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ROBERTSON-WALKER TYPE SOLUTIONS WITH CARTER AND QUINTANA-TYPE ELASTICITY

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ABSTRACT

This paper deals with the elastic solutions having a six parameter symmetry. We show that our previous results obtained by means of Rayner's Hookean formalism, remain valid for certain values of the parameters in Carter and Quintana's more general theory, as non-Hookean special cases. On the other hand, although Carter and Quintana's Hookean approximation is not identical with Rayner's theory, the behaviour of the new solutions does not essentially differ from that of the solutions in Rayner's formalism.

АННОТАЦИЯ

В настоящей статье описываются упругие решения 6-параметровой симметрии. Показано, что полученные ранее методом Райнера результаты действительны для некоторых значений параметров также и в более общем формализме Картера и Квинтана но не описывают упругое поведение типа Хука материала. С другой стороны, полученные новым методом решения типа Хука в незначительной степени отличаются от ранее полученных решений.

KIVONAT

E cikkben a 6 paraméteres szimmetriáju rugalmas megoldásokkal foglalkozunk. Megmutatjuk, hogy korábbi, a Rayner-féle módszerrel kapott eredményeink a paraméterek bizonyos értékeire az általánosabb, Carter és Quintana alkotta formalizmusban is érvényesek maradnak, de nem Hooke-féle rugalmas viselkedést irnak le. Másrészt az uj módszer szerinti Hooke-tipusu megoldások nem térnek el lényegesen a korábban kapottaktól.

1. INTRODUCTION

In 1972, Carter and Quintana modified Rayner's formalism for relativistic elasticity [1], [2]. The most important point of this modification is the density-dependence of the elastic coefficients. In addition to this, the new formalism is more general, it can describe the non-Hookean cases /which are out of Rayner's formalism/, and contains the case of perfect fluid as a limit of zero shear. Of course, Rayner's formalism, which is good from the rheological point of view, remains valid for a special type of elastic matter.

In Ref. 3 we dealt with the solutions describing Rayner-type elastic matter in gravitational field having a six parameter symmetry. Here we repeat the calculation by means of the more general theory. We shall show that previous results remain valid if we choose the equation of state specially. The Hookean limit of the two theories differ from one another but there is no essential difference between the behaviours of the solutions.

We do not want to deal again with the physical meaning of elastic solutions having a six parameter symmetry [3], but note that in the very dense stage of the Universe some type of elasticity may be imagined /though it is not probable/.

2. THE SYMMETRIES

The Robertson-Walker metrics have a six parameter symmetry, namely SO/4/ for k=1, E/3/ for k=0 and SO/3,1/ for k=-1, and we will require these symmetries to hold for the characteristic components of the energy-momentum tensor too, i.e.

$$ds^2 = a^2/x^0/\left[-dx^{0^2} + dx^{1^2} + f^2/x^1/\;(dx^{2^2} + \sin^2 x^2\, dx^{3^2})\right] \qquad /2.1a/$$

$$f/x/ = \begin{cases} \sin x & \text{for } k=+1 \\ x & k=0 \\ \mathrm{sh}\, x & k=-1 \end{cases}$$

and $s=s/x^{o}/$ if s is a scalar,

$v^{i}=v/x^{o}/\delta_{o}^{i}$ if v^{i} is a vector, /2.1b/

$t^{ik}=q/x^{o}/g^{ik}+w/x^{o}/\delta_{o}^{i}\delta_{o}^{k}$ if t^{ik} is a symmetric tensor.

This choice of g_{ik} has fixed the coordinates except a trivial translation in x^{o}.

Coordinate x^{o} has no immediate physical meaning. The time "t" measured by a comoving observer can be obtained as

$$t = \int a/x^{o}/dx^{o} \ . \qquad /2.2/$$

Requiring the particle number conservation $(nu^{r})_{;r}=0$ we get

$$n=\frac{N}{2\pi^{2}}\,a^{-3} \ , \qquad /2.3/$$

where N is the total number of particles for $\kappa=+1$ but it is a constant of integration simply for the other cases.

3. THE FIELD EQUATIONS

We will deal with the isotropic case only. Then there exist an equation of state of the form

$$\rho=nm/n,s^{2},w/ \qquad /3.1/$$

where s^{2} and w are the second- and third-order shear invariants. $m/n,s^{2},w/$ may have arbitrary form except the positivity conditions:

$$\rho/n/ = nm/n,0,0/\geq 0,$$

$$\beta/n/ = n^{3}m,_{nn}/n,0,0/+2n^{2}m,_{n}/n,0,0/\geq 0, \qquad /3.2/$$

$$\mu/n/ = nm,_{s}2/n,0,0/\geq 0.$$

The deformations can be written as

$$\varepsilon_{ik} = \frac{1}{2}\left(h_{ik}-h^{o}_{ik}\right) \ ;$$

$$h_{ik} = g_{ik}+u_{i}u_{k} \ ; \qquad /3.3/$$

$$\mathcal{L}_{u}h^{o}_{ik} = h^{o}_{ik}u^{k} = 0.$$

where u^{i} is the velocity and h^{o}_{ik} is connected to the unstrained state and has rank 3. The shears can be obtained similarly but h^{o}_{ik} is to be replaced by a tensor having a determinant equal to that of g_{ik} and connected to the unsheared state.

In our case the only possible form for h^{o}_{ik} of the mentioned symmetry is

$$h^{o}_{ik} = \frac{A^2}{a^2} h_{ik} \; ; \qquad h_{ik} = g_{ik} - g_{oo}\delta^{o}_{i}\delta^{o}_{k} \; . \qquad /3.4/$$

where A is a constant. Thus the shear vanishes, and the energy-momentum tensor is identical with that of a perfect fluid of the following pressure:

$$p = n^2 m,_{n}/n/, \qquad /3.5/$$
$$m/n/ = m/n,0,0/.$$

Thus the field equations are:

$$-\kappa nm/n/ = 3a^{-2}(k + \frac{\dot{a}^2}{a^2}) + \lambda \; ,$$

$$\kappa n^2 m,_{n}/n/ = (k + 2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}) + \lambda, \qquad /3.6/$$

$$n = \frac{N}{2\Pi^2} \; \frac{1}{a^3} \; .$$

It is easy to see that second equation is a consequence of the first when $\dot{a}\neq 0$. The solution can be explicitly written as

$$t = t_o + \int_{a_o}^{a} \frac{da}{\sqrt{-3k - \lambda a^2 - \kappa a^2 nm/n/}} \qquad /3.7/$$

For the static case there are two algebraic equations:

$$-\kappa n_o m/n_o/ = 3ka_o^{-2} + \lambda \; , \qquad n_o = \frac{N}{2\Pi^2}\,\frac{1}{a_o^3} \qquad /3.8/$$
$$\kappa n_o^2 \, m,_{n}/n_o/ = ka_o^{-2} + \lambda \; ,$$

If the form of the function $m/n/$ is given, a_o and λ can be calculated.

The choice

$$m/n/ = m_o + (\mu + \tfrac{3}{2}\nu)(n^{-1} - 3n^{-1/3} n_o^{-2/3});$$
$$n_o = \frac{N}{2\Pi^2} A^{-3}; \; \mu \text{ and } \nu \text{ are constant} \qquad /3.9/$$

gives the solutions of Ref. 3 with $C = -(6\Pi^2)^{-1} \kappa m_o N$.

The Hookean limit of Carter and Quintana's formalism is the choice [2]

$$m/n/ = m_o + \frac{1}{8n_o} \frac{(a^2-A^2)^2}{A^4} (9\nu_o + 6\mu_o) \qquad /3.10/$$

because

$$\varepsilon_{ik} = \frac{1}{2} \frac{(a^2-A^2)}{a^2} \begin{pmatrix} g_{IK} & 0 \\ 0 & 0 \end{pmatrix}; \quad I = 1,2,3 \qquad /3.11/$$

This function fulfils the positivity conditions /3.2/ for any n /of course, except the third, whose left hand side is undefined now, because we have not assumed anything for the s-dependence of m/. This function is to be substituted into eq. /3.7/ or /3.8/. Since the /3.7/ integral cannot be analytically calculated, in the following section we shall deal with the approximation of the solutions.

4. THE MOTION

The time-radius function can be obtained as

$$t-t_o = -KA^{1/2} \int_{a_o}^{a} \frac{a^{1/2}\, da}{\sqrt{P_4(a)}} \; ; \quad q^2 \equiv \frac{4NA^{-3}}{\Pi^2} \frac{m_o}{9\nu_o + 6\mu_o} \; ; \quad K^2 \equiv - \frac{8}{\kappa(3\nu_o+2\mu_o)}$$

$$P_4/a/ = a^4 - \lambda AK^{-2}a^3 - 2A^2a^2 - 3kAK^{-2}a + A^4(1+q^2) \qquad /4.1/$$

The polinomial $P_4/a/$ may have zero, one /double/ or two positive roots. Since $P_4/a/>0$, there are the following possibilities:

A/ Two positive roots:	Motion is possible either between zero and the smaller root /Case 1/ or between the greater root and infinity /Case 2/
B/ One double positive root	The situation is similar, but there is also a third possibility: static behaviour according to eq. /3.8/.
C/ There is no positive root	Radius monotinously varies between zero and infinity. /Case 3/

Since the correct function t/a/ cannot be analytically calculated, we are going to demonstrate its behaviour by means of an approximating function, which is correct at the limiting points and can be obtained by ignoring some part of $P_4/a/$.

Case 1: Motion between 0 and a_1 /the smaller positive root of P_4/.

$$t-t_1 - KA^{1/2}\sqrt{P_4\left(\frac{a_1}{2}\right)\cdot\frac{a_1}{2}}\left(\text{arc}\ \sin\left(1-2\frac{a}{a_1}\right)+\sqrt{1-\left(1-2\frac{a}{a_1}\right)^2}\right)$$

Case 2: Motion between a_2 /the greater root/ and infinity.

$$a \sim a_2 \cos^{-2}\left(\sqrt{\frac{a_2}{A}}\ \frac{t-t_2}{2K}\right) \qquad /4.2/$$

Case 3: Motion between 0 and ∞.

$$t \approx \frac{2}{3}\ \frac{K}{1+q^2}\ \frac{a^2 t_\infty - Ka^{3/2}A^{1/2}}{2Ka^2/3(1+q^2)-A^{3/2}t_\infty a^{1/2}-2KA^2}\ ;$$

$$\text{where } t_\infty = K(1+q^2)^{-1/2}\int_0^\infty x^{1/2}(1+x^4)^{-1/2}\,dx.$$

It can be seen that the time of a half cycle is finite for every case.

Static solutions exist for special values of λ only, which can be obtained from eq. /3.8/. A simple consideration shows that this solution is unstable: since there is no further positive root, "a" cannot vanish at any other points, thus an oscillation with small amplitude is not possible.

6. THE VISCO-ELASTIC PROBLEM

One of the visco-elastic media, the Kelvin-Voigt system, can be described by a simple addition of the energy-momentum tensors of an elastic medium and a pressureless viscous fluid. It might be obvious to sum up these known terms. However, there is a substantial difficulty.

Carter and Quintana's paper has explicitly supposed that T_{ik} does not depend on anything except the deformations. But, when temperature plays essential role, the elastic coefficients /and also h^c_{ik}/ may depend on the temperature. For this case the formalism has been not worked out. /In Rayner's theory, where the constancy conditions for h^o_{ik} and C_{iklm} are required ab ovo, this problem is formally out of question./

Nevertheless, if we were to use the simple sum of the elastic term for a cold Hookean matter [2] and the viscous term for a fluid having viscous coefficients proportional to $n^{1/3}$ [3], we would obtain the following equations:

$$k - E\frac{\dot{a}}{a} - \frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + \kappa\frac{3\nu+2\mu}{2\Lambda}\left(1-\frac{\Lambda^2}{a^2}\right) + \lambda a^2 = 0 \; ;$$

$$-\kappa\rho = \frac{3}{a^2}\left(k+\frac{\dot{a}^2}{a^2}\right) + \lambda, \qquad /5.1/$$

$$E = \text{const.}$$

However now we will not deal with the investigation of the solutions of /5.1/ because even the approximating solutions analogous with /4.3/can be obtained by means of numerical integration only.

6. CONCLUSIONS

The obtained solutions have a character similar to the previous series of the Rayner-type Hookean ones [3]: there are solutions in which the radius varies between zero and a finite value, or between a finite value and infinity, there are solutions with a radius varying from zero to infinity, and there are unstable static solutions. The motion needs a finite time. The only essential difference in the behaviour of the solutions is that now the stable static solution of Ref. 3 /for $k = -1$/ has not been obtained, because the density for this solutions was negative. /In Ref. 3 we did not investigate the behaviour of ρ because it was not possible for the approximated cases./

Since the matter fulfils a well-behaving and realistic equation of state $p=p/n/$, we will not check the reality of the solutions by means of Hawking's energy conditions.

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