \varphi_{11}] \ ,$$

$$J_\varphi = \frac{i}{r\sqrt{2}}[\varphi_{1,-1} + \varphi_{11}] \ ,$$

$$J_z = \varphi_{10} \ .$$

Analogically for the scalar flux there may be written

$$\Phi(\vec{r}) = \int_{4\pi} d\vec{\Omega}\,\varphi(\vec{r},\vec{\Omega}) = \int_{4\pi} d\vec{\Omega}\,P_0^{0*}(\vec{\Omega})\varphi(\vec{r},\vec{\Omega}) = \varphi_{00}(\vec{r}) \ .$$

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