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MODELS FOR HADRON-NUCLEUS SCATTERING

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Abstract

Models for hadron-nucleus scattering at intermediate and high energies are shown to depend sensitively on the mutual compatibility of the basic approximations such as non-local interactions, fixed scatterers, and estimal amplitudes.

For hadron-nucleus scattering at intermediate and high energies most realistic models are based on a few basic approximations which I will briefly outline in Part I, together with some relations among them. One now understands these approximations sufficiently well (basic ideas, domains of validity, etc...) so as to be able to study their mutual compatibility. The results of such studies are sometimes surprising : some commonly employed models are based on mutually incompatible approximations whereas others involve several approximations which together are better than any one taken alone, as we shall see in Part II.

The consequences in practical calculations can be rather dramatic, and provide some amusing guidelines for model builders who wish to transcend the folklore of scattering theory.

Part I. APPROXIMATIONS SCHEMES

Details of the approximations very briefly outlined here may be found throughout the scattering theory literature ; their mutual relationships are less often considered.

I. Born approximation

If one iterates the Lippmann-Schwinger (L-S) equation

$$\begin{aligned} T &= V + V\, G_0\, T \\ &= V + V\, G_0\, V + \ldots\, (V\, G_0)^n V + \ldots \end{aligned} \tag{1}$$

one obtains a power series in the coupling constant $V = \lambda V$. The first term of this series is the real, non-unitary Born amplitude $T^B \equiv V$. The Born series has the following well known properties [1]

i) $\lim\limits_{E\to\infty} \quad T \to V = T^B$

ii) $\lim\limits_{\lambda\to 0} \quad T \to V$

iii) T contains poles (in E) at bound states and resonances ; at these energies the series diverges.

The latter difficulty may be overcome by the the introduction of quasiparticles based on the Abspaltungsverfahren (separation process) [2] whereby the kernel of the L-S equation is approximated by a separable term

$$VG_0 \equiv K(x,y) = P(x,y)$$
$$P(x,y) = \sum_n g_n(x)\, g_n(y) \tag{2}$$

where the g_n are (optimally) chosen to be eigenfunctions of the kernel. By such methods a "reduced" potential may be calculated for which the Born series will be convergent.

Frequently one simply takes a single separable term for the kernel (or potential) for the practical reason that the L-S equation is then reduced to one dimension and therefore is soluble in closed form. The physical consequences of such an approximation are never negligible.

For scattering problems, a convenient (but rough) criteria for the convergence of the Born series is given [3] in terms of the potential V, the velocity v, and a, a length characteristic of the range of the potential by

$$\frac{V\,a}{\hbar\,v} << 1. \tag{3}$$

II. Semiclassical (WKB) approximation

The "semiclassical" aspect of this approximation comes from the fact that the wavefunction is expressed in powers of $\hbar$ and thus the zeroth order term is independent of Planck's constant. We may express the Schrödinger wave function

$$\psi\,(\bar{x},t) = \exp\left[\frac{i}{\hbar}\; S\;(\bar{x},t)\right] \quad , \tag{4}$$

$$\frac{\partial S}{\partial t} = \frac{1}{2m}\left[(\bar{\nabla} S)^2 + \frac{\hbar}{i}\;\nabla^2 S\right] + V \tag{5}$$

Under the assumption that S is slowly varying ($\nabla^2 S << |\bar{\nabla} S|$), one obtains the Hamilton-Jacobi equations of classical mechanics from the expansion in $\hbar$:

$$S = S_0 + \frac{\hbar}{i} S_1 + \left(\frac{\hbar}{i}\right)^2 S_2 + \ldots$$

$$(\hbar)^0 : \quad \frac{2S_0}{2t} + \frac{1}{2m}(\nabla S_0)^2 + V = 0 \tag{6}$$

$$(\hbar)^1: \quad \frac{\partial S_1}{\partial t} - \frac{1}{m}(\bar{\nabla} S_0 . \bar{\nabla} S_1 + \frac{1}{2}\nabla^2 S_0) = 0$$

$\vdots$

etc...

In one dimension we obtain

$$\Psi \approx \left\{ \frac{i}{\hbar} S_0 + S_1 + \ldots \right\}$$

$$S_0(x) = \pm \int^x dx' \sqrt{[2m\ E - V(x)]} \qquad (7)$$

A rough criteria for the convergence [3] is given by $\frac{Va}{\hbar v} >> 1$. For example the necessary condition ka>>1 for smooth potentials is obviously satisfied for the typical heavy ion reactions $^{16}O + ^{16}O$ at 50 MeV or $\alpha + ^{58}Ni$ at 150MeV where $\lambda \approx 0.2$ fm.

III. Eikonal approximation

The eikonal approximation in classical optics consists in assuming the phase of a beam to be modified in traversing a medium of index n(z) along a straight-line trajectory

$$\Psi \approx \exp[ikz + \chi]$$

$$\chi \approx \int dz[n(z) - 1] \quad .$$

The Glauber-Molière approximation [3] to the Schrödinger equation is obtained by assuming the wave function to vary slowly relative to a plane wave :

$$\Psi(\bar{r}) = e^{i\bar{k}.\bar{r}} \, \phi(\bar{r})$$

$$\left(\frac{1}{2m}\nabla^2 + \frac{k^2}{2m} - V \right) \Psi = 0 \ . \qquad (8)$$

Under the sufficient but not necessary conditions

$$\frac{V}{E} << 1 \qquad ka << 1 \qquad \theta^2 ka << 1 \qquad (9)$$

one obtains the familiar expressions in terms of the impact parameter b :

$$\psi \sim \exp i \{\bar{k}.\bar{r} + \chi\}$$
$$\chi = -\frac{1}{4k} \int_{-\infty}^{\infty} V(b^2+z^2)\,dz \tag{10}$$
$$f(q) = \frac{ik}{2\pi} \int d^2b\; e^{i\bar{q}.\bar{b}}\; [e^{2i\chi} - 1] \quad .$$

In this approximation the convergence parameter $Va/\hbar v$ is not constrained. One sees that in the case of intermediate energy proton-hadron scattering, the condition $ka \ll 1$ is satisfied since $R_A \approx A^{1/3}$fm and $\lambda = 0.2p^{-1}$ fm (with p in GeV/c). For extreme values of the convergence parameter, the average value of the angle behaves as in the case of the WKB or Born approximations [3] :

$$\frac{Va}{\hbar v} \ll 1 \qquad \langle\theta\rangle \sim O\left(\frac{1}{\sqrt{ka}}\right) \qquad \text{Born} : \langle\theta\rangle \sim O\left(\frac{1}{ka}\right)$$

$$\frac{Va}{\hbar v} \gg 1 \qquad \langle\theta\rangle \sim O\left(\sqrt{\frac{V}{E}}\right) \qquad \text{WKB} : \langle\theta\rangle \sim O\left(\frac{V}{E}\right)$$

A more systematic approach for obtaining the eikonal amplitude consists in expanding (linearizing) the free Green's function about a preferred direction k_i (usually the average of initial and final directions) :

$$\bar{p} = \bar{p}_i + (\bar{p} - \bar{k}_i)$$
$$G_0^{-1} = (2m)^{-1}\,[k^2+p^2+i\epsilon] \tag{11}$$
$$= m^{-1}\,[k - \bar{k}_i.\bar{p} + i\epsilon] + \underbrace{(2m)^{-1}(\bar{p}-\bar{k}_i)}_{\approx 0}$$

To lowest order, the Green's function manifests straight-line propagation with constant impact parameter :

$$G_i = \frac{im}{k}\,\delta^{(2)}\,(\bar{b}-\bar{b}_i)\,\theta(z-z')\,e^{ik\,z} \quad , \tag{12}$$

where $\bar{r} = (\bar{b}, z)$.

Systematic corrections may be applied to this approximation [4], where the perturbation parameter is V/E. One may likewise include effects due to Fermi motion, target nucleon overlap, and non-eikonal propagation between multiple scatterings. Such corrections turn out to be of siginficant importance at intermediate energies [5].

Relativistic Eikonal

We merely mention here that similar eikonal methods in relativistic quantum field theory (4) give models with analytic amplitudes representing the sum of an infinite class of ladder diagrams, e.q. for the exchange of spin 0 particles of mass μ and coupling λ,

$$T = 2i\, s \int d^2b \;\; e^{-i\bar{q}.\bar{b}} \;(e^{i\delta_0} - 1)$$

$$\delta_0 = \frac{1}{2s} \int \frac{d^2q}{(2\pi)^2} \;\; e^{-i\bar{q}.\bar{b}} \;\; \frac{\lambda^2}{q^2+\mu^2} \tag{13}$$

or for the exchange of a Regge pole with the trajectory functions α, γ :

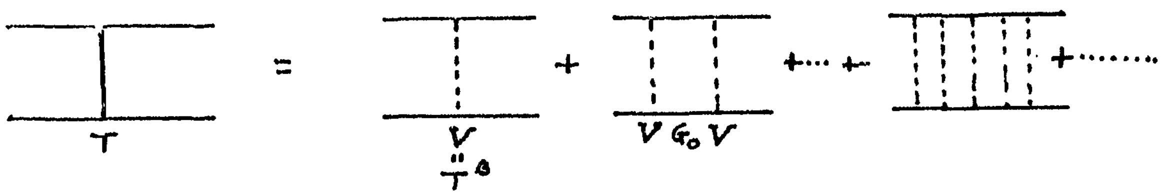
$$T = 2is \int d^2b \; e^{-i\bar{q}.\bar{b}} \;(e^{i\delta_R} - 1)$$

$$\delta_R = \frac{1}{2s} \int \frac{d^2b}{(2\pi)^2} \; e^{-i\bar{q}.\bar{b}} \;\; \gamma(-q^2)\, s^{\alpha(-q)} \tag{14}$$

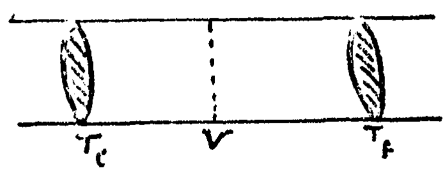
IV. DWBI : Distorted wave Born approximation

Eikonal methods provide an interesting analogy between relativistic calculations involving radiative corrections (or soft pions) and the DWBA familiar from nuclear physics.

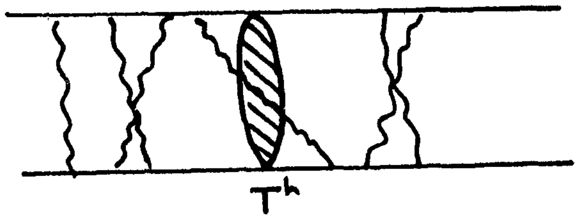
The Born series, diagrammatically represented by



may be modified to employ, instead of plane waves, interacting wave functions in the initial or final state



In relativistic calculations for strong interactions (4) a "hard" interaction may be modified by "soft" meson emission and absorption



The propagators may be linearized as in Eq.(11) to express the fact that their dominant contribution comes from exchanged momenta small compared with that of the initial and final particles. In the resulting amplitude, the hard interaction, e.g. for scalar exchange

$$T^{(h)} = -\frac{\lambda^2}{k^2-\mu^2+i\epsilon}$$

is modified for n soft exchanges

$$T^{n+1} = \int d^4x\, e^{-i\bar{q}.\bar{x}}\, T^{(h)}(x)\, \frac{(i\chi)^n}{n!}$$

$$T = \sum_{n=0}^{\infty} T^{n+1} = \int d^4x\; e^{-i\bar{q}.\bar{x}}\, T^{(h)}(x)\; e^{i\chi} \tag{15}$$

$$\chi \sim \frac{\lambda^2}{(2\pi)^4} \int \frac{d^4k}{k^2-\mu^2+i\epsilon}\; e^{ik.\chi}$$

Various methods [6] have been employed to calculate such amplitudes : i) functional methods (Abarbanel and Itzykson, ii) infinite momentum methods (Cheng, Ma, Chang, Wu) and iii) Feynman propagator perturbations (Lévy and Sucher).

Relations among approximations

I. Eikonal-WKB (Semiclassical)

At high energies are may expand the WKB and eikonal phase functions using the parameter [4]

$$\epsilon \equiv \frac{V_0}{kv}$$

(k is the wave number, v the velocity, $V(r)=V_0U(r)$)

<u>WKB</u> :
$$\chi^W(b) = k \int_{-\infty}^{\infty} dz \; \{[1-2m\,V/k^2]-1\} \tag{16}$$
$$= k \int_{0}^{\infty} dz \; [-2\,\epsilon\,U - \epsilon^2\,U^2 + \ldots]$$
$$= \sum_{n=0}^{\infty} \chi_n^W(b)$$

<u>Eikonal</u> :
$$\chi^E(b) = \sum_n \chi_n^E(b) \tag{17}$$
$$\chi_0^E(b) = -\frac{1}{v} \int_{-\infty}^{\infty} dz \; V(r) = -2k\epsilon \int_0^{\infty} dz \; U(r)$$

The two series may be subsequently related by matching powers of V_0 and k :

$$\chi_0^E = 2\chi_0^W$$
$$\chi_1^E = 2\chi_1^W \tag{18}$$
$$\chi_2^E = 2\chi_2^W - \frac{b}{3k^2}\,[\chi_0^W]^2$$

etc.

It is seen that at very high energies the WKB amplitude reduces to the lowest order eikonal term ; both give the Born term in the limit $E\to\infty$. (In some relativistic models this may not be true [6]).

II. <u>Eikonal-partial wave</u>

An intuitive way [3] to see the passage from the discreet sum of partial waves to the continuum integral over the impact parameter is the following : in the scattering amplitude

$$f(q) = \frac{1}{2ik} \sum_{\ell} (2\ell+1)\,[e^{i\delta_\ell}-1]\,P_\ell(\cos\theta) \tag{19}$$

we take the limit $p \to \infty$, $\ell_{max} \to \infty$, $\Delta\ell/\ell \to 0$ and introduce the impact parameter phase function

$$kb = \ell + \frac{1}{2} \qquad\qquad \delta_\ell \leftrightarrow 2\chi(b)$$

to obtain

$$\begin{aligned} f(q) &\approx -ik \int db\, [e^{2i\chi} - 1]\, P_{kb-\frac{1}{2}}(\cos\theta) \\ &\approx ik \int b\, db\, [e^{2i\chi} - 1]\, J_0\, (2kb \sin\theta/2) \end{aligned} \qquad (20)$$

Such representations are frequently useful for electron scattering where, for E>200 MeV, more than 100 partial waves may contribute and substantial cancellations occur at diffraction minima.

More formally Wallace [4] showed that using the Euler formula for converting the sum in Eq.(19) to an integral and an asymptotic expression for P_ℓ gives the Fourier-Bessel amplitudes (at high energies)

$$f(q) - ik \int b\, db\, J_0(qb)\, [S_F(b) - 1] \qquad (21)$$

Islam [6] has shown that this representation is valid is dependently of any approximation at all angles and energies.

Part II. MODEL MODELS

It is amusing to examine two models where is the basic approximations "interact" in one case to make the model worse (for some purposes) than intended, in the second better than one would expect.

Model I

A model for hadron-nucleus scattering, developed by Foldy and Walecka [7], using separable interactions and fixed scatterers has the advantages of i) expressing the amplitudes in closed form ii) using only on-shell two-particle amplitudes and iii) very conveniently leading to an optical potential limit.

Recently [8] this model was studied for its utility in numerical calculations in the one case where an exact reference calculation is possible, the scattering from a two-particle bound state. A somewhat unexpected result was the observation that in general fixed scatterers and separable interactions are mutually incompatible and moreover the numerical consequences may be very important. Fortunately, for the three body problem, new methods recently developed [8] for exact solutions (with modest calculational requirements) obviate the need for either approximation (although closure is often convenient and accurate).

Fixed scatterers

The fixed scatterer approximation (FSA) is equivalent to the closure approximation for the Green's function of the Faddeev equations :

$$\psi = \phi_{30} + G_3(V_1+V_2)\psi$$

$$G_3 = \sum_n \frac{|\phi_{3n}\rangle\langle\phi_{3n}|}{E-E_{3n}-T+i\epsilon} \qquad (22)$$

Here T_3 is the projectile kinetic energy, ϕ_{3n} is the wavefunction for the n_{th} excited state of the target, E_{3n} its energy :

$$h_3|\phi_{3n}\rangle = E_{3n}|\phi_{3n}\rangle \quad .$$

For incident energies much greater than target excitations, we obtain the approximate Green's function

$$E-E_{3n} = E-E_{30} - (E_{30}-E_{3n}) \simeq E-E_{30} \qquad (23)$$

$$G_3 \approx g_3 \equiv (E-E_{30}-T_3+i\epsilon)^{-1} \sum |\phi_{3n}\rangle\langle\phi_{3n}|$$

$$= (E-E_{30}-T_3+i\epsilon)^{-1} \quad .$$

In this approximation, the kernel of the L-S equation no longer contains the relative separation of the target particles and thus this dynamical variable becomes merely a parameter.

Separable operators (non-local)

In the usual notation [8] for three particle systems, (x_3=the separation of particles 1 and 2, y_3=the projectile-target c.m. separation), the (non-local) interaction between particles 2 and 3 is given by

$$\langle\bar{x}_1\bar{y}_1|V_1|\bar{x}_1'\bar{y}_1'\rangle = V_1(\bar{x}_1,\bar{x}_1')\,\delta(\bar{y}_1-\bar{y}_1') \qquad (24)$$

or, in terms of the x_3, y_3 ,

$$<\bar{x}_3\bar{y}_3|V_1|\bar{x}_3'\bar{y}_3'> =$$

$$= V_1\left(\bar{y}_3 - \frac{\bar{x}_3}{2},\ \bar{y}_3' - \frac{\bar{x}_3'}{2}\right)\ \delta\left[\frac{(\bar{x}_3-\bar{x}_3')(m+2M)}{2(m+M)} + \frac{(\bar{y}_3-\bar{y}_3')m}{m+M}\right]$$

Obviously integration over the primed variables in the L-S equation will not leave $\bar{x}_3=\bar{x}_3'$ and the target particles cannot be fixed. Fortunately for those who use these approximations for pion-nucleus scattering, the FSA is recovered when the projectile becomes lighter than the target particles : for $R \equiv m/M$, the δ-function of Eq.(24) becomes

$$\lim_{R\to 0} \delta \to \delta(\bar{x}_3-\bar{x}_3') \ .$$

Numerical calculations for nucleon-deuteron scattering at 100 MeV shows that imposing these two incompatible approximations gives amplitudes far from the exact Faddeev results.

A somewhat intuitive way to understand the difficulty is to note that the FSA (closure) treats all states as equivalent whereas a separable operators singles cut one particular state.

Model II

The model of Glauber [(3)] for hadron-nucleus multiple scattering contains the approximations i) eikonal amplitudes for the two-particle scatterings ii) neglect of Fermi motion and iii) two-particle kinematics neglecting binding ("sudden passage" approximation for high-energy projectiles).

We now have methods for systematic perturbative corrections for each of the above approximations [(4)], which have been shown [(5)] to provide convenient and non-negligible improvements to the simplest Glauber model for systems such as p-^4He scattering at intermediate and high energies.

One finds that to lowest order in $(kR)^{-1}$ (k is the momentum, R a characteristic dimension), all three corrections cancel exactly. For this model, the somewhat surprising conclusions are the following :

i) The particular ensemble of three approximations is better than any single approximation.

ii) Any effort to improve only one of the approximations will probably worsen the model !

iii) The domain of validity (energy, angle) for a model may be vastly different from that of the individual approximations employed.

Simple models are not without amusing surprises.

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