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II.

CIRCULAR VACUUM - CAVITY - BOUNDARY IN CYLINDRICAL SYMMETRY

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CIRCULAR VACUUM-CAVITY-BOUNDARY IN CYLINDRICAL SYMMETRY

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ABSTRACT

In this paper the numerical analysis of the boundary condition on a finite circular cylindrical cavity with cylindrical symmetry is given. The physical and mathematical formulation has been given in the previous paper ZJE - 173, 1975. The formulation is given in both the P_1 - approximation and the diffusion approximation. The (r, z) - P_1 - approximation of the problem is defined in an effective diffusion model with effective constants and sources. Replacing these effective constants and sources by the usual definitions we get the classic (r, z) - diffusion approximation. This way of the formulation is convenient for programming, as the P_1 - approximation code gives a possibility to use it immediately as a diffusion code. The description of the complete problem for a cylindrical cell with a cavity together with a sample of calculation linked to the experimental research of the problem made in the ŠKODA WORKS will be published. In this mentioned paper we give the formulation with a tensor diffusion coefficient (D_r, D_z), although into the programme the scalar diffusion coefficient $D = D_r = D_z = 1/(3\Sigma_{tr})$ will be introduced due to the temporary infamiliarities of the tensor formulae. We are going to study this problem for the case of a cavity later.

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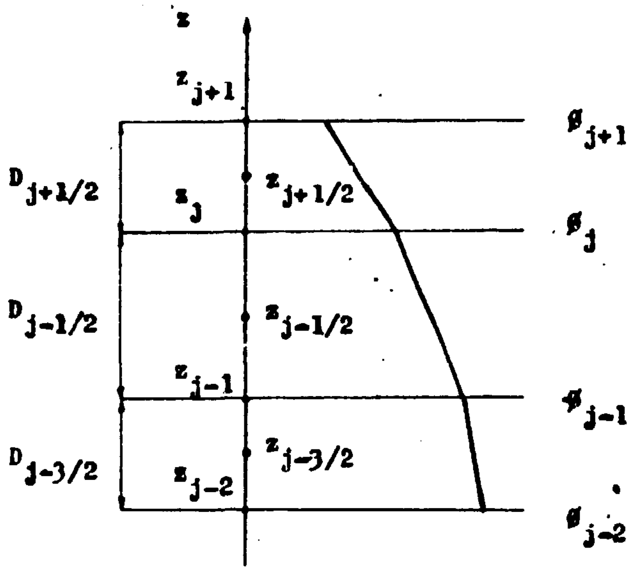
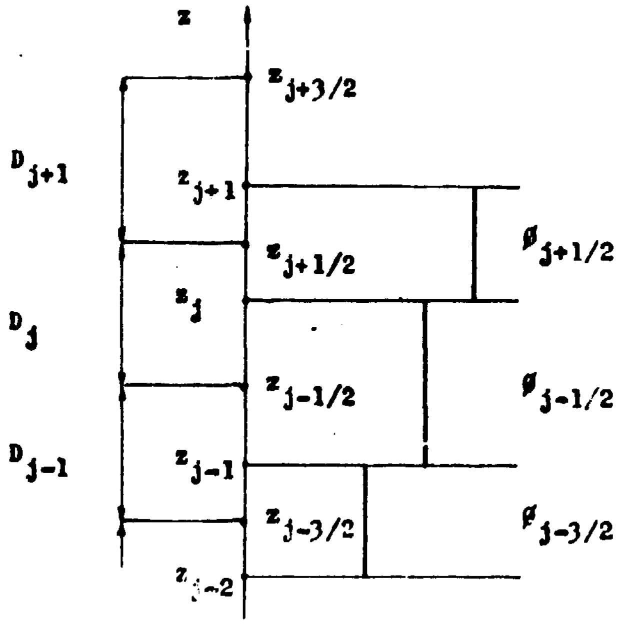
I n t r o d u c t i o n

The boundary condition must be linked into the problem defined for a selected system of mesh points. Hence the height H of the cavity is to be devided into a system of subintervals $\langle z_{j-1}, z_j \rangle$ for $j = 1, 2, \ldots, H$ by a system of mesh points z_j $(j = 0, 1, 2, \ldots, H)$. The flux $\phi(R, z)$ between two neighbouring mesh points is supposed to be either linear or constant in accordance with the following pictures:



A possible boundary between two different materials is chosen as a mesh point z_j $(j = 1, 2, \ldots, H-1)$.



A possible boundary between two different materials is chosen as a mesh point $z_{j-1/2}$ $(j = 1, 2, \ldots, H)$.

1. List of discussed formulations of the problem

I. Finite cavity

1. P_1-approximation

$$\left[\frac{\partial}{\partial r}+\frac{1}{r}\right]J_r^g+\frac{\partial}{\partial x}J_x^g+\Sigma_{rem}^g\phi^g=\sum_{\substack{h=1\\ g\neq 1}}^{g-1}\Sigma_p^{h\to g}\phi^h+\sum_{h=1}^{M}\mathcal{F}^g\nu_f^h\Sigma_f^h\phi^h+S^g, \tag{1.1}$$

$$\frac{1}{3}\frac{\partial}{\partial r}\phi^g+\Sigma_{tr}^g J_r^g=\sum_{\substack{h=1\\ g\neq 1}}^{g-1}\mu_p^{h\to g}\Sigma_p^{h\to g}J_r^h+S_r^g, \tag{1.2}$$

$$\frac{1}{3}\frac{\partial}{\partial x}\phi^g+\Sigma_{tr}^g J_x^g=\sum_{\substack{h=1\\ g\neq 1}}^{g-1}\mu_p^{h\to g}\Sigma_p^{h\to g}J_x^h+S_x^g, \tag{1.3}$$

$$\phi^g(R,x)+2J_r^g(R,x)=\int_0^H K_1(x,x')\phi^g(R,x')\,dx'+\int_0^H K_2(x,x')J_r^g(R,x')\,dx'+\int_0^H K_3(x,x')J_x^g(R,x')\,dx', \tag{1.4}$$

$$K_1(x,x')=\frac{16R^3}{\pi}\int_0^{\pi/2}\frac{\cos^4\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^2}\,d\psi, \tag{1.5}$$

$$K_2(x,x')=-\frac{96R^4}{\pi}\int_0^{\pi/2}\frac{\cos^6\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{5/2}}\,d\psi, \tag{1.6}$$

$$K_3(x,x')=\frac{48R^3}{\pi}(x-x')\int_0^{\pi/2}\frac{\cos^4\psi}{\left[(x-x')^2+4R^2\cos^2\psi\right]^{5/2}}\,d\psi. \tag{1.7}$$

This formulation may be written in an equivalent effective diffusion approximation, eliminating the components of the vector current $\mathbf{J}$:

$$-D^g \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right]\Phi^g + \Sigma^g_{rem}\Phi^g = \sum_{\substack{h=1\\ g\neq 1}}^{g-1} [\Sigma_p^{h\to g}]_{ef}\,\Phi^h + \sum_{h=1}^{M}[\mathcal{F}^g \nu_f^h \Sigma_f^h]_{ef}\,\Phi^h + S^g_{ef} , \tag{1.8}$$

$$[\Sigma_p^{h\to g}]_{ef} = \Sigma_p^{h\to g} + 3D^g \mu_p^{h\to g}\Sigma_p^{h\to g}\Sigma^h_{rem} - 3D^g \sum_{\substack{\beta=h+1\\ h\neq g-1}}^{g-1} \mu_p^{\beta\to g}\Sigma_p^{\beta\to g}\Sigma_p^{h\to\beta} , \tag{1.9}$$

$$[\mathcal{F}^g\nu_f^h\Sigma_f^h]_{ef} = \mathcal{F}^g\nu_f^h\Sigma_f^h - 3D^g\sum_{\substack{\beta=1\\ g\neq 1}}^{g-1}\mu_p^{\beta\to g}\Sigma_p^{\beta\to g}\mathcal{F}^\beta\nu_f^h\Sigma_f^h , \tag{1.10}$$

$$S^g_{ef} = S^g - 3D^g\sum_{\substack{h=1\\ g\neq 1}}^{g-1}\mu_p^{h\to g}\Sigma_p^{h\to g}S^h - 3D^g\left[\left[\frac{\partial}{\partial r}+\frac{1}{r}\right]S_r^g + \frac{\partial}{\partial x}S_x^g\right] . \tag{1.11}$$

$$\Phi^g(R,x) = \sum_{h=1}^{g} K^{h\to g}\left[\frac{\partial}{\partial r}\Phi^h(r,x)\right]_R + \int_0^H K_1(x,x')\Phi^g(R,x')\,dx' +$$

$$+\sum_{h=1}^g \int_0^H K_2^{h\to g}(x,x')\left[\frac{\partial}{\partial r}\Phi^h(r,x')\right]_R dx' + \sum_{h=1}^g\int_0^H K_3^{h\to g}(x,x')\left[\frac{\partial}{\partial x'}\Phi^h(R,x')\right]dx' +$$

$$+F^g(R,x) \tag{1.12}$$

$$K^{h\to g} = -2D^h k^{h\to g} , \tag{1.13}$$

$$K_2^{h\to g}(x,x') = -D^h k^{h\to g} K_2(x,x') , \tag{1.14}$$

$$K_3^{h\to g}(x,x') = -D^h k^{h\to g} K_3(x,x') , \tag{1.15}$$

$$F^g(R,x) = -3\sum_{h=1}^g K^{h\to g}S^h(R,x) + K^g(R,x) , \tag{1.16}$$

$$K^g(R,x) = 3\sum_{h=1}^g\left[\int_0^H K_2^{h\to g}(x,x')S_r^h(R,x')\,dx' +\right.$$

$$\left. + \int_0^H K_3^{h\to g}(x,x')S_x^h(R,x')\,dx'\right] , \tag{1.17}$$

$$\left.\begin{aligned} &k^{g \to g} = 1 , \\ &k^{g-\alpha \to g} = - C^{g-\alpha \to g} - \sum_{\beta=1}^{\alpha-1} C^{g-\alpha \to g-\beta} \, k^{g-\beta \to g} , \\ &\alpha = 1, 2, \dots, g-1 \end{aligned}\right| \tag{1.18}$$

$$C^{h \to g} = -3 D^g \mu_p^{h \to g} \Sigma_p^{h \to g} , \tag{1.19}$$

$$D^g = \frac{1}{3 \Sigma_{tr}^g} . \tag{1.20}$$

2. Diffusion approximation

$$-D^g \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} \right] \Phi^g + \Sigma_{rem}^g \Phi^g = \sum_{\substack{h=1 \\ g \neq h}}^{g-1} \Sigma_p^{h \to g} \Phi^h + \sum_{h=1}^{M} \mathcal{F}^g \nu_f^h \Sigma_f^h \Phi^h + S^g , \tag{1.21}$$

$$\Phi^g(R,x) + K^{g \to g} \left[\frac{\partial}{\partial r} \Phi^g(r,x) \right]_R = \int_0^H K_1(x,x') \Phi^g(R,x') dx' +$$

$$+ \int_0^H K_2^{g \to g}(x,x') \left[\frac{\partial}{\partial r} \Phi^g(r,x') \right]_R dx' + \int_0^H K_3^{g \to g}(x,x') \left[\frac{\partial}{\partial x'} \Phi^g(R,x') \right] dx' + F^g(R,x) , \tag{1.22}$$

$$K^{g \to g} = -2 D^g , \tag{1.23}$$

$$K_2^{g \to g}(x,x') = -D^g K_2(x,x') , \tag{1.24}$$

$$K_3^{g \to g}(x,x') = -D^g K_3(x,x') , \tag{1.25}$$

$$F^g(R,x) = -3 K^{g \to g} S^g(R,x) , \tag{1.26}$$

$$D^g = \frac{1}{3 \Sigma_{tr}^g} . \tag{1.27}$$

II. Infinite cavity

1. P_1-approximation

$$\left[\frac{\partial}{\partial r}+\frac{1}{r}\right]J_r^g+\Sigma_{rem}^g\phi^g=\sum_{\substack{h=1\\g\neq 1}}^{g-1}\Sigma_s^{h\to g}\phi^h+\sum_{h=1}^{M}\mathcal{F}^g\nu_f^h\Sigma_f^h\phi^h+S^g,\tag{1.28}$$

$$\frac{1}{3}\frac{\partial}{\partial r}\phi^g+\Sigma_{tr}^g J_r^g=\sum_{\substack{h=1\\g\neq 1}}^{g-1}\mu_s^{h\to g}\Sigma_s^{h\to g}J_r^h+S_r^g,\tag{1.29}$$

$$J_r^g(R)=0\;.\tag{1.30}$$

P_1-approximation in the equivalent effective diffusion model

$$-D^g\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\phi^g+\Sigma_{rem}^g\phi^g=\sum_{\substack{h=1\\g\neq 1}}^{g-1}[\Sigma_s^{h\to g}]_{ef}\phi^h+\sum_{h=1}^{M}[\mathcal{F}^g\nu_f^h\Sigma_f^h]_{ef}\phi^h+S_{ef}^g,\tag{1.31}$$

$$\sum_{h=1}^{g}D^h k^{h\to g}\left[\left[\frac{\partial}{\partial r}\phi^h(r)\right]_R+3S_r^h(R)\right]=0\;.\tag{1.32}$$

2. Diffusion approximation

$$-D^g\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\phi^g+\Sigma_{rem}^g\phi^g=\sum_{\substack{h=1\\g\neq 1}}^{g-1}\Sigma_s^{h\to g}\phi^h+\sum_{h=1}^{M}\mathcal{F}^g\nu_f^h\Sigma_f^h\phi^h+S^g,\tag{1.33}$$

$$\left[\frac{\partial}{\partial r}\phi^g(r)\right]_R=0\;.\tag{1.34}$$

Note: In (1.31) the effective source function S_{ef}^g is defined by the (1.11) with the exception that in (1.31) there is $\frac{d}{dz}S_z^g=0$.

2. Numerical analysis of the boundary condition in the P_1-approximation

The P_1-approximation of the problem is considered in the form of the equivalent effective diffusion approximation (1.8) to (1.20), and (1.5), (1.6), (1.7). All operations are performed analytically with the only exception that the neutron flux between any two neighbouring mesh points is considered to be either linear or constant. Then the integrations (1.5), (1.6), (1.7) according to the azimuthal angle Ψ may be performed by using the elliptic integrals

$$E(\pi/2,\lambda) = \int_0^{\pi/2} \sqrt{1-\lambda^2\sin^2\Psi}\, d\Psi , \tag{2.1}$$

$$F(\pi/2,\lambda) = \int_0^{\pi/2} \frac{d\Psi}{\sqrt{1-\lambda^2\sin^2\Psi}} , \tag{2.2}$$

the values of which are given either in tables or by some analytical approximations. In / 3 / there are given the following approximations:

$$E(\pi/2,\lambda) = \sum_{l=0}^{4} \beta_l (1-\lambda^2)^l - \sum_{l=1}^{4} \gamma_l (1-\lambda^2)^l \ln(1-\lambda^2) + \varepsilon , \tag{2.3}$$

$$|\varepsilon| \leq 2 \cdot 10^{-8}$$

$\beta_0 = 1$	
$\beta_1 = .44325\ 141463$	$\gamma_1 = .24998\ 368310$
$\beta_2 = .06260\ 601220$	$\gamma_2 = .09200\ 180037$
$\beta_3 = .04757\ 383546$	$\gamma_3 = .04069\ 697526$
$\beta_4 = .01736\ 506451$	$\gamma_4 = .00526\ 449639$

$$F(\pi/2,\lambda) = \sum_{l=0}^{4} \beta_l (1-\lambda^2)^l - \sum_{l=0}^{4} \gamma_l (1-\lambda^2)^l \ln(1-\lambda^2) + \varepsilon , \tag{2.4}$$

$$|\varepsilon| \leq 2 \cdot 10^{-8}$$

$\beta_0 = 1.38629\ 436112$	$\gamma_0 = .5$
$\beta_1 = .09666\ 344259$	$\gamma_1 = .12498\ 593597$
$\beta_2 = .03590\ 092383$	$\gamma_2 = .06880\ 248576$
$\beta_3 = .03742\ 563713$	$\gamma_3 = .03328\ 355346$
$\beta_4 = .01451\ 196212$	$\gamma_4 = .00441\ 787012$

A) L i n e a r   a p p r o x i m a t i o n   o f   t h e   n e u t r o n   f l u x

The height of the channel is to be divided into subintervals $\langle z_{j-1}, z_j \rangle$, $j = 1, 2, \ldots, H$ by a system of mesh points $0 = z_0 < z_1 < \ldots < z_H = H$. The only limitation of the selection of the mesh points is that a possible boundary between two different materials (perpendicular to the channel) is to be chosen as a mesh point. Furthermore we define another system of subintervals $\langle z_{j-1/2}, z_{j+1/2} \rangle = \left\langle \frac{z_{j-1} + z_j}{2}, \frac{z_j + z_{j+1}}{2} \right\rangle$ by a system of mesh points $z_{j-1/2} = \frac{z_{j-1} + z_j}{2}$ for $j = 1, 2, \ldots, H$. The designation $\Delta_{j,j-1} = z_j - z_{j-1}$ is used. The course of the neutron flux ϕ and of the anisotropic terms of the source function between any two neighbouring mesh points is supposed to be linear :

$$\phi(R, z') = k_1^j z' + q_1^j ,$$

$$S_r(R, z') = k_2^j z' + q_2^j ,$$

$$S_z(R, z') = k_3^j z' + q_3^j ,$$

$$k_1^j = \frac{1}{z_j - z_{j-1}} \phi^j - \frac{1}{z_j - z_{j-1}} \phi^{j-1} ,$$

$$q_1^j = -\frac{z_{j-1}}{z_j - z_{j-1}} \phi^j + \frac{z_j}{z_j - z_{j-1}} \phi^{j-1} ,$$

$$k_2^j = \frac{1}{z_j - z_{j-1}} S_r^j - \frac{1}{z_j - z_{j-1}} S_r^{j-1} ,$$

$$q_2^j = -\frac{z_{j-1}}{z_j - z_{j-1}} S_r^j + \frac{z_j}{z_j - z_{j-1}} S_r^{j-1} ,$$

$$k_3^j = \frac{1}{z_j - z_{j-1}} S_z^j - \frac{1}{z_j - z_{j-1}} S_z^{j-1} ,$$

$$q_3^j = -\frac{z_{j-1}}{z_j - z_{j-1}} S_z^j + \frac{z_j}{z_j - z_{j-1}} S_z^{j-1} .$$

The constant diffusion parameters within the subinterval $\langle z_{j-1}, z_j \rangle$ are indexed with $j-1/2$:

$$D^g_{j-1/2} = \frac{1}{3 \Sigma^g_{tr, j-1/2}} , \tag{2.5}$$

$$C^{h \to g}_{j-1/2} = -3 D^g_{j-1/2} \left[\mu^{h \to g} \Sigma_s^{h \to g} \right]_{j-1/2} , \tag{2.6}$$

$k^{g \to g}_{j-1/2} = 1$,

$$k^{g-\alpha \to g}_{j-1/2} = -C^{g-\alpha \to g}_{j-1/2} - \sum_{\beta=1}^{\alpha-1} C^{g-\alpha \to g-\beta}_{j-1/2} \, k^{g-\beta \to g}_{j-1/2} , \tag{2.7}$$

$\alpha = 1, 2, \ldots, g-1$.

As any of the mesh points z_j for $j = 1, 2, \ldots, H-1$ may separate two different materials, an effective value of the constant $K^{h \to g}$ in the point z_j is defined:

$$K^{h \to g}_{j,ef} = \frac{\Delta_{j,j-1} K^{h \to g}_{j-1/2} + \Delta_{j+1,j} K^{h \to g}_{j+1/2}}{\Delta_{j,j-1} + \Delta_{j+1,j}} . \tag{2.8}$$

We define two points z_{-1} (symmetrical to z_1 according to $z_o = 0$), z_{H+1} (symmetrical to z_{H-1} according to z_H). Now the formula (2.8) may be used also in the points z_o and z_H, defining $K^{h \to g}_{-1/2} = K^{h \to g}_{1/2}$, $K^{h \to g}_{H+1/2} = K^{h \to g}_{H-1/2}$. Therefore $K^{h \to g}_{o,ef} = K^{h \to g}_{1/2}$, $K^{h \to g}_{H,ef} = K^{h \to g}_{H-1/2}$.

For the definition of the boundary condition the neutron fluxes ϕ^g_{-1} in the point z_{-1} and ϕ^g_{H+1} in the point z_{H+1} are needed. These must be defined using the boundary conditions on the boundaries $z = 0$ and $z = H$:

$$\Phi^g(r,0) + 2 J^g_z(r,0) = 0 , \tag{2.9}$$

$$\Phi^g(r,H) - 2 J^g_z(r,H) = 0 ; \tag{2.10}$$

according to / 2 / there is

$$J^g_z(r,z) = -\sum_{h=1}^{g} D^h k^{h \to g} \left[\frac{\partial \Phi^h(r,z)}{\partial z} + 3 S^h_z(r,z) \right] ; \tag{2.11}$$

hence the boundary conditions (2.9), (2.10) may be written as

$$\Phi^g(R,0) - 2 \sum_{h=1}^{g} D^h_{1/2} k^{h \to g}_{1/2} \left[\left[\frac{\partial \Phi^h(R,z)}{\partial z} \right]_o + 3 S^h_z(R,0) \right] = 0 , \tag{2.12}$$

$$\Phi^g(R,H) + 2 \sum_{h=1}^{g} D^h_{H-1/2} k^{h \to g}_{H-1/2} \left[\left[\frac{\partial \Phi^h(R,z)}{\partial z} \right]_H + 3 S^h_z(R,H) \right] = 0 ; \tag{2.13}$$

the following definitions are being used :

$$\Phi^g(R,0) = \frac{\Phi^g_1(R) + \Phi^g_{-1}(R)}{2} , \tag{2.14}$$

$$\left[\frac{\partial \Phi^g(R,z)}{\partial z} \right]_o = \frac{\Phi^g_1(R) - \Phi^g_{-1}(R)}{2 \Delta_{1o}} , \tag{2.15}$$

$$\Phi^g(R,H) = \frac{\Phi^g_{H+1}(R) + \Phi_{H-1}(R)}{2} , \tag{2.16}$$

$$\left[\frac{\partial \Phi^g(R,z)}{\partial z}\right]_H = \frac{\Phi^g_{H+1}(R) - \Phi^g_{H-1}(R)}{2\,\Delta_{H,H-1}} , \tag{2.17}$$

where $\Delta_{10} = z_1 - z_0$, $\Delta_{H,H-1} = z_H - z_{H-1}$; substituting this into (2.12) and (2.13) we get after some rearangements two systems of algebraic equations

$$\sum_{h=1}^{g} A_0^{h\to g}\, \Phi^h_{-1}(R) = F_0\left[\Phi^1_1, \Phi^2_1, \ldots, \Phi^g_1\right] , \tag{2.18}$$

$$\sum_{h=1}^{g} A_H^{h\to g}\, \Phi^h_{H+1}(R) = F_H\left[\Phi^1_{H-1}, \Phi^2_{H-1}, \ldots, \Phi^g_{H-1}\right] , \tag{2.19}$$

$$\left.\begin{aligned} A_0^{g\to g} &= 1 , \\ A_0^{h\to g} &= \frac{\frac{2}{\Delta_{10}} D^h_{1/2}\, k^{h\to g}_{1/2}}{1 + \frac{2}{\Delta_{10}} D^g_{1/2}\, k^{g\to g}_{1/2}} \qquad \text{for} \quad h < g , \end{aligned}\right| \tag{2.20}$$

$$F_0\left[\Phi^1_1, \Phi^2_1, \ldots, \Phi^g_1\right] = -\frac{1 - \frac{2}{\Delta_{10}} D^g_{1/2}\, k^{g\to g}_{1/2}}{1 + \frac{2}{\Delta_{10}} D^g_{1/2}\, k^{g\to g}_{1/2}}\, \Phi^g_1(R) +$$

$$+ \sum_{\substack{h=1 \\ g\neq 1}}^{g+1} \frac{\frac{2}{\Delta_{10}} D^h_{1/2}\, k^{h\to g}_{1/2}}{1 + \frac{2}{\Delta_{10}} D^g_{1/2}\, k^{g\to g}_{1/2}} \left[\Phi^h_1(R) + 6\,\Delta_{10}\, S^h_{z,0}(R)\right] , \tag{2.21}$$

$$\left.\begin{aligned} A_H^{g\to g} &= 1 , \\ A_H^{h\to g} &= \frac{\frac{2}{\Delta_{H,H-1}} D^h_{H-1/2}\, k^{h\to g}_{H-1/2}}{1 + \frac{2}{\Delta_{H,H-1}} D^g_{H-1/2}\, k^{g\to g}_{H-1/2}} \qquad \text{for} \quad h < g , \end{aligned}\right| \tag{2.22}$$

$$F_H\left[\Phi^1_{H-1}, \Phi^2_{H-1}, \ldots, \Phi^g_{H-1}\right] = -\frac{1 - \frac{2}{\Delta_{H,H-1}} D^g_{H-1/2}\, k^{g\to g}_{H-1/2}}{1 + \frac{2}{\Delta_{H,H-1}} D^g_{H-1/2}\, k^{g\to g}_{H-1/2}}\, \Phi^g_{H-1}(R) +$$

$$+ \sum_{\substack{h=1 \\ g\neq 1}}^{g+1} \frac{\frac{2}{\Delta_{H,H-1}} D^h_{H-1/2}\, k^{h\to g}_{H-1/2}}{1 + \frac{2}{\Delta_{H,H-1}} D^g_{H-1/2}\, k^{g\to g}_{H-1/2}} \left[\Phi^h_{H-1}(R) - 6\,\Delta_{H,H-1}\, S^h_{z,H}(R)\right] , \tag{2.23}$$

being $\| B_0^{h-g} \|$, $\| B_H^{h-g} \|$ the inverse matrices of $\| A_0^{h-g} \|$, $\| A_H^{h-g} \|$, the solutions of (2.18), (2.19) may be written as

$$\Phi_{-1}^{g}(R) = \sum_{h=1}^{g} B_0^{h-g} F_0 \left[\Phi_1^1 , \Phi_1^2 , \ldots , \Phi_1^h \right] , \tag{2.24}$$

$$\Phi_{H+1}^{g}(R) = \sum_{h=1}^{g} B_H^{h-g} F_H \left[\Phi_{H-1}^1 , \Phi_{H-1}^2 , \ldots , \Phi_{H-1}^h \right] , \tag{2.25}$$

$$\left. \begin{aligned} & B_0^{g-g} = 1 , \\ & B_0^{g-\alpha-g} = - A_0^{g-\alpha-g} - \sum_{\beta=1}^{\alpha-1} A_0^{g-\alpha-g-\beta} B_0^{g-\beta-g} , \\ & \alpha = 1, 2, \ldots, g-1 , \end{aligned} \right\} \tag{2.26}$$

$$F_0 \left[\Phi_1^1 , \Phi_1^2 , \ldots , \Phi_1^h \right] = - \frac{1 - \frac{2}{\Delta_{10}} D_{1/2}^h k_{1/2}^{h-h}}{1 + \frac{2}{\Delta_{10}} D_{1/2}^h k_{1/2}^{h-h}} \Phi_1^h(R) + $$
$$ + \sum_{\substack{\sigma=1 \\ h \neq 1}}^{h-1} \frac{\frac{2}{\Delta_{10}} D_{1/2}^{\sigma} k_{1/2}^{\sigma-h}}{1 + \frac{2}{\Delta_{10}} D_{1/2}^h k_{1/2}^{h-h}} \left[\Phi_1^{\sigma}(R) + 6 \Delta_{10} S_{x,0}^{\sigma}(R) \right] , \tag{2.27}$$

$$\left. \begin{aligned} & B_H^{g-g} = 1 , \\ & B_H^{g-\alpha-g} = - A_H^{g-\alpha-g} - \sum_{\beta=1}^{\alpha-1} A_H^{g-\alpha-g-\beta} B_H^{g-\beta-g} , \\ & \alpha = 1, 2, \ldots, g-1 , \end{aligned} \right\} \tag{2.28}$$

$$F_H \left[\Phi_{H-1}^1 , \Phi_{H-1}^2 , \ldots , \Phi_{H-1}^h \right] = - \frac{1 - \frac{2}{\Delta_{H,H-1}} D_{H-1/2}^h k_{H-1/2}^{h-h}}{1 + \frac{2}{\Delta_{H,H-1}} D_{H-1/2}^h k_{H-1/2}^{h-h}} \Phi_{H-1/2}^h(R) + $$
$$ + \sum_{\substack{\sigma=1 \\ h \neq 1}}^{h-1} \frac{\frac{2}{\Delta_{H,H-1}} D_{H-1/2}^{\sigma} k_{H-1/2}^{\sigma-h}}{1 + \frac{2}{\Delta_{H,H-1}} D_{H-1/2}^h k_{H-1/2}^{h-h}} \left[\Phi_{H-1}^{\sigma}(R) - 6 \Delta_{H,H-1} S_{x,H}^{\sigma}(R) \right] . \tag{2.29}$$

Thus
$$\phi_{-1}^{g} = f(\phi_1^1 , \phi_1^2 , \ldots , \phi_1^g) , \tag{2.30}$$
$$\phi_{H+1}^{g} = f(\phi_{H-1}^1 , \phi_{H-1}^2 , \ldots , \phi_{H-1}^g) . \tag{2.31}$$

Now the boundary condition (1.12) may be written as

$$\phi_i^g(R) + \sum_{h=1}^{g} K_{i,el}^{h\to g}\left[\frac{\partial \phi_i^h(r)}{\partial r}\right]_R + 3\sum_{h=1}^{g} K_{i,ef}^{h\to g}\, S_i^h(R) =$$

$$= \sum_{j=0}^{N}\sum_{h=1}^{g}\left[\alpha_{ij}^{h\to g}\left[\frac{\partial\phi_j^h(r)}{\partial r}\right]_R + \beta_{ij}^{h\to g}\,\phi_j^h(R)\right] +$$

$$+ 3\sum_{j=0}^{N}\sum_{h=1}^{g}\left[\alpha_{ij}^{h\to g}\, S_{r,j}^h(R) + \gamma_{ij}^{h\to g}\, S_{x,j}^h(R)\right], \tag{2.32}$$

$$\beta_{ij}^{h\to g} = [1-(1-\delta_{hg})]\,\alpha_{ij} + {}^*\beta_{ij}^{h\to g}, \tag{2.33}$$

$$\alpha_{ij} = (1-\delta_{j0})\, M_{ij} + (1-\delta_{jN})\, N_{i,j+1}, \tag{2.34}$$

$$\alpha_{ij}^{h\to g} = (1-\delta_{j0})\,{}^1M_{ij}^{h\to g} + (1-\delta_{jN})\,{}^1N_{i,j+1}^{h\to g}, \tag{2.35}$$

$${}^*\beta_{ij}^{h\to g} = (1-\delta_{j0})\,{}^2M_{ij}^{h\to g} + (1-\delta_{jN})\,{}^2N_{i,j+1}^{h\to g}, \tag{2.36}$$

$$\gamma_{ij}^{h\to g} = (1-\delta_{j0})\,{}^3M_{ij}^{h\to g} + (1-\delta_{jN})\,{}^3N_{i,j+1}^{h\to g}, \tag{2.37}$$

$$M_{ij} = -\frac{x_{j-1}}{\Delta_{j,j-1}} K_1(x_i,x_j) + \frac{1}{\Delta_{j,j-1}} K_1^*(x_i,x_j), \tag{2.38}$$

$$N_{ij} = \frac{x_j}{\Delta_{j,j-1}} K_1(x_i,x_j) - \frac{1}{\Delta_{j,j-1}} K_1^*(x_i,x_j), \tag{2.39}$$

$${}^1M_{ij}^{h\to g} = -\frac{x_{j-1}}{\Delta_{j,j-1}} K_2^{h\to g}(x_i,x_j) + \frac{1}{\Delta_{j,j-1}} K_2^{*h\to g}(x_i,x_j), \tag{2.40}$$

$${}^1N_{ij}^{h\to g} = \frac{x_j}{\Delta_{j,j-1}} K_2^{h\to g}(x_i,x_j) - \frac{1}{\Delta_{j,j-1}} K_2^{*h\to g}(x_i,x_j), \tag{2.41}$$

$${}^2M_{ij}^{h\to g} = \frac{1}{\Delta_{j,j-1}} K_3^{h\to g}(x_i,x_j) \tag{2.42}$$

$${}^2N_{ij}^{h\to g} = -\frac{1}{\Delta_{j,j-1}} K_3^{h\to g}(x_i,x_j), \tag{2.43}$$

$${}^{3}M_{ij}^{h+g} = -\frac{x_{j-1}}{\Delta_{j,j-1}}\, K_3^{h+g}(x_i, x_j) + \frac{1}{\Delta_{j,j-1}}\, K_3^{*h+g}(x_i, x_j) \,, \tag{2.44}$$

$${}^{3}N_{ij}^{h+g} = \frac{x_j}{\Delta_{j,j-1}}\, K_3^{h+g}(x_i, x_j) - \frac{1}{\Delta_{j,j-1}}\, K_3^{*h+g}(x_i, x_j) \,, \tag{2.45}$$

$$K_1(x_i, x_j) = \int_{x_{j-1}}^{x_j} K_1(x_i, x')\, dx' \,, \tag{2.46}$$

$$K_1^*(x_i, x_j) = \int_{x_{j-1}}^{x_j} x' K_1(x_i, x')\, dx' \,, \tag{2.47}$$

$$\begin{aligned} K_2^{h+g}(x_i, x_j) = \int_{x_{j-1}}^{x_j} K_2^{h+g}(x_i, x')\, dx' &= -D^h_{j-1/2}\, k^{h+g}_{j-1/2} \int_{x_{j-1}}^{x_j} K_2(x_i, x')\, dx' = \\ &= -D^h_{j-1/2}\, k^{h+g}_{j-1/2}\, K_2(x_i, x_j) \,, \end{aligned} \tag{2.48}$$

$$\begin{aligned} K_2^{*h+g}(x_i, x_j) = \int_{x_{j-1}}^{x_j} x' K_2^{h+g}(x_i, x')\, dx' &= -D^h_{j-1/2}\, k^{h+g}_{j-1/2} \int_{x_{j-1}}^{x_j} x' K_2(x_i, x')\, dx' = \\ &= -D^h_{j-1/2}\, k^{h+g}_{j-1/2}\, K_2^*(x_i, x_j) \,, \end{aligned} \tag{2.49}$$

$$\begin{aligned} K_3^{h+g}(x_i, x_j) = \int_{x_{j-1}}^{x_j} K_3^{h+g}(x_i, x')\, dx' &= -D^h_{j-1/2}\, k^{h+g}_{j-1/2} \int_{x_{j-1}}^{x_j} K_3(x_i, x')\, dx' = \\ &= -D^h_{j-1/2}\, k^{h+g}_{j-1/2}\, K_3(x_i, x_j) \,, \end{aligned} \tag{2.50}$$

$$\begin{aligned} K_3^{*h+g}(x_i, x_j) = \int_{x_{j-1}}^{x_j} x' K_3^{h+g}(x_i, x')\, dx' &= -D^h_{j-1/2}\, k^{h+g}_{j-1/2} \int_{x_{j-1}}^{x_j} x' K_3(x_i, x')\, dx' = \\ &= -D^h_{j-1/2}\, k^{h+g}_{j-1/2}\, K_3^*(x_i, x_j) \,, \end{aligned} \tag{2.51}$$

$$K_1(x_i,x_j) = \frac{1}{\pi}\left[K_1^{i,j-1} - K_1^{ij}\right] + \frac{2R}{\pi}\left[\Delta_{i,j-1}K_2^{i,j-1} - \Delta_{ij}K_2^{ij}\right], \tag{2.52}$$

$$K_1^*(x_i,x_j) = \frac{x_i}{\pi}\left[K_1^{i,j-1} - K_1^{ij}\right] - \frac{2R}{\pi}\left[(\Delta^2_{i,j-1} - x_i\Delta_{i,j-1})K_2^{i,j-1} - (\Delta^2_{ij} - x_i\Delta_{ij})K_2^{ij}\right], \tag{2.53}$$

$$K_2(x_i,x_j) = -\frac{4}{\pi}\left[\Delta_{i,j-1}K_3^{i,j-1} - \Delta_{ij}K_3^{ij}\right] - \frac{8R^2}{\pi}\left[\Delta_{i,j-1}K_4^{i,j-1} - \Delta_{ij}K_4^{ij}\right], \tag{2.54}$$

$$\begin{aligned} K_2^*(x_i,x_j) = {} & \frac{2}{\pi}\left[(\Delta^2_{i,j-1} - 2x_i\Delta_{i,j-1})K_3^{i,j-1} - (\Delta^2_{ij} - 2x_i\Delta_{ij})K_3^{ij}\right] + \\ & + \frac{8R^2}{\pi}\left[(\Delta^2_{i,j-1} - 2x_i\Delta_{i,j-1})K_4^{i,j-1} - (\Delta^2_{ij} - 2x_i\Delta_{ij})K_4^{ij}\right] - \\ & - \frac{2}{\pi}\left[K_5^{i,j-1} - K_5^{ij}\right], \end{aligned} \tag{2.55}$$

$$\begin{aligned} K_3(x_i,x_j) = {} & -\frac{1}{\pi R}\left[K_5^{i,j-1} - K_5^{ij}\right] + \frac{1}{\pi R}\left[\Delta^2_{i,j-1}K_6^{i,j-1} - \Delta^2_{ij}K_6^{ij}\right] + \\ & + \frac{4R}{\pi}\left[\Delta^2_{i,j-1}K_7^{i,j-1} - \Delta^2_{ij}K_7^{ij}\right], \end{aligned} \tag{2.56}$$

$$\begin{aligned} K_3^*(x_i,x_j) = {} & -\frac{x_i}{\pi R}\left[K_5^{i,j-1} - K_5^{ij}\right] + \frac{x_i}{\pi R}\left[\Delta^2_{i,j-1}K_6^{i,j-1} - \Delta^2_{ij}K_6^{ij}\right] - \\ & - \frac{4R}{\pi}\left[(\Delta^3_{i,j-1} - x_i\Delta^2_{i,j-1})K_7^{i,j-1} - (\Delta^3_{ij} - x_i\Delta^2_{ij})K_7^{ij}\right], \end{aligned} \tag{2.57}$$

where

$$K_1^{ij} = \int_0^{\pi/2} \cos\Psi \ \mathrm{arctg}\,\frac{\Delta_{ij}}{2R\cos\Psi}\, d\Psi, \tag{2.58}$$

$$K_2^{ij} = \int_0^{\pi/2} \frac{\cos^2\Psi}{\Delta^2_{ij} + 4R^2\cos^2\Psi}\, d\Psi, \tag{2.59}$$

$$K_3^{ij} = \int_0^{\pi/2} \frac{\cos^2\Psi}{\left[\Delta^2_{ij} + 4R^2\cos^2\Psi\right]^{1/2}}\, d\Psi, \tag{2.60}$$

$$K_4^{\lambda\gamma} = \int_0^{\pi/2} \frac{\cos^4\psi}{\left[\Delta_{\lambda\gamma}^2 + 4R^2\cos^2\psi\right]^{3/2}}\, d\psi, \tag{2.61}$$

$$K_5^{\lambda\gamma} = \int_0^{\pi/2} \left[\Delta_{\lambda\gamma}^2 + 4R^2\cos^2\psi\right]^{1/2} d\psi, \tag{2.62}$$

$$K_6^{\lambda\gamma} = \int_0^{\pi/2} \frac{d\psi}{\left[\Delta_{\lambda\gamma}^2 + 4R^2\cos^2\psi\right]^{1/2}}, \tag{2.63}$$

$$K_7^{\lambda\gamma} = \int_0^{\pi/2} \frac{\cos^2\psi}{\left[\Delta_{\lambda\gamma}^2 + 4R^2\cos^2\psi\right]^{3/2}}\, d\psi; \tag{2.64}$$

these integrals may be expressed as

$$K_1^{\lambda\gamma} = \frac{\pi}{2}\,\mathrm{sgn}\,\Delta_{\lambda\gamma}\left[1 - \frac{2R}{\sqrt{4R^2+\Delta_{\lambda\gamma}^2} + |\Delta_{\lambda\gamma}|}\right], \tag{2.65}$$

$$K_2^{\lambda\gamma} = \frac{\pi}{2}\,\frac{1}{\sqrt{4R^2+\Delta_{\lambda\gamma}^2}\left[\sqrt{4R^2+\Delta_{\lambda\gamma}^2} + |\Delta_{\lambda\gamma}|\right]}, \tag{2.66}$$

$$K_3^{\lambda\gamma} = \frac{1}{\sqrt{4R^2+\Delta_{\lambda\gamma}^2}}\,\frac{1}{\lambda_{\lambda\gamma}^2}\left[E(\pi/2,\lambda_{\lambda\gamma}) - (1-\lambda_{\lambda\gamma}^2)F(\pi/2,\lambda_{\lambda\gamma})\right], \tag{2.67}$$

$$K_4^{\lambda\gamma} = \frac{1}{\left[4R^2+\Delta_{\lambda\gamma}^2\right]^{3/2}}\,\frac{1}{\lambda_{\lambda\gamma}^4}\left[(2-\lambda_{\lambda\gamma}^2)E(\pi/2,\lambda_{\lambda\gamma}) - 2(1-\lambda_{\lambda\gamma}^2)F(\pi/2,\lambda_{\lambda\gamma})\right], \tag{2.68}$$

$$K_5^{\lambda\gamma} = \sqrt{4R^2+\Delta_{\lambda\gamma}^2}\; E(\pi/2,\lambda_{\lambda\gamma}), \tag{2.69}$$

$$K_6^{\lambda\gamma} = \frac{1}{\sqrt{4R^2+\Delta_{\lambda\gamma}^2}}\, F(\pi/2,\lambda_{\lambda\gamma}), \tag{2.70}$$

$$K_7^{\lambda\gamma} = \frac{1}{\left[4R^2+\Delta_{\lambda\gamma}^2\right]^{3/2}}\,\frac{1}{\lambda_{\lambda\gamma}^2}\left[F(\pi/2,\lambda_{\lambda\gamma}) - E(\pi/2,\lambda_{\lambda\gamma})\right]. \tag{2.71}$$

where $\lambda_{ij}^2 = \dfrac{4R^2}{4R^2 + \Delta_{ij}^2}$,

$$\operatorname{sgn} \Delta_{ij} = \operatorname{sgn}(z_i - z_j) = \begin{cases} 1 & \text{for } \Delta_{ij} > 0 \\ -1 & \text{for } \Delta_{ij} < 0 \end{cases},$$

$$|\Delta_{ij}| = \begin{cases} \Delta_{ij} & \text{for } \Delta_{ij} > 0 \\ -\Delta_{ij} & \text{for } \Delta_{ij} < 0 \end{cases};$$

so far the $\Delta_{ij} = z_i - z_j$ was supposed to be non-equal to zero; for $\Delta_{ii} = z_i - z_i = 0$, i.e. $\lambda_{ii} = 1$ the following table and the limit-formulae are to be used:

K_1^{ii}	K_2^{ii}	K_3^{ii}	K_4^{ii}	K_5^{ii}	K_6^{ii}	K_7^{ii}
0	$\frac{\pi}{8R^2}$	$\frac{1}{2R}$	$\frac{1}{8R^3}$	2R	∞	∞

(2.72)

$$E(\pi/2,\ 1) = 1 \ , \tag{2.73}$$

$$F(\pi/2,\ 1) \to \infty \ , \tag{2.74}$$

$$\lim_{\substack{\Delta \to 0 \\ (\lambda \to 1)}} F(\pi/2,\ \lambda)\,\Delta^n = 0 \ , \quad n = 2;\ 3 \ . \tag{2.75}$$

Therefore, to get the coefficients in the boundary condition (2.32), the following must be calculated:

1) (2.65) to (2.71), using (2.3), (2.4);
2) (2.52) to (2.57); for $j = i$ the table and the limits are to be used;
3) (2.46) to (2.51);
4) (2.38) to (2.54);
5) (2.33) to (2.37).

In the boundary condition (2.32) both the ϕ_0 and the ϕ_N are eliminated using (2.24), (2.26), (2.27), (2.30), and (2.25), (2.28), (2.29), (2.31) :

$$\phi_0^g(R) = \frac{1}{2}\left(\phi_1^g + f(\phi_1^1,\ \phi_1^2,\ \ldots,\ \phi_1^g) \right) ,$$

$$\phi_N^g(R) = \frac{1}{2}\left(\phi_{N-1}^g + f(\phi_{N-1}^1,\ \phi_{N-1}^2,\ \ldots,\ \phi_{N-1}^g) \right) .$$

B) Constant approximation of the neutron flux

The height of the channel is to be divided into subintervals $\langle z_{j-1}, z_j \rangle$, $j = 1, 2, \ldots, N$ by a system of mesh points $0 = z_0 < z_1 < \ldots < z_j < \ldots < z_N = H$. Furthermore we define another system of subintervals

$\langle z_{j-1/2}, z_{j+1/2} \rangle = \left\langle \frac{z_{j-1} + z_j}{2}, \frac{z_j + z_{j+1}}{2} \right\rangle$ by a system of mesh points

$z_{j-1/2} = \frac{z_{j-1} + z_j}{2}$ for $j = 1, 2, \ldots, N$ with the only limitation of the selection of the mesh points that a possible boundary between two different materials (perpendicular to the channel) is to be chosen as a mesh point. The neutron flux $\emptyset$ and the anisotropic terms S_r, S_z of the source function are supposed to be constant, and that

$$\emptyset(r, z) = \emptyset(r, z_{j-1/2}) = \emptyset_{j-1/2}(r) ,$$

$$S_r(r, z) = S_r(r, z_{j-1/2}) = S_{r,j-1/2}(r) ,$$

$$S_z(r, z) = S_z(r, z_{j-1/2}) = S_{z,j-1/2}(r)$$

within the subinterval $\langle z_{j-1}, z_j \rangle$ for $j = 1, 2, \ldots, N$. The constant diffusion parameters within the subinterval $\langle z_{j-1/2}, z_{j+1/2} \rangle$ are indexed with j.

$$D_j^g = \frac{1}{3\Sigma_{t,j}^g} \qquad (2.76)$$

$$C_j^{h\to g} = -3 D_j^g \left[\mu^{h\to g} \Sigma_s^{h\to g} \right]_j \qquad (2.77)$$

$$\left.\begin{aligned} & K_j^{g\to g} = 1 , \\ & K_j^{g-\alpha\to g} = - C_j^{g-\alpha\to g} - \sum_{\beta=1}^{\alpha-1} C_j^{g-\alpha\to g-\beta} K_j^{g-\beta\to g} \\ & \alpha = 1, 2, \ldots, g-1 . \end{aligned}\right\} \qquad (2.78)$$

As any of the mesh points $z_{j-1/2}$ for $j = 1, 2, \ldots, N$ may separate two different materials, an effective value of the constant $K^{h\to g}$ in the point $z_{j-1/2}$ is defined:

$$K_{j-1/2}^{h\to g} = \frac{\Delta_{j-1/2,j-1} K_{j-1}^{h\to g} + \Delta_{j,j-1/2} K_j^{h\to g}}{\Delta_{j-1/2,j-1} + \Delta_{j,j-1/2}} , \qquad (2.79)$$

where $\Delta_{j-1/2,j-1} = z_{j-1/2} - z_{j-1}$ etc. Further we define two points $z_{-1/2}$ (symmetrical to $z_{1/2}$ according to $z_0 = 0$), $z_{N+1/2}$ (symmetrical to $z_{N-1/2}$ according to $z_N = H$). The diffusion constants $K^{h\to g}$ in the points $z = 0$ and $z = H$ are equal to those within the subintervals $\langle 0, z_{1/2} \rangle$,

($z_{H+1/2}$, R), the neutron fluxes $\phi^g_{-1/2}$, $\phi^g_{H+1/2}$ in the points $z_{-1/2}$, $z_{H+1/2}$ are defined by using the boundary conditions on the boundaries $z = 0$ and $z = H$:

$$\phi^g(r,0) + 2\,J_z^g(r,0) = 0\ ; \tag{2.80}$$

$$\phi^g(r,H) - 2\,J_z^g(r,H) = 0\ ; \tag{2.81}$$

using (2.11) the (2.80), (2.81) may be written as

$$\phi^g(R,0) - 2\sum_{h=1}^{g} D_0^h\, k_0^{h\to g}\left[\left[\frac{\partial \phi^h(R,z)}{\partial z}\right]_0 + 3\,S_z^h(R,0)\right] = 0\ , \tag{2.82}$$

$$\phi^g(R,H) + 2\sum_{h=1}^{g} D_H^h\, k_H^{h\to g}\left[\left[\frac{\partial \phi^h(R,z)}{\partial z}\right]_H + 3\,S_z^h(R,H)\right] = 0\ ; \tag{2.83}$$

The following definitions are being used ;

$$\phi^g(R,0) = \frac{\phi^g_{1/2}(R) + \phi^g_{-1/2}(R)}{2}\ , \tag{2.84}$$

$$\left[\frac{\partial \phi^g(R,z)}{\partial z}\right]_0 = \frac{\phi^g_{1/2}(R) - \phi_{-1/2}(R)}{\Delta_{10}}\ , \tag{2.85}$$

$$\phi^g(R,H) = \frac{\phi^g_{H+1/2}(R) + \phi_{H-1/2}(R)}{2}\ , \tag{2.86}$$

$$\left[\frac{\partial \phi^g(R,z)}{\partial z}\right]_H = \frac{\phi^g_{H+1/2}(R) - \phi^g_{H-1/2}(R)}{\Delta_{H,H-1}}\ , \tag{2.87}$$

where $\Delta_{10} = z_1 - z_0$, $\Delta_{H,H-1} = z_H - z_{H-1}$; substituting this into (2.82) and (2.83) we get after some rearrangement two systems of algebraic equations

$$\sum_{h=1}^{g} A_0^{h\to g}\,\phi^h_{-1/2}(R) = F_0\left[\phi^1_{1/2}, \phi^2_{1/2}, \dots, \phi^g_{1/2}\right]\ , \tag{2.88}$$

$$\sum_{h=1}^{g} A_H^{h\to g}\,\phi^h_{H+1/2}(R) = F_H\left[\phi^1_{H-1/2}, \phi^2_{H-1/2}, \dots, \phi^g_{H-1/2}\right]\ , \tag{2.89}$$

$$A_0^{g\to g} = 1$$

$$A_0^{h\to g} = \frac{\frac{4}{\Delta_{10}} D_0^h k_0^{h\to g}}{1 + \frac{4}{\Delta_{10}} D_0^g k_0^{g\to g}} \qquad \text{for} \quad h < g \tag{2.90}$$

$$F_0\left[\phi_{1/2}^1, \phi_{1/2}^2, \ldots, \phi_{1/2}^g\right] = -\frac{1 - \frac{4}{\Delta_{10}} D_0^g k_0^{g\to g}}{1 + \frac{4}{\Delta_{10}} D_0^g k_0^{g\to g}} \phi_{1/2}^g(R) +$$

$$+ \sum_{\substack{h=1 \\ g\neq 1}}^{g-1} \frac{\frac{4}{\Delta_{10}} D_0^h k_0^{h\to g}}{1 + \frac{4}{\Delta_{10}} D_0^g k_0^{g\to g}} \left[\phi_{1/2}^h(R) + 3\Delta_{10} S_{X,1/2}^h(R)\right], \tag{2.91}$$

$$A_N^{g\to g} = 1$$

$$A_N^{h\to g} = \frac{\frac{4}{\Delta_{N,N-1}} D_N^h k_N^{h\to g}}{1 + \frac{4}{\Delta_{N,N-1}} D_N^g k_N^{g\to g}} \qquad \text{for} \quad h < g \tag{2.92}$$

$$F_N\left[\phi_{N-1/2}^1, \phi_{N-1/2}^2, \ldots, \phi_{N-1/2}^g\right] = -\frac{1 - \frac{4}{\Delta_{N,N-1}} D_N^g k_N^{g\to g}}{1 + \frac{4}{\Delta_{N,N-1}} D_N^g k_N^{g\to g}} \phi_{N-1/2}^g(R) +$$

$$+ \sum_{\substack{h=1 \\ g\neq 1}}^{g-1} \frac{\frac{4}{\Delta_{N,N-1}} D_N^h k_N^{h\to g}}{1 + \frac{4}{\Delta_{N,N-1}} D_N^g k_N^{g\to g}} \left[\phi_{N-1/2}^h(R) - 3\Delta_{N,N-1} S_{X,N-1/2}^h(R)\right], \tag{2.93}$$

being $\| B_0^{h\to g} \|$, $\| B_N^{h\to g} \|$ the inverse matrices of $\| A_0^{h\to g} \|$, $\| A_N^{h\to g} \|$, the solutions of (2.88), (2.89) may be written as:

$$\phi_{1/2}^g(R) = \sum_{h=1}^{g} B_0^{h\to g} F_0\left[\phi_{1/2}^1, \phi_{1/2}^2, \ldots, \phi_{1/2}^h\right] \tag{2.94}$$

$$\phi_{N-1/2}^g(R) = \sum_{h=1}^{g} B_N^{h\to g} F_N\left[\phi_{N-1/2}^1, \phi_{N-1/2}^2, \ldots, \phi_{N-1/2}^h\right] \tag{2.95}$$

$$\left.\begin{aligned}
&B_0^{g\to g} = 1\,,\\
&B_0^{g-\alpha\to g} = -A_0^{g-\alpha\to g} - \sum_{\beta=1}^{\alpha-1} A_0^{g-\alpha\to g-\beta}\, B_0^{g-\beta\to g}\\
&\alpha = 1, 2, \ldots, g-1\,,
\end{aligned}\right\} \qquad (2.96)$$

$$F_0\left[\Phi^1_{1/2}, \Phi^2_{1/2}, \ldots, \Phi^h_{1/2}\right] = -\frac{1-\dfrac{4}{\Delta_{10}} D_0^h\, k_0^{h\to h}}{1+\dfrac{4}{\Delta_{10}} D_0^h\, k_0^{h\to h}}\,\Phi^h_{1/2}(R) + \sum_{\substack{\sigma=1\\ h\neq 1}}^{h-1} \frac{\dfrac{4}{\Delta_{10}} D_0^\sigma\, k_0^{\sigma\to h}}{1+\dfrac{4}{\Delta_{10}} D_0^h\, k_0^{h\to h}} \left[\Phi^\sigma_{1/2}(R) + 3\Delta_{10}\, S^\sigma_{x,1/2}(R)\right], \qquad (2.97)$$

$$\left.\begin{aligned}
&B_N^{g\to g} = 1\,,\\
&B_N^{g-\alpha\to g} = -A_N^{g-\alpha\to g} - \sum_{\beta=1}^{\alpha-1} A_N^{g-\alpha\to g-\beta}\, B_N^{g-\beta\to g}\,,\\
&\alpha = 1, 2, \ldots, g-1\,,
\end{aligned}\right\} \qquad (2.98)$$

$$F_N\left[\Phi^1_{N-1/2}, \Phi^2_{N-1/2}, \ldots, \Phi^h_{N-1/2}\right] = -\frac{1-\dfrac{4}{\Delta_{N,N-1}} D_N^h\, k_N^{h\to h}}{1+\dfrac{4}{\Delta_{N,N-1}} D_N^h\, k_N^{h\to h}}\,\Phi^h_{N-1/2}(R) + \sum_{\substack{\sigma=1\\ h\neq 1}}^{h-1} \frac{\dfrac{4}{\Delta_{N,N-1}} D_N^\sigma\, k_N^{\sigma\to h}}{1+\dfrac{4}{\Delta_{N,N-1}} D_N^h\, k_N^{h\to h}} \left[\Phi^\sigma_{N-1/2}(R) - 3\Delta_{N,N-1}\, S^\sigma_{x,N-1/2}(R)\right]. \qquad (2.99)$$

Thus

$$\phi^g_{-1/2} = f(\phi^1_{1/2}, \phi^2_{1/2}, \ldots, \phi^g_{1/2}), \qquad (2.100)$$

$$\phi^g_{N+1/2} = f(\phi^1_{N-1/2}, \phi^2_{N-1/2}, \ldots, \phi^g_{N-1/2}). \qquad (2.101)$$

Now the boundary condition (1.12) may be written as

$$\Phi^{g}_{i-1/2}(R) + \sum_{h=1}^{g} K^{h\to g}_{i-1/2,ef}\left[\frac{\partial\Phi^{h}_{i-1/2}(r)}{\partial r}\right]_R + 3\sum_{h=1}^{g} K^{h\to g}_{i-1/2,ef}\, S^{h}_{i-1/2}(R) =$$

$$= \sum_{j=0}^{N+1}\sum_{h=1}^{g}\left[\alpha^{h\to g}_{ij}\left[\frac{\partial\Phi^{h}_{j-1/2}(r)}{\partial r}\right]_R + \beta^{h\to g}_{ij}\,\Phi^{h}_{j-1/2}(R)\right] +$$

$$+ 3\sum_{j=0}^{N+1}\sum_{h=1}^{g}\left[\alpha^{h\to g}_{ij}\, S^{h}_{r,j-1/2}(R) + \gamma^{h\to g}_{ij}\, S^{h}_{x,j-1/2}(R)\right], \tag{2.102}$$

$$\beta^{h\to g}_{ij} = \left[1-(1-\delta_{hg})\right]\alpha_{ij} + {}^{*}\beta^{h\to g}_{ij}, \tag{2.103}$$

$$\alpha_{ij} = (1-\delta_{j0})(1-\delta_{jN})\, M_{ij}, \tag{2.104}$$

$$\alpha^{h\to g}_{ij} = (1-\delta_{j0})(1-\delta_{jN})\, {}^{1}M^{h\to g}_{ij}, \tag{2.105}$$

$${}^{*}\beta^{h\to g}_{ij} = (1-\delta_{j0})(1-\delta_{jN})\, {}^{2}M^{h\to g}_{ij} + (1-\delta_{j0})(1-\delta_{j1})\, N^{h\to g}_{ij} + (1-\delta_{jN})(1-\delta_{j,N+1})\, L^{h\to g}_{ij}, \tag{2.106}$$

$$\gamma^{h\to g}_{ij} = (1-\delta_{j0})(1-\delta_{jN})\, {}^{3}M^{h\to g}_{ij}, \tag{2.107}$$

$$M_{ij} = \int_{x_{j-1}}^{x_j} K_1(x_{i-1/2}, x')\, dx', \tag{2.108}$$

$${}^{1}M^{h\to g}_{ij} = -\left[D^{h}_{j-1}\, k^{h\to g}_{j-1}\int_{x_{j-1}}^{x_{j-1/2}} K_2(x_{i-1/2}, x')\, dx' + D^{h}_{j}\, k^{h\to g}_{j}\int_{x_{j-1/2}}^{x_j} K_2(x_{i-1/2}, x')\, dx'\right], \tag{2.109}$$

$${}^{2}M^{h\to g}_{ij} = -\left[D^{h}_{j-1}\, k^{h\to g}_{j-1}\, K_3(x_{i-1/2}, x_{j-1}) - D^{h}_{j}\, k^{h\to g}_{j}\, K_3(x_{i-1/2}, x_j)\right], \tag{2.110}$$

$$N^{h\to g}_{ij} = -D^{h}_{j-1}\, k^{h\to g}_{j-1}\, K_3(x_{i-1/2}, x_{j-1}), \tag{2.111}$$

$$L^{h\to g}_{ij} = D^{h}_{j}\, k^{h\to g}_{j}\, K_3(x_{i-1/2}, x_j), \tag{2.112}$$

$$ {}^{3}M^{h\to g}_{ij} = -\left[D^{h}_{j-1}\, k^{h\to g}_{j-1} \int_{x_{j-1}}^{x_{j-1/2}} K_3(x_{i-1/2}, x')\,dx' + D^{h}_{j}\, k^{h\to g}_{j} \int_{x_{j-1/2}}^{x_{j}} K_3(x_{i-1/2}, x')\,dx' \right], \tag{2.113} $$

$$ \int_{x_{j-1}}^{x_j} K_1(x_{i-1/2}, x')\,dx' = \frac{1}{\pi}\left[K_1^{i-1/2,j-1} - K_1^{i-1/2,j} \right] + \frac{2R}{\pi}\left[\Delta_{i-1/2,j-1} K_2^{i-1/2,j-1} - \Delta_{i-1/2,j} K_2^{i-1/2,j} \right], \tag{2.114} $$

$$ \int_{x_{j-1}}^{x_{j-1/2}} K_2(x_{i-1/2}, x')\,dx' = -\frac{4}{\pi}\left[\Delta_{i-1/2,j-1} K_3^{i-1/2,j-1} - \Delta_{i-1/2,j-1/2} K_3^{i-1/2,j-1/2} \right] - \frac{8R^2}{\pi}\left[\Delta_{i-1/2,j-1} K_4^{i-1/2,j-1} - \Delta_{i-1/2,j-1/2} K_4^{i-1/2,j-1/2} \right], \tag{2.115} $$

$$ \int_{x_{j-1/2}}^{x_j} K_2(x_{i-1/2}, x')\,dx' = -\frac{4}{\pi}\left[\Delta_{i-1/2,j-1/2} K_3^{i-1/2,j-1/2} - \Delta_{i-1/2,j} K_3^{i-1/2,j} \right] - \frac{8R^2}{\pi}\left[\Delta_{i-1/2,j-1/2} K_4^{i-1/2,j-1/2} - \Delta_{i-1/2,j} K_4^{i-1/2,j} \right], \tag{2.116} $$

$$ \int_{x_{j-1}}^{x_{j-1/2}} K_3(x_{i-1/2}, x')\,dx' = -\frac{1}{\pi R}\left[K_5^{i-1/2,j-1} - K_5^{i-1/2,j-1/2} \right] + \frac{1}{\pi R}\left[\Delta^2_{i-1/2,j-1} K_6^{i-1/2,j-1} - \Delta^2_{i-1/2,j-1/2} K_6^{i-1/2,j-1/2} \right] + \frac{4R}{\pi}\left[\Delta^2_{i-1/2,j-1} K_7^{i-1/2,j-1} - \Delta^2_{i-1/2,j-1/2} K_7^{i-1/2,j-1/2} \right], \tag{2.117} $$

$$ \int_{x_{j-1/2}}^{x_j} K_3(x_{i-1/2}, x')\,dx' = -\frac{1}{\pi R}\left[K_5^{i-1/2,j-1/2} - K_5^{i-1/2,j} \right] + \frac{1}{\pi R}\left[\Delta^2_{i-1/2,j-1/2} K_6^{i-1/2,j-1/2} - \Delta^2_{i-1/2,j} K_6^{i-1/2,j} \right] + \frac{4R}{\pi}\left[\Delta^2_{i-1/2,j-1/2} K_7^{i-1/2,j-1/2} - \Delta^2_{i-1/2,j} K_7^{i-1/2,j} \right], \tag{2.118} $$

$$K_3(x_{i-1/2},\ x_{j-1}) = \frac{4 \varepsilon R^3}{\pi} \Delta_{i-1/2,\,j-1}\ K_8^{i-1/2,\,j-1}\ , \tag{2.119}$$

$$K_3(x_{i-1/2},\ x_j\ ) = \frac{4 \varepsilon R^3}{\pi} \Delta_{i-1/2,\,j}\ K_8^{i-1/2,\,j}\ , \tag{2.120}$$

$$K_1^{i-1/2,\nu} = \int_0^{\pi/2} \cos\Psi\ \operatorname{arctg} \frac{\Delta_{i-1/2,\nu}}{2R\cos\Psi}\, d\Psi\ , \tag{2.121}$$

$$K_2^{i-1/2,\nu} = \int_0^{\pi/2} \frac{\cos^2\Psi}{\Delta^2_{i-1/2,\nu} + 4R^2\cos^2\Psi}\, d\Psi\ , \tag{2.122}$$

$$K_3^{i-1/2,\nu} = \int_0^{\pi/2} \frac{\cos^2\Psi}{\left[\Delta^2_{i-1/2,\nu} + 4R^2\cos^2\Psi\right]^{1/2}}\, d\Psi\ , \tag{2.123}$$

$$K_4^{i-1/2,\nu} = \int_0^{\pi/2} \frac{\cos^4\Psi}{\left[\Delta^2_{i-1/2,\nu} + 4R^2\cos^2\Psi\right]^{3/2}}\, d\Psi\ , \tag{2.124}$$

$$K_5^{i-1/2,\nu} = \int_0^{\pi/2} \left[\Delta^2_{i-1/2,\nu} + 4R^2\cos^2\Psi\right]^{1/2} d\Psi\ , \tag{2.125}$$

$$K_6^{i-1/2,\nu} = \int_0^{\pi/2} \frac{d\Psi}{\left[\Delta^2_{i-1/2,\nu} + 4R^2\cos^2\Psi\right]^{1/2}}\ , \tag{2.126}$$

$$K_7^{i-1/2,\nu} = \int_0^{\pi/2} \frac{\cos^2\Psi}{\left[\Delta^2_{i-1/2,\nu} + 4R^2\cos^2\Psi\right]^{3/2}}\, d\Psi\ , \tag{2.127}$$

$$K_8^{i-1/2,\nu} = \int_0^{\pi/2} \frac{\cos^4\Psi}{\left[\Delta^2_{i-1/2,\nu} + 4R^2\cos^2\Psi\right]^{5/2}}\, d\Psi\ ; \tag{2.128}$$

these integrals may be expressed as

$$K_1^{i-1/2,\nu} = \frac{\pi}{2}\, \mathrm{sgn}\, \Delta_{i-1/2,\nu} \left[1 - \frac{2R}{\sqrt{4R^2 + \Delta^2_{i-1/2,\nu}} + \left| \Delta_{i-1/2,\nu} \right|} \right], \tag{2.129}$$

$$K_2^{i-1/2,\nu} = \frac{\pi}{2} \frac{1}{\sqrt{4R^2 + \Delta^2_{i-1/2,\nu}} \left[\sqrt{4R^2 + \Delta^2_{i-1/2,\nu}} + \left| \Delta_{i-1/2,\nu} \right| \right]}, \tag{2.130}$$

$$K_3^{i-1/2,\nu} = \frac{1}{\sqrt{4R^2 + \Delta^2_{i-1/2,\nu}}} \frac{1}{\lambda^2_{i-1/2,\nu}} \left[E(\pi/2, \lambda_{i-1/2,\nu}) - (1 - \lambda^2_{i-1/2,\nu}) F(\pi/2, \lambda_{i-1/2,\nu}) \right], \tag{2.131}$$

$$K_4^{i-1/2,\nu} = \frac{1}{[4R^2 + \Delta^2_{i-1/2,\nu}]^{3/2}} \frac{1}{\lambda^4_{i-1/2,\nu}} \Big[(2 - \lambda^2_{i-1/2,\nu}) E(\pi/2, \lambda_{i-1/2,\nu}) -$$
$$- 2(1 - \lambda^2_{i-1/2,\nu}) F(\pi/2, \lambda_{i-1/2,\nu}) \Big], \tag{2.132}$$

$$K_5^{i-1/2,\nu} = \sqrt{4R^2 + \Delta^2_{i-1/2,\nu}}\; E(\pi/2, \lambda_{i-1/2,\nu}), \tag{2.133}$$

$$K_6^{i-1/2,\nu} = \frac{1}{\sqrt{4R^2 + \Delta^2_{i-1/2,\nu}}} F(\pi/2, \lambda_{i-1/2,\nu}), \tag{2.134}$$

$$K_7^{i-1/2,\nu} = \frac{1}{[4R^2 + \Delta^2_{i-1/2,\nu}]^{3/2}} \frac{1}{\lambda^2_{i-1/2,\nu}} \left[F(\pi/2, \lambda_{i-1/2,\nu}) - E(\pi/2, \lambda_{i-1/2,\nu}) \right], \tag{2.135}$$

$$K_8^{i-1/2,\nu} = \frac{1}{[4R^2 + \Delta^2_{i-1/2,\nu}]^{5/2}} \frac{1}{3\lambda^4_{i-1/2,\nu}} \Big[(2 + \lambda^2_{i-1/2,\nu}) F(\pi/2, \lambda_{i-1/2,\nu}) -$$
$$- 2(1 + \lambda^2_{i-1/2,\nu}) E(\pi/2, \lambda_{i-1/2,\nu}) \Big], \tag{2.136}$$

where $\lambda^2_{i-1/2,\nu} = \dfrac{4R^2}{4R^2 + \Delta^2_{i-1/2,\nu}}$,

$$\mathrm{sgn}\, \Delta_{i-1/2,\nu} = \mathrm{sgn}(x_{i-1/2} - x_\nu) = \begin{cases} 1 & \text{for} \quad \Delta_{i-1/2,\nu} > 0 \\ -1 & \text{for} \quad \Delta_{i-1/2,\nu} < 0 , \end{cases}$$

$$\left| \Delta_{i-1/2,\nu} \right| = \begin{cases} \Delta_{i-1/2,\nu} & \text{for} \quad \Delta_{i-1/2,\nu} > 0 \\ -\Delta_{i-1/2,\nu} & \text{for} \quad \Delta_{i-1/2,\nu} < 0 , \end{cases}$$

so far the $\Delta_{i-1/2,\nu} = z_{i-1/2} - z_\nu$ was supposed to be non-equal to zero; for $\Delta_{i-1/2,i-1/2} = z_{i-1/2} - z_{i-1/2} = 0$, i.e. $\lambda_{i-1/2,i-1/2} = 1$ the following table and the limit formulae are used:

$K_1^{i-1/2,i-1/2}$	$K_2^{i-1/2,i-1/2}$	$K_3^{i-1/2,i-1/2}$	$K_4^{i-1/2,i-1/2}$	$K_5^{i-1/2,i-1/2}$
0	$\frac{\pi}{8R^2}$	$\frac{1}{2R}$	$\frac{1}{8R^3}$	$2R$

$K_6^{i-1/2,i-1/2}$	$K_7^{i-1/2,i-1/2}$	$K_8^{i-1/2,i-1/2}$
∞	∞	∞

$$E(\pi/2,\ 1) = 1\ ,$$

$$F(\pi/2,\ 1) \to \infty\ ,$$

$$\lim_{\substack{\Delta \to 0 \\ (\lambda \to 1)}} F(\pi/2,\ \lambda)\,\Delta^n = 0\ , \quad n = 1;\ 2\ .$$

To get the coefficients in the boundary condition (2.102), the following must be used:

1) (2.129) to (2.136), using (2.3), (2.4);
2) (2.114) to (2.120); for $\nu = i-1/2$ the table and the limits must be used;
3) (2.108) to (2.113);
4) (2.103) to (2.107).

3. Numerical analysis of the boundary condition in the diffusion approximation

Now the boundary condition (1.22) to (1.27) will be discussed. Again the formulae (2.1) to (2.4) are used.

A) Linear approximation of the neutron flux

We get the formulation immediately from the problem 1. A) , putting $S_r^g = 0$, $S_\gamma^g = 0$, and $\mu^{h \to g} = 0$ for $h < g$. Then the boundary condition (1.22) may be written for $i = 0, 1, 2, \ldots, H$ as

$$\phi_i^g(R) - 2D_{i,ef}^g \left[\frac{\partial \phi_i^g(r)}{\partial r}\right]_R - 6D_{i,ef}^g\, S_i^g(R) = \sum_{j=0}^{H} \left[\alpha_{ij}^g \left[\frac{\partial \phi_j^g(r)}{\partial r}\right]_R + \beta_{ij}^g\, \phi_j^g(R)\right], \tag{3.1}$$

$$D_{i,ef}^g = \frac{\Delta_{i,i-1} D_{i-1/2}^g + \Delta_{i+1,i} D_{i+1/2}^g}{\Delta_{i,i-1} + \Delta_{i+1,i}}, \quad (i = 1, 2, \ldots, H-1), \tag{3.2}$$

$$\text{and} \quad D_{0,ef}^g = D_{1/2}^g, \quad D_{H,ef}^g = D_{H-1/2}^g, \tag{3.3}$$

$$\beta_{ij}^g = \alpha_{ij} + {}^*\beta_{ij}^g, \tag{3.4}$$

$$\alpha_{ij} = (1-\delta_{j0})\, M_{ij} + (1-\delta_{jH})\, N_{i,j+1}, \tag{3.5}$$

$$\alpha_{ij}^g = (1-\delta_{j0})\, {}^1M_{ij}^g + (1-\delta_{jH})\, {}^1N_{i,j+1}^g, \tag{3.6}$$

$${}^*\beta_{ij}^g = (1-\delta_{j0})\, {}^2M_{ij}^g + (1-\delta_{jH})\, {}^2N_{i,j+1}^g, \tag{3.7}$$

$$M_{ij} = -\frac{x_{j-1}}{\Delta_{j,j-1}} K_1(x_i, x_j) + \frac{1}{\Delta_{j,j-1}} K_1^*(x_i, x_j), \tag{3.8}$$

$$N_{ij} = \frac{x_j}{\Delta_{j,j-1}} K_1(x_i, x_j) - \frac{1}{\Delta_{j,j-1}} K_1^*(x_i, x_j), \tag{3.9}$$

$${}^1M_{ij}^g = -\frac{x_{j-1}}{\Delta_{j,j-1}} K_2^g(x_i, x_j) + \frac{1}{\Delta_{j,j-1}} K_2^{*g}(x_i, x_j), \tag{3.10}$$

$${}^1N_{ij}^g = \frac{x_j}{\Delta_{j,j-1}} K_2^g(x_i, x_j) - \frac{1}{\Delta_{j,j-1}} K_2^{*g}(x_i, x_j), \tag{3.11}$$

$$ {}^2M^g_{ij} = \frac{1}{\Delta_{j,j-1}} K^g_3(x_i, x_j) , \tag{3.12} $$

$$ {}^2N^g_{ij} = -\frac{1}{\Delta_{j,j-1}} K^g_3(x_i, x_j) , \tag{3.13} $$

$$ K_1(x_i, x_j) = \int_{x_{j-1}}^{x_j} K_1(x_i, x')\,dx' , \tag{3.14} $$

$$ K^*_1(x_i, x_j) = \int_{x_{j-1}}^{x_j} x' K_1(x_i, x')\,dx' , \tag{3.15} $$

$$ K^g_2(x_i, x_j) = -D^g_{j-1/2} \int_{x_{j-1}}^{x_j} K_2(x_i, x')\,dx' = -D^g_{j-1/2} K_2(x_i, x_j) , \tag{3.16} $$

$$ K^{*g}_2(x_i, x_j) = -D^g_{j-1/2} \int_{x_{j-1}}^{x_j} x' K_2(x_i, x')\,dx' = -D^g_{j-1/2} K^*_2(x_i, x_j) ; \tag{3.17} $$

further the formulae (2.52) to (2.75) are used.

In the boundary condition (3.1) both the ϕ_0 and ϕ_N are eliminated. Immediately from (2.14), (2.16), (2.24), (2.25), (2.26), (2.27), (2.28), (2.29) follows

$$ \phi^g_{-1} = -\frac{1 - \frac{2}{\Delta_{10}} D^g_{1/2}}{1 + \frac{2}{\Delta_{10}} D^g_{1/2}} \phi^g_1 = -\frac{1 - \frac{1}{\Delta_{10}} \frac{2}{3} \lambda^g_{tr,1/2}}{1 + \frac{1}{\Delta_{10}} \frac{2}{3} \lambda^g_{tr,1/2}} \phi^g_1 = -\frac{1 - \frac{d^g_0}{\Delta_{10}}}{1 + \frac{d^g_0}{\Delta_{10}}} \phi^g_1 , \tag{3.18} $$

$$ \phi^g_{N+1} = -\frac{1 - \frac{2}{\Delta_{N,N-1}} D^g_{N-1/2}}{1 + \frac{2}{\Delta_{N,N-1}} D^g_{N-1/2}} \phi^g_{N-1} = -\frac{1 - \frac{1}{\Delta_{N,N-1}} \frac{2}{3} \lambda^g_{tr,N-1/2}}{1 + \frac{1}{\Delta_{N,N-1}} \frac{2}{3} \lambda^g_{tr,N-1/2}} \phi^g_{N-1} = -\frac{1 - \frac{d^g_N}{\Delta_{N,N-1}}}{1 + \frac{d^g_N}{\Delta_{N,N-1}}} \phi^g_{N-1} , \tag{3.19} $$

where $d^g_0 = \frac{2}{3}\lambda^g_{tr,1/2} = 2\,D^g_{1/2}$, $d^g_N = \frac{2}{3}\lambda^g_{tr,N-1/2} = 2\,D^g_{N-1/2}$ is the extrapolated length; from the corrections of the diffusion theory in the vicinity of a boundary follows

$$d_o^g = 0.7104\,\lambda^g_{tr,1/2} = 2.1312\ D^g_{1/2}\ , \qquad (3.20)$$

$$d_N^g = 0.7104\,\lambda^g_{tr,N-1/2} = 2.1312\ D^g_{N-1/2}\ ; \qquad (3.21)$$

now the $\phi^g_o = \frac{1}{2}(\phi^g_1 + \phi^g_{-1})$, $\phi^g_N = \frac{1}{2}(\phi^g_{N-1} + \phi^g_{N-1})$ may be written as

$$\Phi^g_o = \frac{2.1312\ D^g_{1/2}}{\Delta_{1o} + 2.1312\ D^g_{1/2}}\ \Phi^g_1 \qquad (3.22)$$

$$\Phi^g_N = \frac{2.1312\ D^g_{N-1/2}}{\Delta_{N,N-1} + 2.1312\ D^g_{N-1/2}}\ \Phi^g_{N-1} \qquad (3.23)$$

the diffusion approximation gives the following relations:

$\Delta_{1o} < d^g_o$: $\phi^g_{-1} > 0$, $\phi^g_o > \frac{1}{2}\phi^g_1$,

$\Delta_{1o} = d^g_o$: $\phi^g_{-1} = 0$, $\phi^g_o = \frac{1}{2}\phi^g_1$,

$\Delta_{1o} > d^g_o$: $\phi^g_{-1} < 0$, $\phi^g_o < \frac{1}{2}\phi^g_1$,

$\Delta_{N,N-1} < d^g_N$: $\phi^g_{N+1} > 0$, $\phi^g_N > \frac{1}{2}\phi^g_{N-1}$,

$\Delta_{N,N-1} = d^g_N$: $\phi^g_{N+1} = 0$, $\phi^g_N = \frac{1}{2}\phi^g_{N-1}$,

$\Delta_{N,N-1} > d^g_N$: $\phi^g_{N+1} < 0$, $\phi^g_N < \frac{1}{2}\phi^g_{N-1}$;

the $\phi^g_{-1} < 0$, $\phi^g_{N+1} < 0$ are negative fictitious fluxes; in fact this is not true, as the fluxes are positive throughout the space; it is only an incorrectness due to the diffusion approximation.

B) Constant approximation of the neutron flux

The boundary condition (2.102) may be written for $i = 1, 2, \ldots, N$ as

$$\Phi^g_{i-1/2}(R) - 2D^g_{i-1/2,ef}\left[\frac{\partial \Phi^g_{j-1/2}(r)}{\partial r}\right]_R - 6D^g_{i-1/2,ef}\, S^g_{i-1/2}(R) =$$

$$= \sum_{j=0}^{N+1}\left[\alpha^g_{ij}\left[\frac{\partial \Phi^g_{j-1/2}(r)}{\partial r}\right]_R + \beta^g_{ij}\,\Phi^g_{j-1/2}\right], \tag{3.24}$$

$$D^g_{i-1/2,ef} = \frac{\Delta_{i-1/2,i-1}\, D^g_{i-1} + \Delta_{i,i-1/2}\, D^g_i}{\Delta_{i-1/2,i-1} + \Delta_{i,i-1/2}}, \quad (\, i = 1, 2, \ldots, N\,), \tag{3.25}$$

and $D^g_{o,ef} = D^g_o$, $D^g_{N,ef} = D^g_N$, (3.26)

$$\beta^g_{ij} = \alpha_{ij} + {}^*\beta^g_{ij}, \tag{3.27}$$

$$\alpha_{ij} = (1-\delta_{j0})(1-\delta_{jN})\; M_{ij}, \tag{3.28}$$

$$\alpha^g_{ij} = (1-\delta_{j0})(1-\delta_{jN})\; {}^1M^g_{ij}, \tag{3.29}$$

$${}^*\beta^g_{ij} = (1-\delta_{j0})(1-\delta_{jN})\; {}^2M^g_{ij} + (1-\delta_{j0})(1-\delta_{j1})\; N^g_{ij} + (1-\delta_{jN})(1-\delta_{j,N+1})\; L^g_{ij}, \tag{3.30}$$

$$M_{ij} = \int_{x_{j-1}}^{x_j} K_1(x_{i-1/2}, x')\, dx', \tag{3.31}$$

$${}^1M^g_{ij} = -\left[D^g_{j-1}\int_{x_{j-1}}^{x_{j-1/2}} K_2(x_{i-1/2}, x')\, dx' + D^g_j \int_{x_{j-1/2}}^{x_j} K_2(x_{i-1/2}, x')\, dx'\right], \tag{3.32}$$

$${}^2M^g_{ij} = -\left[D^g_{j-1}\; K_3(x_{i-1/2}, x_{j-1}) - D^g_j\; K_3(x_{i-1/2}, x_j)\right], \tag{3.33}$$

$$N^g_{ij} = -D^g_{j-1}\; K_3(x_{i-1/2}, x_{j-1}), \tag{3.34}$$

$$L^g_{ij} = D^g_j\; K_3(x_{i-1/2}, x_j), \tag{3.35}$$

further the (2.114) to (2.140) are to be used.

The fluxes $\phi^g_{-1/2}$, $\phi^g_{H+1/2}$ are defined as

$$\phi^g_{-1/2} = -\frac{1-\frac{4}{\Delta_{10}}D^g_0}{1+\frac{4}{\Delta_{10}}D^g_0}\,\phi^g_{1/2} = -\frac{1-\frac{2}{\Delta_{10}}\frac{2}{3}\lambda^g_{tr,0}}{1+\frac{2}{\Delta_{10}}\frac{2}{3}\lambda^g_{tr,0}}\,\phi^g_{1/2} = -\frac{1-\frac{2d^g_0}{\Delta_{10}}}{1+\frac{2d^g_0}{\Delta_{10}}}\,\phi^g_{1/2}\,, \tag{3.36}$$

$$\phi^g_{N+1/2} = -\frac{1-\frac{4}{\Delta_{N,N-1}}D^g_N}{1+\frac{4}{\Delta_{N,N-1}}D^g_N}\,\phi^g_{N-1/2} = -\frac{1-\frac{2}{\Delta_{N,N-1}}\frac{2}{3}\lambda^g_{tr,N}}{1+\frac{2}{\Delta_{N,N-1}}\frac{2}{3}\lambda^g_{tr,N}}\,\phi^g_{N-1/2} = -\frac{1-\frac{2d^g_N}{\Delta_{N,N-1}}}{1+\frac{2d^g_N}{\Delta_{N,N-1}}}\,\phi^g_{N-1/2}\,, \tag{3.37}$$

$$d^g_o = 0.7104\,\lambda^g_{tr,o} = 2.1312\,D^g_o\,, \tag{3.38}$$

$$d^g_H = 0.7104\,\lambda^g_{tr,H} = 2.1312\,D^g_H\,.$$

Appendix 1

To get the equation (2.102), the integration of the (1.12) within the interval $\langle z_{j-1}, z_j \rangle$ is to be performed. To express the integral including the derivative according to z', the following theorem is used:

The symbolic derivative of a differetiable function with discontinuities of the first kind is given by the ordinary derivative in the points, where it exists, including the sum of differences multiplied by corresponding delta-functions in all points of discontinuities.

Thence

$$\int_{x_{j-1}}^{x_j} K_3^{h\to g}(x_{i-1/2}, x') \frac{\partial \phi^h(R, x')}{\partial x'} \, dx' =$$

$$= [\phi^h_{j-1/2} - \phi^h_{j-3/2}] \int_{x_{j-1}}^{x_j} [-D^h(x') k^{h\to g}(x')] K_3(x_{i-1/2}, x') \, \delta(x' - x_{j-1}) \, dx' +$$

$$+ [\phi^h_{j+1/2} - \phi^h_{j-1/2}] \int_{x_{j-1}}^{x_j} [-D^h(x') k^{h\to g}(x')] K_3(x_{i-1/2}, x') \, \delta(x' - x_j) \, dx' =$$

$$= -[\phi^h_{j-1/2} - \phi^h_{j-3/2}] D^h_{j-1} k^{h\to g}_{j-1} K_3(x_{i-1/2}, x_{j-1}) -$$

$$- [\phi^h_{j+1/2} - \phi^h_{j-1/2}] D^h_j k^{h\to g}_j K_3(x_{i-1/2}, x_j) .$$

Appendix 2

The following integrals according to / 4 / had been used:

Designation: Gr. : Gröbner; B. : Band; p. : page; f. : formula.

Gr.; B. 1; p. 14; f. 15-11a :

$$\int \frac{dx}{(a^2x^2+b^2)^k} = \frac{1}{2(k-1)b^2}\,\frac{x}{(a^2x^2+b^2)^{k-1}} + \frac{2k-3}{2(k-1)b^2}\int \frac{dx}{(a^2x^2+b^2)^{k-1}} \,, \tag{1}$$

Gr.; B. 1; p. 14; f. 15-12 :

$$\int \frac{dx}{a^2x^2+b^2} = \frac{1}{ab}\,\operatorname{arctg}\frac{ax}{b} + C \,, \tag{2}$$

Gr.; B. 1; p. 43; f. 233-11a :

$$\int \frac{dx}{(ax^2+c)^{k+1/2}} = \frac{1}{(2k-1)c}\,\frac{x}{(ax^2+c)^{k-1/2}} + \frac{2k-2}{(2k-1)c}\int \frac{dx}{(ax^2+c)^{k-1/2}} \,, \quad k \geq 1 \,, \tag{3}$$

Gr.; B. 1; p. 43; f. 233-9a :

$$\int \frac{x^m}{(ax^2+c)^{k+1/2}}\,dx = \frac{1}{(2k-1)c}\,\frac{x^{m+1}}{(ax^2+c)^{k-1/2}} + \frac{2k-m-2}{(2k-1)c}\int \frac{x^m}{(ax^2+c)^{k-1/2}}\,dx \,, \tag{4}$$

Gr.; B. 1; p. 45; f. 234-5b :

$$\int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2} + C \,, \tag{5}$$

Gr.; B. 1; p. 14; f. 15-13a :

$$\int \frac{x^m}{(x^2+a^2)^k}\,dx = \frac{1}{(2k-2)a^2}\,\frac{x^{m+1}}{(x^2+a^2)^{k-1}} - \frac{(m-2k+3)}{(2k-2)a^2}\int \frac{x^m}{(x^2+a^2)^{k-1}}\,dx \,, \tag{6}$$

Gr.; B. 1; p. 63; f. 241-13a :

$$\int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\,d\varphi = E(\pi/2,k) \,, \tag{7}$$

Gr.; B. 1; p. 64; f. 241-12a :

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = F(\pi/2,k) \,, \tag{8}$$

Gr.; B. 1; p. 67; f. 241-12l :

$$\int_0^{\pi/2} \frac{\cos^2\varphi}{\sqrt{1-K^2\sin^2\varphi}}\,d\varphi = -\frac{1-K^2}{K^2}F(\pi/2,K) + \frac{1}{K^2}E(\pi/2,K) \quad, \tag{9}$$

Gr.; B. 1; p. 67; f. 241-14b :

$$\int \frac{\sin^r\varphi\cos^s\varphi}{(1-K^2\sin^2\varphi)^{n+1/2}}\,d\varphi = \frac{-(1-K^2)}{K^2}\int \frac{\sin^r\varphi\cos^{s-2}\varphi}{(1-K^2\sin^2\varphi)^{n+1/2}}\,d\varphi + \frac{1}{K^2}\int \frac{\sin^r\varphi\cos^{s-2}\varphi}{(1-K^2\sin^2\varphi)^{n-1/2}}\,d\varphi \quad, \tag{10}$$

Gr.; B. 1; p. 67; f. 241-18b :

$$\int_0^{\pi/2} \frac{d\varphi}{(1-K^2\sin^2\varphi)^{3/2}} = \frac{1}{1-K^2}E(\pi/2,K) \quad, \tag{11}$$

Gr.; B. 1; p. 67; f. 241-18c :

$$\int_0^{\pi/2} \frac{d\varphi}{(1-K^2\sin^2\varphi)^{5/2}} = -\frac{1}{3(1-K^2)}F(\pi/2,K) + \frac{2(2-K^2)}{3(1-K^2)^2}E(\pi/2,K) \quad, \tag{12}$$

Gr.; B. 2; p. 102; f. 331-56b :

$$\int_0^{\pi/2} \frac{\sin^2 x}{a^2\cos^2 x + b^2\sin^2 x}\,dx = \frac{\pi}{2b(a+b)} \quad, \qquad a>0,\ b>0, \tag{13}$$

Gr.; B. 2; p. 102; f. 331-56c :

$$\int_0^{\pi/2} \frac{\cos^2 x}{a^2\cos^2 x + b^2\sin^2 x}\,dx = \frac{\pi}{2a(a+b)} \quad, \qquad a>0,\ b>0. \tag{14}$$

The integrations according to z' and Ψ in chapters 2A, 2B had been calculated by using the following formulae :

Integrations according to z' :

(2.52) : (1), (2);

(2.53) : (6), (2.52);

(2.54) : (3).

(2.55) : (4), (5), (2.54);

(2.56) : (4), (5);

(2.57) : (4), (2.56).

Integrations according to Ψ :

(2.65) : (13) - appendix 3 ;

(2.66) : (14) - appendix 3 ;

(2.67) : (9);

(2.68) : (10);

(2.69) : (7);

(2.70) : (8);

(2.71) : (10), (11);

(2.136) : (10), (11), (12).

Appendix 3

[illegible] integral (2.[illegible]8) is a little hard and we are going to do [illegible] below:

$$K_1^{ij} = \int_0^{\pi/2} \cos\Psi \, \mathrm{arctg}\,\frac{k}{\cos\Psi}\, d\Psi \,, \quad k = \frac{\Delta_{ij}}{2R} \;;$$

in [illegible] the following relations are true: $\mathrm{arctg}(x) = \mathrm{arccotg}(1/x)$ for $x > 0$, $\mathrm{arccotg}(y) = \frac{\pi}{2} - \mathrm{arctg}(y)$; we may suppose $0 \le \Psi < \pi/2$;

1) $\Delta_{ij} > 0 \quad (k > 0)$: $$\mathrm{arctg}\,\frac{k}{\cos\Psi} = \mathrm{arccotg}\,\frac{\cos\Psi}{k} = \frac{\pi}{2} - \mathrm{arctg}\,\frac{\cos\Psi}{k} \;;$$

2) $\Delta_{ij} < 0 \quad (k < 0)$: $$\mathrm{arctg}\,\frac{k}{\cos\Psi} = -\,\mathrm{arctg}\,\frac{-k}{\cos\Psi} = -\,\mathrm{arccotg}\,\frac{\cos\Psi}{-k} =$$

$$= -\left[\frac{\pi}{2} - \mathrm{arctg}\,\frac{\cos\Psi}{-k}\right] = -\left[\frac{\pi}{2} + \mathrm{arctg}\,\frac{\cos\Psi}{k}\right] ;$$

1) $k > 0$: $$K_1^{ij} = \int_0^{\pi/2} \cos\Psi\left[\frac{\pi}{2} - \mathrm{arctg}\,\frac{\cos\Psi}{k}\right] d\Psi = \frac{\pi}{2} + J \;;$$

$$J = -\int_0^{\pi/2} \cos\Psi \, \mathrm{arctg}\,\frac{\cos\Psi}{k}\, d\Psi \;;$$ substitution: $\mathrm{arctg}\,\frac{\cos\Psi}{k} = t$;

$$J = -k^2 \int_0^{\mathrm{arctg}\,1/k} \frac{t \;\mathrm{tg}\,t\,(1 + \mathrm{tg}^2 t)}{\sqrt{1 - k^2\,\mathrm{tg}^2 t}}\, dt \;;$$ per partes: $u = t$, $v' = f(\mathrm{tg}\,t)$,

$$v = \int \frac{\mathrm{tg}\,t\,(1 + \mathrm{tg}^2 t)}{\sqrt{1 - k^2\,\mathrm{tg}^2 t}}\, dt \;;$$ substitution: $\mathrm{tg}\,t = x$; $$v = \int \frac{x}{\sqrt{1 - k^2 x^2}}\, dx =$$

$$= -\frac{1}{k^2}\sqrt{1 - k^2\,\mathrm{tg}^2 t} + C \;; \quad J = -\int_0^{\mathrm{arctg}\,1/k} \sqrt{1 - k^2\,\mathrm{tg}^2 t}\; dt \;;$$ substitution:

$\mathrm{tg}\,t = x$; $$J = -\int_0^{1/k} \frac{\sqrt{1 - k^2 x^2}}{1 + x^2}\, dx \;;$$ substitution: $k\,x = \cos\Psi$;

$$J = -k \int_0^{\pi/2} \frac{\sin^2\varphi}{k^2 + \cos^2\varphi}\, d\varphi = -k \int_0^{\pi/2} \frac{\sin^2\varphi}{(1+k^2)\cos^2\varphi + k^2 \sin^2\varphi}\, d\varphi \ ,$$

Since (13), appendix 2 : $J = -\frac{\pi}{2} \frac{1}{k + \sqrt{1+k^2}} = -\frac{\pi}{2} \frac{2R}{\Delta_{ij} + \sqrt{4R^2 + \Delta_{ij}^2}}$

and $$K_1^{ij} = \frac{\pi}{2} + J = \frac{\pi}{2}\left[1 - \frac{2R}{\sqrt{4R^2 + \Delta_{ij}^2} + \Delta_{ij}}\right] .$$

2) $k < 0$: $$K_1^{ij} = -\int_0^{\pi/2} \cos\psi \left[\frac{\pi}{2} - \operatorname{arctg}\frac{\cos\psi}{-k}\right] d\psi = -\frac{\pi}{2} + J \ ;$$

$$J = \int_0^{\pi/2} \cos\psi \operatorname{arctg}\frac{\cos\psi}{-k}\, d\psi$$; analogously to the case 1) we get

$$J = \frac{\pi}{2} \frac{1}{\sqrt{1+k^2} - k} = \frac{\pi}{2} \frac{2R}{\sqrt{4R^2 + \Delta_{ij}^2} - \Delta_{ij}}$$, and

$$K_1^{ij} = -\frac{\pi}{2} + J = -\frac{\pi}{2}\left[1 - \frac{2R}{\sqrt{4R^2 + \Delta_{ij}^2} - \Delta_{ij}}\right] .$$

For $\Delta_{ii} = 0$ there is $K_1^{ii} = 0$. Now for all three cases $\Delta_{ij} \gtreqless 0$ there may be written a uniform formula (2.65) :

$$K_1^{ij} = \frac{\pi}{2} \operatorname{sgn}(\Delta_{ij}) \left[1 - \frac{2R}{\sqrt{4R^2 + \Delta_{ij}^2} + \operatorname{abs}(\Delta_{ij})}\right] .$$

Analogously we get the formula (2.66) :

$$K_2^{ij} = \frac{\pi}{2} \frac{1}{\sqrt{4R^2 + \Delta_{ij}^2}\left[\sqrt{4R^2 + \Delta_{ij}^2} + \operatorname{abs}(\Delta_{ij})\right]} .$$

CONCLUSION

In the nuclear reactor core physics, shielding physics, e.t.c., it is often necessary to calculate the neutron fields in the vicinity of an empty technological channel. Therefore we meet with the problem of the formulation of the boundary condition on a cylindrical cavity. In general the neutron flux in the vicinity of a circular cylindrical cavity is not cylindrically symmetric. The mathematical formulation of this problem is rather difficult and it is to be defined using the cylindrical functions. In many cases there may be supposed a cylindrical symmetry of the flux to get a more easy formulation. We have given the formulation of the latter problem in a set of reports ZJE - 168, 1977; ZJE - 173, 1975; ZJE - 197 ; e.t.c. (the remaining are to be published). The final job of this task is to verify mathematically the results of the experimental research of the problem, which has been already mentioned.

When this task finished, there will remain some other open problems to us, which we are possibly going to discuss. First, when a useful method for calculation of neutron fields in the vicinity of a cylindrical cavity with a circular cross section is developed, there may be supposed that this method will be convenient also for a cylindrical cavity with an annular cross section, which is also often used in both the reactor core and the shielding physics, e.t.c. . Second, for both mentioned cases, i.e. the circular and the annular cross section of the cavity, the problem with cylindrical asymmetry is to be solved.

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