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KFKI-76-20

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Hungarian Academy of Sciences

CENTRAL
RESEARCH
INSTITUTE FOR
PHYSICS

BUDAPEST

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CUT-OFF SCALING AND MULTIPLICATIVE RENORMALIZATION IN THE THEORY OF CRITICAL PHENOMENA

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** Supported by the NSF program DMR-72-03026

ISBN 963 371 122 3

ABSTRACT

A new simplified renormalization group procedure which was developed to treat logarithmic problems in solid state physics is applied to the critical phenomena in $4-\varepsilon$ dimensions. It combines the physics of the Kadanoff scaling idea as developed by Wilson to calculate critical indices in the ε-expansion and the simple mathematical structure of the Lie equations of the Gell-Mann-Low multiplicative renormalization. In order to show the connection of the new method to the conventional Gell-Mann-Low renormalization a large class of the normalization conditions are considered including typical examples. The new procedure is found to be equivalent with the Gell-Mann-Low method with a specially chosen normalization condition. The new procedure has its advantage in its considerably simpler applicability and, furthermore, it is closer to the underlying physics. The critical indices ν, η, γ and α the anomalous dimensions d_φ, d_{φ^2}, and $d_{\varphi\varphi}$ as well, as correction to scaling have been determined at least up to second order in ε and the expressions obtained agree with the result of other calculations. In four dimensions logarithmic corrections are obtained to the specific heat.

АННОТАЦИЯ

Применяется упрощенный метод группы ренормировок к исследованию критических явлений в $4-\varepsilon$ размерности. Предложенный метод может быть применен к изучению логарифмических проблем в теории твердых тел. Наш метод удачно объединяет физическую картину гипотезы подобия Каданофа в форме развитой Вильсоном для определения критических индексов на основе ε- разложения и простую математическую структуру уравнений Ли метода мультипликативной ренормировки Гелл-Мана-Лоу. Для изучения взаимосвязи обычного метода Гелл-Мана-Лоу и предложенного нового метода исследуется обширный класс условий нормировки. Новый метод равносильный методу Гелл-Мана-Лоу в том случае, если в последнем специально подберем условие нормировки. Преимуществом нового метода является то, что значительно легче можно применять и, кроме этого, наш метод ближе к реальной физической картине. Критические индексы ν, η, γ и α, а также аномальные размерности d_φ, d_{φ^2} и $d_{\varphi\varphi}$ и поправки к масштабному поведению определены по ε не менее чем во втором порядке и наши результаты совпадают с раньше полученными результатами. В случае 4-х мерных систем получены логарифмические поправки к теплоемкости.

KIVONAT

Egy uj, egyszerüsitett renormálási csoport eljárást alkalmazunk a kritikus jelenségekre $4-\varepsilon$ dimenzióban. Az eljárást szilárdtestfizikai logaritmikus problémák tárgyalására lehet használni. A módszer egyesiti Kadanoff skálahipotézisének fizikáját, ugy, ahogy azt Wilson továbbfejlesztette a kritikus indexek meghatározására az ε sorfejtéssel és a Gell-Mann-Low multiplikativ renormálás Lie egyenleteinek egyszerü matematikai strukturáját. Az uj módszer és a szokásos Gell-Mann-Low renormálás kapcsolatának vizsgálatára a normálási feltételek egy széles osztályát tanulmányozzuk. Az uj eljárás ekvivalens a Gell-Mann-Low módszerrel, ha ott a normálási feltételt speciálisan választjuk.
Az uj módszer előnye a lényegesen egyszerübb alkalmazhatóság, és ezen felül a fizikai képhez is közelebb áll. Az η, ν, γ és α kritikus indexeket és a d_φ, d_{φ^2} és $d_{\varphi\varphi}$ anomális dimenziókat és a skálaviselkedéshez adódó korrekciót ε-ban legalább másodrendig meghatároztuk és az eredmények egyeznek mások számolásaival. Négy dimenziós rendszereknél a fajhőhöz logaritmikus korrekciókat kaptunk.

I. Introduction

The theory of critical fluctuations in systems around the phase transition point was the subject of many investigations in the last several years. A review of the many early attempts to account for these fluctuations and to describe the underlying physics can be found in Stanley's book [1]. It became clear that due to the divergence of the coherence length the short distance behaviour is irrelevant for the critical behaviour of the system and only the long range fluctuations play an important role. As a consequence of this statement Kadanoff [2] suggested that the critical behaviour of a magnetic system can be studied by grouping the individual spins on the lattice sites into blocks. These blocks can then be considered as new entities and the system of blocks behaves similarly as the individual spins do. In this way the original system is scaled into a new, similar system with the same free energy. Using this scaling property Kadanoff could derive relations between the critical exponents which describe the singular behaviour of thermodynamic quantities. In this approach it was, however, not possible to determine the critical exponents themselves.

A quite new development started in this problem with Wilson's [3] renormalization group treatment of critical phenomena. Formulating Kadanoff's scaling idea in reciprocal space and relying on that the physically interesting properties near the critical point are determined by the long wavelength (small wave vector) fluctuations Wilson noticed that the large wave vectors can be eliminated successively by a simultaneous renormalization of the interaction parameters. This renormalization transformation is performed with the requirement that the free energy be invariant under it. If the renormalization procedure leads to a fixed point Hamiltonian and a fixed point coupling constant, the critical exponents are determined by the eigenvalues of the renormalization transformation around the fixed point or by the fixed point coupling constant.

This method allows to calculate numerical values for the critical exponents either in power series of $\varepsilon = 4 - d$ where d is the dimensionality of the system or in powers of $1/n$ where n is the number of components. This length scaling renormalization procedure has already been reviewed by several authors. We refer here only to the paper by Wilson and Kogut [4] where all the essential ideas can be found.

A very similar procedure was introduced independently by Anderson, Yuval and Hammann [5] and by Anderson [6] in the treatment of the Kondo problem. Scaling of the system into an equivalent one was achieved through a variation of the short time cutoff or the energy cutoff (band width). The coupling constants of the transformed system were determined from the requirement that the free energy or the scattering matrix be invariant. This transformation produced scaling laws from which some features of the Kondo problem could be obtained.

Wilson's renormalization group treatment and in general this scaling argument is very different from the usual multiplicative renormalization group. The latter was discovered by Stückelberg and Peterman [7] , and applied successfully by Gell-Mann and Low [8] in quantum electrodynamics. Later it was extended to other renormalizable field theories. The conventional Gell-Mann-Low renormalization is well described in the book by Bogoliubov and Shirkov [9] for field theories. It was shown by Di Castro and Jona--Lasinio [10] - [12] that this method can be used to study critical phenomena, and similarly to Wilson's theory, the critical exponents can be calculated in the ε-expansion. The multiplicative renormalization can be formulated as resulting from the scaling of a reference momentum, but the simple physical picture of Kadanoff's scaling is absent. The scaling reference momentum has no simple physical meaning. Using the Gell-Mann-Low multiplicative renormalization also the Kondo problem was studied by Fowler and Zawadowski [13] and, as well, by Abrikosov and Migdal [14] , with the same result as the above mentioned papers. This indicates that the Wilson type so-called "modern version of renormalization" and the conventional multiplicative renormalization are in some sense equivalent.

Recently Jona-Lasinio [15] has proposed a general definition of renormalization transformations. In this definition "a renormalization group is a set of transformations acting on the arguments of a thermodynamical functional and leaving this functional invariant in value". This general definition includes both Gell-Mann-Low and Wilson type renormalization, showing that the expression renormalization group is not unique and has many different realizations. Furthermore as it is shown in the present work, even in the Gell-Mann-Low method there is an arbitrariness connected with the normalization condition. Even if we stick to multiplicative renormalization there are two distinct approaches using different differential equations for the Green's function and vertices. One version relies on the Lie equation of the group, the other uses the Callan-Symanzik equation [16]. Although the formulation is different, in both cases first the invariant coupling and its fixed point value have to be determined and then the critical exponents expressed in terms of the fixed point coupling can be calculated. Making use of the Callan-Symanzik equation Brézin et al [17], [18] have calculated the critical exponents to order ε^2 and ε^3, respectively.

There are many other methods to calculate critical exponents, not relying on renormalization group arguments. Feynman graph expansion [19], skeleton graph expansion [20], [21] and parquet diagram summation [22] have been successfully applied to calculate various critical exponents such as α, β, γ, δ, ν, η etc.

In this paper we present a new method which is a combination of Kadanoff's scaling idea or Wilson's method of eliminating degrees of freedom on the one hand

and multiplicative renormalization of Green's functions and vertices on the other hand. In solid state or statistical physical problems we usually have to deal with systems where there is a natural scale of the momentum, namely the cut-off momentum. This can serve as a natural scaling parameter whose change will eliminate the degrees of freedom in momentum space. But in contrast to Wilson's approach a multiplicative renormalization group will be generated by the cut-off scaling.

This fomulation of multiplicative renormalization via cut-off scaling was proposed by one of the authors and was applied to the x-ray absorption problem and Kondo problem [23] and to a one-dimensional Fermi gas model [24]. It was shown that in logarithmic problems this method provides us with a convenient procedure to do better than leading logarithmic approximation. In the theory of critical phenomena in 4 dimensions the situation is similar, and this method is expected to work correctly. Near 4 dimensions the problem is not simply logarithmic in the sense that the higher power of logarithm is not always associated with higher power in the coupling - the basic element is $1/\varepsilon \cdot (x^{\varepsilon}-1)$ instead of $\ln x$ - and it is questionable whether this simple renormalization procedure can be extended to calculate the critical exponents in ε-expansion or not. The aim of the present paper is to show that the critical exponents can be calculated correctly applying this procedure. Since, as was mentioned, this procedure is based on the physical picture of Kadanoff's scaling idea, but uses the mathematical formalism of the Gell--Mann-Low renormalization, this new approach may give a better insights into the physics underlying the critical phenomena. It is shown that the new method is a particular case of the Gell-Mann-Low renormalization with an adequately chosen normalization condition.

The paper is organized as follows. Since the conventional Gell-Mann-Low multiplicative renormalization is not as popular as Wilson's renormalization group approach, a short recapitulation of the main steps seems to us necessary. In Sec. II. the method is presented with emphasis on the difference between field theoretical and statistical physical applications and on the requirement that the renormalization transformation be a non-trivial transformation. Our new renormalization procedure is developed in Sec. III. The basic idea is that a change in the physical cut-off can be compensated by an effective coupling in such a way that the Green's function and vertex in the original and transformed system differ only by a multiplicative factor. It is shown here that this is true in perturbation theory to that order until the calculations can be done reasonably. Accepting that this procedure is generally true, the critical indices η, ν and γ, the anomalous dimensions d_φ, d_{φ^2}, and $d_{\varphi\varphi}$ and the exponent ω describing the correction to scaling are determined in

Sec. IV. to order ε^2 or ε^3. The specific heat exponent α is not so straightforward to calculate and this will be pr sented in Sec. V. In four dimensions the exponents have mean field values. The effect of fluctuations appears in the form of logarithmic corrections. This will be studied in Sec. V. where a new type of correction is obtained for the specific heat. In the last sections this new method is compared to other ones and the differences are discussed.

II. Conventional Gell-Mann-Low renormalization

2.1 Renormalization transformation

Multiplicative renormalization was invented in quantum electrodynamics to cope with the problem of divergent contributions from some self-energy or vertex diagrams. The method is excellently described in the book of Bogoliubov and Shirkov [5] and we refer the reader to this book for the background in field theory. This paper is, however, self-contained and in this section a brief review of the Gell-Mann-Low method is given. The approach using the Callan-Symanzik equation will not be reviewed since our method to be developed in the next section has more resemblance to the conventional treatment. We will put emphasis on those points which will allow us to make a direct comparison with our method.

The critical phenomena in $d=4-\varepsilon$ dimensions can be studied as a field theoretical problem where the Lagrangian or Hamiltonian is of the form of the Ginzburg-

Landau-Wilson functional

$$H = \int d^d x \left\{ \frac{\tau_0}{2} \varphi^2(x) + \frac{1}{2} \left[\nabla \varphi(x)\right]^2 + \frac{g_0}{4!} \left[\varphi^2(x)\right]^2 \right\} \qquad /2.1/$$

with

$$\varphi^2(x) = \sum_{i=1}^{n} \varphi_i^2(x) \quad \text{and} \quad \left[\nabla \varphi(x)\right]^2 = \sum_{i=1}^{n} \left(\nabla \varphi_i(x)\right)^2 \qquad /2.2/$$

n being the number of components. τ_0 is proportional to the temperature and g_0 has the dimensionality $\varkappa^{-\varepsilon}$. In this paper only the static critical phenomena will be studied and therefore no time-dependence is considered.

Renormalization of the theory means that instead of working with this Hamiltonian, subtraction terms are introduced, which - except for the mass renormalization - can be considered as multiplicative renormalization of the different terms of the Hamiltonian. According to this first a "mass renormalization" is performed by an identical transformation, introducing a quantity $\varkappa^2$ instead of τ_0 in the free Hamiltonian

$$H = \int d^d x \left\{ \frac{\varkappa^2}{2} \varphi^2(x) + \frac{1}{2} \left[\nabla \varphi(x)\right]^2 + \frac{g_0}{4!} \left[\varphi^2(x)\right]^2 + \frac{\delta m^2}{2} \varphi^2(x) \right\} \qquad /2.3/$$

with $\delta m^2 = \tau_0 - \varkappa^2$.

In the perturbational calculation the first two terms will serve as the unperturbed part of the Hamiltonian while the third and fourth terms of eq. /2.3/ are considered as perturbations. The unperturbed Green's function in momentum representation is

$$G^{(0)}(q) = -\frac{1}{q^2+\kappa^2} \qquad /2.4/$$

κ^2 is still undetermined and its value will be fixed later on.

In a second step the multiplicative renormalization will be done in a straightforward manner by inspecting the diagrams, without the intoduction of the subtraction terms. The total Green's function and the vertex can be calculated in perturbation theory. The diagrammatic representation of the successive contributions is given in Fig. 1. and 2. for the Green's function and vertex, respectively. The crosses in these diagrams represent the $\delta m^2 \varphi^2$ insertions into the Green's functions.

The Green's function G and the reduced vertex $\tilde{\Gamma} = \Gamma / g_o$ are functions of the momenta q_i^2, the renormalized mass κ^2, the coupling constant g_o and the strength of the φ^2 insertion δm^2. Using a sharp cut-off in momentum space, G and $\tilde{\Gamma}$ depend on the cut-off Λ. When the diagrammatic contributions are written in a formal way, these quantities depend on their unperturbed value $G^{(0)}$ and $\tilde{\Gamma}^{(0)}$ and, therefore, we can write the Green's function and vertex as $G = G(q^2, \kappa^2, \Lambda^2, g_o, \delta m^2, \tilde{\Gamma}_o, G^{(0)})$ and $\tilde{\Gamma} = \tilde{\Gamma}(q_i^2, \kappa^2, \Lambda^2, g_o, \delta m^2, \tilde{\Gamma}_o, G^{(0)})$. Looking at the corresponding diagrams in Fig. 1. and 2. and considering the simple procedure how higher and higher order diagrams are constructed, it is easy to check in any order that under the transformation

$$\tilde{\Gamma}_0 \to \tilde{\Gamma}_0 z_1 , \qquad /2.5/$$

$$G^{(0)} \to G^{(0)} z_3^{-1} , \qquad /2.6/$$

$$\delta m^2 \to \delta m^2 z_3 , \qquad /2.7/$$

$$g_0 \to g_0 z_1^{-1} z_3^2 \qquad /2.8/$$

all the diagrams contributing to the total Green's function carry the same multiplicative factor z_3^{-1} and similarly all the vertex diagrams have an overall multiplicative factor z_1. Compensating these factors we get the following invariance property:

$$G(q^2, \kappa^2, \Lambda^2, g_0, \delta m^2, \tilde{\Gamma}_0, G^{(0)}) = \\ = z_3\, G(q^2, \kappa^2, \Lambda^2, g, \delta m^2 z_3, \tilde{\Gamma}_0 z_1, G^{(0)} z_3^{-1}), \qquad /2.9/$$

$$\tilde{\Gamma}(q_i^2, \kappa^2, \Lambda^2, g_0, \delta m^2, \tilde{\Gamma}_0, G^{(0)}) = \\ = z_1^{-1}\, \tilde{\Gamma}(q_i^2, \kappa^2, \Lambda^2, g, \delta m^2 z_3, \tilde{\Gamma}_0 z_1, G^{(0)} z_3^{-1}), \qquad /2.10/$$

$$g = g_0 z_1^{-1} z_3^2 . \qquad /2.11/$$

This transformed Green's function and vertex could have been obtained from the following Hamiltonian

$$H = \int d^4x \left\{ z_3 \left[\frac{\kappa^2}{2} \varphi^2(x) + \frac{1}{2} [\nabla \varphi(x)]^2 \right] + \frac{g}{4!} z_1 [\varphi^2(x)]^2 + z_3 \frac{\delta m^2}{2} \varphi^2(x) \right\} \qquad /2.12/$$

From now on we will work with this Hamiltonian and the physical system will be recovered by taking $z_1 = z_3 = 1$ only in a final step. This is the same Hamiltonian as that used by Brezin et al. [17] to investigate Wilson's theory of critical phenomena by the Callah-Symanzik equation.

Until now Z_1 and Z_3 are arbitrary multiplicative factors which generate a two parameter continuous group, the renormalization group. They transform the original physical system into an equivalent one where the Green's function and vertex have the same momentum and temperature dependence. This is a very essential feature of the renormalization group and that will enable us to get information on the momentum and temperature dependence of these quantities.

As a next step in renormalization theory, using either the Callan-Symanzik équation or Lie equations, the values of the multiplicative factors Z_1 and Z_3 and the "renormalized mass" κ^2 have to be fixed by imposing normalization conditions on the Green's function and vertex. At this point the distinction between field theoretical and statistical mechanical treatment of the problem has to be made clear. For the field theoretical approach the transformed Hamiltonian /2.12/ is the basic Hamiltonian and the normalization is done in such a way that there are no divergences in the Green's functions and vertices of this transformed system when the cut-off Λ goes to infinity. It is not required that Z_1 and Z_3 could be set equal to unity in a final step, they might even be equal to zero or infinity.

In the statistical physical formulation the basic Hamiltonian is that of eq. /2.1/. It should be kept in mind that there is a minimal distance here, /the lattice constant/ or put in another way the integration over momenta has a large cut-off Λ and therefore there is no inherent ultraviolet divergence in the problem. The auxiliary Hamiltonian /2.12/ and the introduction of the multiplicative factors Z_1 and Z_3 are only useful tools to study the behaviour of the system but in the final step we have to return to the original system by taking $Z_1 = Z_3 = 1$.

The field theoretical approach is only briefly presented here to see the relation to the statistical mechanical treatment. In the approach using Callan-Symanzik equations the theory is conveniently renormalized at zero momentum [17] or at the symmetry point [18] and the standard field theoretical arguments can be used to derive the invariant coupling and the exponents.

2.2 Multiplicative renormalization with field theoretical normalization

In the conventional Gell-Mann-Low theory the multiplicative factors are fixed by requiring the normalization of the Green's function and vertex at a momentum $q_i^2 = \lambda^2$ and λ will serve as a scaling parameter. Following Di Castro [7] the normalization is required for

$$d\left(q^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_0 z_1, G^{(0)} z_3^{-1}\right) = \frac{G\left(q^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_0 z_1, G^{(0)} z_3^{-1}\right)}{G^{(0)}\left(q^2, \kappa^2\right)} \qquad /2.13/$$

and for $\tilde{\Gamma}(q_i^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_0 z_1, G^{(0)} z_3^{-1})$ in the form

$$d(q_i^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_0 z_1, G^{(0)} z_3^{-1})_{q_i^2=\lambda^2} = 1 \qquad /2.14/$$

and

$$\tilde{\Gamma}(q_i^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_0 z_1, G^{(0)} z_3^{-1})_{q_i^2=\lambda^2} = 1 \qquad /2.15/$$

and the new "mass" κ^2 is fixed from the condition that the Green's function be singular at $q^2 = -\kappa^2$, i.e.

$$G^{-1}(q^2, \kappa^2, g_0, \delta m^2, \tilde{\Gamma}_0, G^{(0)})_{q^2=-\kappa^2} = \qquad /2.16/$$
$$= G^{-1}(q^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_0 z_1, G^{(0)} z_3^{-1})_{q^2=-\kappa^2} = 0.$$

κ^2 can be calculated from this equation step by step in any order of the perturbation theory. Denoting by $\Sigma(q^2, \kappa^2)$ the self-energy corrections coming from the third term of the Hamiltonian given by eq. /2.3/ and by $\Sigma'(q^2, \kappa^2)$ the self-energy corrections which contain also the $\delta m^2 \varphi^2$ insertion on the usual self-energy diagram, the Green's function is

$$G(q^2, \kappa^2) = -\frac{1}{q^2 + \kappa^2 + \Sigma'(q^2, \kappa^2) + \delta m^2} . \qquad /2.17/$$

The requirement /2.16/ can be written in the form

$$\Sigma'(-\kappa^2, \kappa^2) + \delta m^2 = 0 \qquad /2.18/$$

and therefore

$$G(q^2, \kappa^2) = -\frac{1}{q^2 + \kappa^2 + \Sigma'(q^2, \kappa^2) - \Sigma'(-\kappa^2, \kappa^2)} . \qquad /2.19/$$

The difference of Σ' and Σ comes just from the φ^2 insertions and therefore in a step by step calculation it is easy to see that in Σ' on every internal line the same subtraction procedure has been performed.

Solving the system of equations /2.14/ - /2.16/ for z_1, z_3 and $\delta m^2 = r_0 - \kappa^2$, these quantities can be expressed in terms of λ^2, κ^2 and g

$$z_1 = z_1(\lambda^2, \kappa^2, g), \qquad /2.20/$$

$$z_3 = z_3(\lambda^2, \kappa^2, g), \qquad /2.21/$$

$$r_0 = r_0(\lambda^2, \kappa^2, g). \qquad /2.22/$$

Using these functional forms z_1, z_3 and r_0 can be eliminated from the transformed d and $\tilde{\Gamma}$ and the variable λ appears instead of them. Introducing the dimensionless coupling

$$u = g\,\lambda^{-\varepsilon} \qquad /2.23/$$

the dimensionless functions d and $\tilde{\Gamma}$ can be written as functions of the dimensionless variable q^2/λ^2 and κ^2/λ^2 and u.

$$d\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, u\right) = \qquad /2.24/$$

$$= d\left(q^2, \kappa^2, g, \delta m^2 z_3(\lambda^2,\kappa^2,g), \tilde{\Gamma}_0 z_1(\lambda^2,\kappa^2,g), G^{(0)} z_3^{-1}(\lambda^2,\kappa^2,g)\right),$$

$$\tilde{\Gamma}\left(\frac{q_i^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, u\right) = \qquad /2.25/$$

$$= \tilde{\Gamma}\left(q_i^2, \kappa^2, g, \delta m^2 z_3(\lambda^2,\kappa^2,g), \tilde{\Gamma}_0 z_1(\lambda^2,\kappa^2,g), G^{(0)} z_3^{-1}(\lambda^2,\kappa^2,g)\right).$$

These functions are normalized to unity at $q^2 = \lambda^2$ and $q_i^2 = \lambda^2$ respectively /see eqs. /2.14/ and /2.15//.

Changing the normalization momentum λ, a scaling relation can be obtained for these functions. The mathematical procedure of scaling is clearly described in Ref. [7] and here we will only write down the relations

$$d\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, u\right) = d\left(\frac{\lambda'^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, u\right) d\left(\frac{q^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, u'\right), \qquad /2.26/$$

$$\tilde{\Gamma}\left(\frac{q_i^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, u\right) = \tilde{\Gamma}\left(\frac{\lambda'^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, u\right) \tilde{\Gamma}\left(\frac{q_i^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, u'\right), \qquad /2.27/$$

where

$$u' = u\left(\frac{\lambda'^2}{\lambda^2}\right)^{-\varepsilon/2} \tilde{\Gamma}\left(\frac{\lambda'^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, u\right) d^2\left(\frac{\lambda'^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, u\right). \qquad /2.28/$$

The Lie equations or the Callan-Symanzik type equations that have been used by Di Castro [7] are the differential forms of these relations. In this form they express a triviality, namely that eqs. /2.9/ - /2.11/ are valid for any choice of Z_1 and Z_3. These equations contain a non-trivial information for particular problems only, i.e. for renormalizable theories. In a field theoretical problem, where there is no momentum cut-off, the perturbational expression of d and $\tilde{\Gamma}$ contain singular contributions which were non-singular for finite cut-off. Calculating the Green's function and vertex with finite cut-off Λ, the transformed, renormalized functions have to be chosen in such a manner that the limit $\Lambda \to \infty$ should lead to finite results. If the transformed functions d and $\tilde{\Gamma}$ do not depend on Λ, or the dependence is smooth, this limit procedure can be performed and the theory is renormalizable. The particularity of this φ^4 model is that it is renormalizable in this sense. It is shown in the Appendix on the particular form of d and $\tilde{\Gamma}$ that the transformed functions do not depend on Λ. It is also shown there how the critical exponents can be calculated using the Lie equations.

The requirement of renormalizability, i.e. the requirement that the transformed d and $\tilde{\Gamma}$ should not depend on Λ is equivalent to a special relationship between lower and higher order perturbational corrections and this allows us to use a low order perturbational expression in the Lie equation to generate higher order contributions.

In this renormalization procedure the original two parameter continuous group with z_1 and z_3 is reduced to a one parameter group with scaling parameter λ . The reduction of the number of parameters means that only a subset of the equivalent systems can be generated by different choices of λ . This set is, however, fully sufficient for us to determine the momentum and temperature dependence of the Green's function and vertex.

The abovementioned normalization condition is not very useful if temperature dependent quantities, like susceptibility are to be determined. The temperature dependence is incorporated into κ^2 and therefore eqs. /2.14/ and /2.15/ can be used only if the Green's function and vertex are known as a function of both q^2 and κ^2 simultaneously. These expressions are generally not known and this makes a calculation of the exponents γ, ν, α etc. practically impossible with this normalization condition.

A way out of this difficulty is that in the normalization condition κ^2 is also fixed. One possibility is to require

$$d\left(q^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}^{\circ} z_1, G^{(o)} z_3^{-1}\right)_{q^2=\lambda^2,\ \kappa^2=0} = 1, \qquad /2.29/$$

$$\tilde{\Gamma}\left(q^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_o z_1, G^{(o)} z_3^{-1}\right)_{q^2=\lambda^2,\ \kappa^2=0} = 1. \qquad /2.3o/$$

An equally good choice is

$$d\left(q^2, \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_o z_1, G^{(o)} z_3^{-1}\right)_{q^2=0,\ \kappa^2=\lambda^2} = 1, \qquad /2.31/$$

$$\tilde{\Gamma}\left(q^2\ \kappa^2, g, \delta m^2 z_3, \tilde{\Gamma}_o z_1, G^{(o)} z_3^{-1}\right)_{q^2=0,\ \kappa^2=\lambda^2} = 1. \qquad /2.32/$$

In this latter formulation the scaling of λ explicitely corresponds to scaling the temperature and the formalism can be worked out in the same way as for any other normalization condition. There may of course be other normalization conditions as well, in which e.g. q^2 and κ^2 are normalized at the same value, corresponding to the expected similar behaviour in q^2 and κ^2.

2.3 Multiplicative renormalization with physical normalization

Turning now to the statistical mechanical treatment of the problem it becomes clear that the normalization conditions /2.14/, /2.15/ cannot be maintained. As it was emphasized, in this case there is an inherent large momentum cut-off, the problem has no ultraviolet divergences. The multiplicative renormalization is used only as a useful tool to get the low momentum or low κ^2 behaviour of various physical quantities. Therefore the physical situation $z_1 = z_3 = 1$ should be recovered in the final step of the calculation and therefore the scaling momentum λ has to be introduced in such a way that there should be a value λ_{phys}, for which the multiplicative factors are equal to unity, thus

$$Z_1(\lambda^2_{phys}, \kappa^2, u) = 1, \qquad /2.33/$$

and

$$Z_3(\lambda^2_{phys}, \kappa^2, u) = 1. \qquad /2.34/$$

In general, however, the Green's function and vertex cannot be normalized simultaneously to unity at the same momentum, and there exists no such λ_{phys} which would reproduce the physical situation. A similar normalization problem is present in quantum electrodynamics for the electron Green's function /see e. g. Ref. [9]/.

This shows that instead of normalizing d and $\tilde{\Gamma}$ to unity, another normalization has to be chosen. Furthermore we want to avoid the complication coming from the dependence of the invariant coupling on κ^2 and therefore we propose such a normalization of d and $\tilde{\Gamma}$ that both q^2 and κ^2 be fixed to a characteristic value. Since for κ^2 there is no other characteristic value than zero, we will fix it at $\kappa^2 = 0$. The characteristic value of q is the cut-off Λ. Therefore the proposed normalization of d and $\tilde{\Gamma}$ is such that the physical value of the scaling parameter λ be equal to Λ. Denoting by d_{phys} and $\tilde{\Gamma}_{phys}$ the physical value of d and $\tilde{\Gamma}$ calculated from the physical Hamiltonian with $Z_1 = Z_3 = 1$, the normalization condition can be written as

$$d(q^2, \kappa^2, \Lambda^2, g, \delta m^2 Z_3, \tilde{\Gamma}_0 Z_1, G^{(0)} Z_3^{-1})\big|_{q^2=\lambda, \kappa^2=0} = \qquad /2.35/$$

$$= d_{phys}(q^2, \kappa^2, \Lambda^2, g_0 \mathcal{U}(\lambda^2, \Lambda^2, g))\big|_{q^2=\Lambda^2, \kappa^2=0},$$

$$\Gamma(q^2, \kappa^2, \Lambda^2, g, \delta\kappa^2 z_3, \tilde{\Gamma}_0 z_1, G^{(n)} z_3^{-1})_{q^2=\lambda^2,\, \kappa^2=0} = \qquad /2.36/$$

$$= \tilde{\Gamma}_{ren}(q^2, \kappa^2, \Lambda^2, g_0\, u(\lambda^2, \Lambda^2, g))_{q^2=\Lambda^2,\, \kappa^2=0},$$

where $u(\lambda^2, \Lambda^2, g) = u(\frac{\Lambda^2}{\lambda^2}, u)$ is an arbitrary continuous dimensionless function with the boundary condition

$$u(\lambda^2, \Lambda^2, g)_{\lambda=\Lambda} = 1. \qquad /2.37/$$

The analysis of eqs. /2.35/ - /2.37/ shows that when $\lambda = \Lambda$, indeed $z_1 = z_3 = 1$ and $g = g_0$. This means that the physical value of λ is Λ if this normalization condition is accepted. The solution of eqs. /2.35/ and /2.36/ yields the multiplicative factors z_1 and z_3 as

$$z_1 = z_1\left(\frac{\Lambda^2}{\lambda^2}, u\right), \qquad /2.38/$$

$$z_3 = z_3\left(\frac{\Lambda^2}{\lambda^2}, u\right), \qquad /2.39/$$

and the new functions d and $\tilde{\Gamma}$ after elimination of z_1 and z_3 are the functions of q^2/λ^2, κ^2/λ^2 and Λ^2/λ^2

$$d = d\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right), \qquad /2.40/$$

$$\tilde{\Gamma} = \tilde{\Gamma}\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right). \qquad /2.41/$$

Introducing the bare dimensionless [illegible] similarly to eq. /2.23/ by

$$u_0 = g_0 \Lambda^{-\epsilon} \qquad /2.42/$$

the normalization condition can be expressed in terms of the new notation as

$$d\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)_{q^2=\lambda^2,\ \kappa^2=0} =$$
$$= d_{\mathrm{rep}}\left(\frac{q^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0\,\mathcal{U}\left(\frac{\Lambda^2}{\lambda^2}, u\right)\right)_{q^2=\Lambda^2,\ \kappa^2=0}, \qquad /2.43/$$

$$\tilde{\Gamma}\left(\frac{q_i^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)_{q_i^2=\lambda^2,\ \kappa^2=0} =$$
$$= \tilde{\Gamma}_{\mathrm{rep}}\left(\frac{q_i^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0\,\mathcal{U}\left(\frac{\Lambda^2}{\lambda^2}, u\right)\right)_{q_i^2=\Lambda^2,\ \kappa^2=0}, \qquad /2.44/$$

where $\mathcal{U}\left(\frac{\Lambda^2}{\lambda^2}, u\right)$ obeys the boundary condition

$$\mathcal{U}\left(\frac{\Lambda^2}{\lambda^2}, u\right)_{\lambda^2=\Lambda^2} = 1. \qquad /2.45/$$

In what follows we will consider two special choices of the function $\mathcal{U}(\Lambda^2/\lambda^2, u)$. The first choice is

$$\mathcal{U} = 1 \qquad /2.46/$$

while the second choice is

$$\mathcal{U} = u/u_0 \qquad /2.47/$$

It is important to note that the functional form of z_1 and z_3 /see eqs. /2.38/ and /2.39// and consequently the transformed d and $\tilde{\Gamma}$ depend on the explicit form of $\mathcal{U}(\Lambda^2/\lambda^2, u)$ and a large variety of different renormalizations can be generated. These different renormalizations are equivalent in the sense that they lead to the same physical Green's function and vertex, they are, however, inequivalent in that sense that a particular choice of $\mathcal{U}(\Lambda^2/\lambda^2, u)$ is not always suitable to describe scaling and to get useful information from the scaling relations of d and $\tilde{\Gamma}$. This will be discussed later on.

Other choices of the normalization condition are also possible, in which e. g. κ^2 is not fixed. A general form of this type of normalization is

$$d(q^2, \kappa^2, \Lambda^2, g, \delta m^2 Z_3, \tilde{\Gamma}_o Z_1, G^{-1} Z_3^{-1})_{q^2=\lambda^2} =$$

$$= d_{ren}(q^2, \kappa^2 K(\lambda^2, \Lambda^2, \kappa^2, g), \Lambda^2, g\, U(\lambda^2, \Lambda^2, \kappa^2, g))_{q^2=\lambda^2}, \qquad /2.48/$$

$$\tilde{\Gamma}(q^2, \kappa^2, \Lambda^2, g, \delta m^2 Z_3, \tilde{\Gamma}_o Z_1, G^{-1} Z_3^{-1})_{q^2=\lambda^2} =$$

$$= \tilde{\Gamma}_{ren}(q^2, \kappa^2 K(\lambda^2, \Lambda^2, \kappa^2, g), \Lambda^2, g\, U(\lambda^2, \Lambda^2, \kappa^2, g))_{q^2=\lambda^2}, \qquad /2.49/$$

where the dimensionless functions $K(\lambda^2, \Lambda^2, \kappa^2, g)$ and $U(\lambda^2, \Lambda^2, \kappa^2, g)$ obey the boundary condition

$$K(\lambda^2, \Lambda^2, \kappa^2, g)_{\lambda^2=\Lambda^2} = 1 \qquad /2.50/$$

$$U(\lambda^2, \Lambda^2, \kappa^2, g)_{\lambda^2=\Lambda^2} = 1 \qquad /2.51/$$

This normalization condition, similarly to eqs. /2.14/ and /2.15/ has the disadvantage that the multiplicative factors and the new coupling depend explicitely on κ^2 and the calculation of temperature dependent quantities is very tedious. In what follows we will use the normalization conditions given in eqs. /2.35/ and /2.36/.

2.4 Renormalization group and Lie equations

Making use of the new notations the basic transformation given by eqs. /2.9/ - /2.11/ can be written as

$$d_{reg.}\left(\frac{q^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0\right) = z_3\, d\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right), \qquad /2.52/$$

$$\tilde{\Gamma}_{reg.}\left(\frac{q_i^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0\right) = z_1^{-1}\, \tilde{\Gamma}\left(\frac{q_i^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right), \qquad /2.53/$$

and

$$u = u_0\left(\frac{\lambda^2}{\Lambda^2}\right)^{-\varepsilon/2} z_1^{-1}\, z_3^2 . \qquad /2.54/$$

Repeating this transformation with normalization at another value of the momenta $q^2 = \lambda'^2$ and denoting the coupling constant with g' and the dimensionless coupling with u' where $u' = g'\lambda'^{-\varepsilon}$, the Green's functions and vertices in the two cases are very simply related. Using the invariance properties in eqs. /2.52/ - /2.54/, the following identities are obtained

$$z_3\left(\frac{\Lambda^2}{\lambda^2}, u\right) d\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = z_3\left(\frac{\Lambda^2}{\lambda'^2}, u'\right) d\left(\frac{q^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right) \qquad /2.55/$$

$$z_1^{-1}\left(\frac{\Lambda^2}{\lambda^2}, u\right) \tilde{\Gamma}\left(\frac{q_i^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = z_1^{-1}\left(\frac{\Lambda^2}{\lambda'^2}, u'\right) \tilde{\Gamma}\left(\frac{q_i^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right) \qquad /2.56/$$

$$u\,\lambda^{\varepsilon}\, z_1\left(\frac{\Lambda^2}{\lambda^2}, u\right) z_3^{-2}\left(\frac{\Lambda^2}{\lambda^2}, u\right) = u'\,\lambda'^{\varepsilon}\, z_1\left(\frac{\Lambda^2}{\lambda'^2}, u'\right) z_3^{-2}\left(\frac{\Lambda^2}{\lambda'^2}, u'\right) \qquad /2.57/$$

These equations show that the transformation forms a group. A more usual form can be obtained if the factors z_1 and z_3 are expressed in terms of d and $\tilde{\Gamma}$ by a special choice of the momenta $q_i^2 = \lambda^2$. We get

$$d\left(1, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right) d\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = d\left(\frac{\lambda'^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) d\left(\frac{q^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right) \qquad /2.58/$$

$$\tilde{\Gamma}\left(1, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right) \tilde{\Gamma}\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = \tilde{\Gamma}\left(\frac{\lambda'^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) \tilde{\Gamma}\left(\frac{q^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right) \qquad /2.59/$$

$$u' = u\left(\frac{\lambda'^2}{\lambda^2}\right)^{-\varepsilon/2} \frac{\tilde{\Gamma}\left(\frac{\lambda'^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)}{\tilde{\Gamma}\left(1, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right)} \cdot \frac{d^2\left(\frac{\lambda'^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)}{d^2\left(1, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right)} \qquad /2.60/$$

The new coupling u' has to be determined from the self-consistent solution of this equation. It should be stressed *that u' does not depend on κ^2 due to our particular choice of* the normalization condition. The dependence on κ^2 of the right hand side of eq. /2.6o/ is spurious, similarly as u' does not depend on q^2 if /2.6o/ is transcribed into an equivalent form

$$u' = u\,\frac{\lambda'^2}{\lambda^2}\; \frac{\tilde{\Gamma}\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)}{\tilde{\Gamma}\left(\frac{q^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right)}\; \frac{d^2\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)}{d^2\left(\frac{q^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right)} . \qquad /2.61/$$

The real variables of u' are λ'/λ and Λ/λ

$$u' = u'\left(\frac{\lambda'^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right). \qquad /2.62/$$

Introducing the function $u_R(q^2/\lambda^2, \Lambda^2/\lambda^2, u)$ by the definition

$$u_R\left(\frac{q^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = u\left(\frac{q^2}{\lambda^2}\right)^{-\varepsilon/2} \frac{\tilde{\Gamma}\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)}{\tilde{\Gamma}\left(1, \frac{\kappa^2}{q^2}, \frac{\Lambda^2}{q^2}, u_R\left(\frac{q^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)\right)} \cdot$$

$$\cdot \frac{d^2\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)}{d^2\left(1, \frac{\kappa^2}{q^2}, \frac{\Lambda^2}{q^2}, u_R\left(\frac{q^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right)\right)} \qquad /2.63/$$

this function is equal to u' when $q = \lambda'$

$$u_R\left(\frac{\lambda'^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = u'\left(\frac{\lambda'^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = u' . \qquad /2.64/$$

It can easily be shown using the invariance of the combination $u\lambda^{\varepsilon}\tilde{\Gamma} d^{2}$ that u_R obeys the following scaling relation

$$u_R\left(\frac{q^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = u_R\left(\frac{q^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right), \qquad /2.65/$$

where u' is given by eq. /2.64/. Due to this relation this quantity is called invariant coupling.

The transformation properties of the Green's function, vertex and the invariant coupling, eqs. /2.58/, /2.59/ and /2.65/, can be written in the common form

$$A\left(\frac{q^2}{\lambda^2}, \frac{\kappa^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) = Z\left(\frac{\lambda'^2}{\lambda^2}, \frac{\Lambda^2}{\lambda^2}, u\right) A\left(\frac{q^2}{\lambda'^2}, \frac{\kappa^2}{\lambda'^2}, \frac{\Lambda^2}{\lambda'^2}, u'\right). \qquad /2.66/$$

Introducing the variables $x = \frac{q^2}{\lambda^2}$, $y = \frac{\kappa^2}{\lambda^2}$, $v = \frac{\Lambda^2}{\lambda^2}$ and $s = \frac{\lambda'^2}{\lambda^2}$, differentiating with respect to x the logarithm of both sides and finally taking $s = x$ we get

$$\frac{\partial \ln A(x,y,v,u)}{\partial x} = \frac{1}{x}\frac{\partial}{\partial \xi} \ln A\left(\xi, \frac{y}{x}, \frac{v}{x}, u_R(x,v,u)\right)\Big|_{\xi=1} \qquad /2.67/$$

and similarly differentiation with respect to y gives

$$\frac{\partial \ln A(x,y,v,u)}{\partial y} = \frac{1}{y}\frac{\partial}{\partial \eta} \ln A\left(\frac{x}{y}, \eta, \frac{v}{y}, u_R(y,v,u)\right)\Big|_{\eta=1} \qquad /2.68/$$

This is the Lie equation of the group and this will be used later on in the calculations. Everywhere in what follows we use the convention that when in the course of deriving the Lie equation s is replaced by x or y (λ'^2 by q^2 or κ^2), the notation u_R will be applied instead of u'. A Callan-Symanzik-type equation can be obtained from eq. /2.66/ if differentiation is performed with respect to λ'^2 and then putting $\lambda^2 = \lambda'^2$. Di Castro [7] used this equation to determine the critical exponent η . The calculation of η with field theoretical normalization using the Lie equations is presented in the Appendix. We will calculate now the same quantity using the Lie equation with this normalization scheme, to show how the method works.

We have chosen the normalization condition in eqs. /2.35/ and /2.36/ with the aim to be able to calculate the physical Green's function and vertex with the normalization momentum λ chosen appropriately. Therefore we have to calculate the dependence of d and $\tilde{\Gamma}$ not only on x and y but on v as well. It is seen from eq. /2.66/ that a simple Lie equation cannot be derived if differentiation is performed with respect to v , since $z(\lambda'^2/\lambda^2, \Lambda^2/\lambda^2, u)$ also depends on v. In addition the criticism that was mentioned in connection with the field theoretical normalization is valid here as well. The scaling equations /2.58/ - /2.6o/ express a triviality and usually are of no help in calculating the Green's function and vertex. These relations only tell us how the transformed function is generated from the physical function but it contains no special information about the physical function itself. These relations become non-trivial if the number of variables does not increase in the course of the transformation and the variable $v = \Lambda^2/\lambda^2$ does not appear in the Lie equations. This can only happen in particular cases, for particular models where cut-off scaling holds with particular normalization condition. We will show that this is the case with the φ^4 model and the particular normalization will lead to a new formulation of the problem. A comparison with the method of Sec. III. will be possible in this way.

It is important to mention that there are cases /e. g. the Anderson model in the weak coupling limit [25]/ where the v-dependence is very weak, scaling holds approximately and the results obtained in this way represent good approximations.

Supposing now that the variable $v = \Lambda^2/\lambda^2$ is absent from the Lie equations, eqs. /2.67/ and /2.68/ can be used to calculate the q^2 and κ^2 dependence of the Green's function and vertex. First the Lie equation for the invariant coupling itself has to be solved and then this function can be used to determine the Green's function and vertex. When using the Lie differential equation, the right hand side of eq. /2.67/ is usually determined in perturbation theory. The solution of the equation contains higher order terms generated from the low order contributions. Since according to eq. /2.67/ the bare coupling has to be replaced by the invariant coupling on the right hand side, this perturbational calculation is useful only if the invariant coupling is small. This is the case in $d = 4 - \varepsilon$ dimensions as will be shown and that is the reason why this multiplicative renormalization scheme is very appropriate to determine the critical exponents in ε expansion.

2.5 Calculation of the exponents

As a typical example we will show now how the exponent η can be obtained in this scheme. Working at $\kappa^2=0$ the Green's function and vertex can easily be calculated in perturbation theory up to second or third order, considering the diagrams of Fig. 3. and Fig. 4. respectively /due to the subtraction in eq. /2.19/ the Hartree loop gives no contribution at $\kappa^2=0$/

$$G(q^2) = \frac{1}{q^2}\Big\{1 + \frac{n+2}{144}u_o^2 K_d^2 \left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon}\left[\left(1-\frac{\varepsilon}{4}\right)\ln\frac{q^2}{\Lambda^2} - \frac{5}{2} + \frac{\varepsilon}{2}\ln^2\frac{q^2}{\Lambda^2} + \dots\right]$$

$$+ \frac{(n+2)(n+8)}{12^3}u_o^3 K_d^3\left(\frac{q^2}{\Lambda^2}\right)^{-\frac{3}{2}\varepsilon}\left[\ln^2\frac{q^2}{\Lambda^2} - 5\ln\frac{q^2}{\Lambda^2} + \frac{15}{2} + \dots\right] + \dots\Big\} \qquad /2.69/$$

$$\tilde{\Gamma}(q^2) = 1 + \frac{n+8}{12}u_o K_d\left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon/2}\left\{\frac{2}{\varepsilon}\left[\left(\frac{q^2}{\Lambda^2}\right)^{\varepsilon/2} - 1\right] - 1 + \dots\right\}$$

$$+ \frac{n^2+6n+20}{36}u_o^2 K_d^2\left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon}\frac{1}{4}\left\{\frac{2}{\varepsilon}\left[\left(\frac{q^2}{\Lambda^2}\right)^{\varepsilon/2} - 1\right] - 1 + \dots\right\}^2$$

$$+ \frac{5n+22}{9}u_o^2 K_d^2\left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon}\frac{1}{4}\left[\frac{1}{2}\ln^2\frac{q^2}{\Lambda^2} - 2\ln\frac{q^2}{\Lambda^2} + 2 + \dots\right] + \dots, \qquad /2.70/$$

where $K_d = 2^{-(d-1)}\pi^{-d/2}\left[\Gamma\left(\frac{d}{2}\right)\right]^{-1}$. We have chosen the momentum variables on the four legs of the vertex in a special manner so that the vertex depends on one momentum only. This is a restriction only for the fourth diagram of Fig. 4., the others depend anyway on one variable only. In Fig. 5. this choice is indicated for two different orientations of the lines. Since these diagrams with differently oriented lines yield the same contribution due to the structure of the unperturbed Green's function, in general we draw the diagram with non-oriented lines.

The analytic contribution of the different diagrams is given in the Appendix. In the calculation of the relevant integrals in the ε-expansion, only those terms were retained which will contribute to the invariant coupling and to the critical exponents to order ε^2 or ε^3.

Knowing the Green's function and vertex in perturbation theory the multiplicative factors Z_1 and Z_3 can be determined as functions of Λ^2/λ^2 and u using the normalization conditions in eqs. /2.35/ and /2.36/ and expressing g_0 in terms of g / u_0 in terms of u / with the help of eq. /2.11/. The actual calculation is performed with two different choices of $\mathcal{U}(\Lambda^2/\lambda^2, u)$ given in eqs. /2.46/ and /2.47/. After that the transformed Green's function and vertex can be constructed as functions of q^2/λ^2, Λ^2/λ^2 and u.

Using the first normalization condition given by eqs. /2.35/, /2.36/ and /2.46/ the multiplicative factors are

$$\begin{aligned} Z_1 = 1 &- \frac{n+8}{12} u K_d \left(1+\frac{\varepsilon}{2}\right) f\left(\frac{\lambda^2}{\Lambda^2}\right) + 2\left(\frac{n+8}{12}\right)^2 u^2 K_d^2 \left(1+\frac{\varepsilon}{2}\right)^2 f^2\left(\frac{\lambda^2}{\Lambda^2}\right) \\ &- \left(\frac{n+8}{12}\right)^2 u^2 K_d^2 \left(1+\frac{\varepsilon}{2}\right)\left(\frac{\lambda^2}{\Lambda^2}\right)^{\varepsilon/2} f\left(\frac{\lambda^2}{\Lambda^2}\right) \\ &- \frac{n^2+6n+20}{36} u^2 K_d^2 \frac{1}{4}\left[f^2\left(\frac{\lambda^2}{\Lambda^2}\right) - 2 f\left(\frac{\lambda^2}{\Lambda^2}\right) + 1 - \left(\frac{\lambda^2}{\Lambda^2}\right)^{\varepsilon} + \dots\right] \qquad /2.71/ \\ &- \frac{5n+22}{9} u^2 K_d^2 \frac{1}{4}\left[\frac{1}{2}\ln^2\frac{\lambda^2}{\Lambda^2} - 2\ln\frac{\lambda^2}{\Lambda^2} + 2\left(1-\left(\frac{\lambda^2}{\Lambda^2}\right)^{\varepsilon}\right) + \dots\right] + \dots, \end{aligned}$$

$$Z_3 = 1 + \frac{n+2}{144} u^2 K_d^2 \left[\left(1-\frac{\varepsilon}{4}\right)\ln\frac{\lambda^2}{\Lambda^2} - \frac{5}{2}\left(1-\left(\frac{\lambda^2}{\Lambda^2}\right)^{\varepsilon}\right) + \dots\right] + \dots \qquad /2.72/$$

where

$$f(x) = \frac{2}{\varepsilon}\left(x^{\varepsilon/2} - 1\right). \qquad /2.73/$$

The transformed Green's function and vertex after elimination of Z_1 and Z_3 are

$$d\left(\frac{q^2}{\lambda^2},\frac{\Lambda^2}{\lambda^2},u\right) = 1 + \frac{n+2}{144}u^2K_d^2\left[\left(1+\frac{9}{4}\varepsilon\right)\ln\frac{q^2}{\lambda^2} - \frac{5}{2} + \ldots\right] + \frac{(n+2)(n+8)}{12^3}u^3K_d^3\left[\ln^2\frac{q^2}{\lambda^2} - 5\ln\frac{q^2}{\lambda^2} + \frac{15}{2} + \ldots\right] + \ldots \qquad /2.74/$$

$$\tilde{\Gamma}\left(\frac{q^2}{\lambda^2},\frac{\Lambda^2}{\lambda^2},u\right) = 1 + \frac{n+8}{12}uK_d\left(\frac{q^2}{\lambda^2}\right)^{-\varepsilon/2}\left[f\left(\frac{q^2}{\lambda^2}\right) - 1 - \frac{\varepsilon}{2}\left(\frac{q^2}{\lambda^2}\right)^{\varepsilon/2} f\left(\frac{\lambda^2}{\Lambda^2}\right) + \ldots\right] + \left(\frac{n+8}{12}\right)^2u^2K_d^2\left(\ln^2\frac{q^2}{\lambda^2} - 2\ln\frac{q^2}{\lambda^2} - \ln\frac{\Lambda^2}{\lambda^2} + 1 + \ldots\right) \qquad /2.75/$$

$$\frac{5n+22}{36}u^2K_d^2\left(\ln\frac{q^2}{\lambda^2} - \frac{3}{2} + \ldots\right) + \ldots$$

In the second order terms in u the contributions of order ε have been neglected. It should be noted that the function $\tilde{\Gamma}$ depends explicitely on the variable Λ^2/λ^2 and not only on q^2/λ^2. As we have mentioned, an explicit dependence on the variable Λ^2/λ^2, i.e. the appearance of an additional variable in the transformed function makes in general this renormalization procedure useless. In this particular case, however, this variable does not appear in the invariant coupling and this normalization condition can be used to calculate the critical behaviour.

Making use of the expressions /2.69/ and /2.7o/ for d and $\tilde{\Gamma}$, respectively, the self-consistent solution of eq. /2.61/ for u' is

$$u' = u\left(\frac{\lambda'^2}{\lambda^2}\right)^{-\varepsilon/2}\left[1 + \frac{n+8}{12}uK_d\left(1+\frac{\varepsilon}{2}\right)\left(\frac{\lambda'^2}{\lambda^2}\right)^{-\varepsilon/2} f\left(\frac{\lambda'^2}{\lambda^2}\right) + \left(\frac{n+8}{12}\right)^2u^2K_d^2\left(\ln^2\frac{\lambda'^2}{\lambda^2} - \ln\frac{\lambda'^2}{\lambda^2} + \ldots\right) - \frac{9n+42}{72}u^2K_d^2\ln\frac{\lambda'^2}{\lambda^2} + \ldots\right. \qquad /2.76/$$

Using this perturbational expression and replacing λ'^2 by q^2 the Lie equation for the invariant coupling can be obtained in a straightforward manner in the form

$$\frac{\partial u_R(x)}{\partial x} = \frac{u_R(x)}{x}\Big\{ -\frac{\varepsilon}{2} + \frac{n+8}{12}\, u_R(x) K_d \left(1+\frac{\varepsilon}{2}\right) - \left[\left(\frac{n+8}{12}\right)^2 + \frac{9n+42}{72}\right] u_R^2(x) K_d^2 + \dots \Big\} \qquad /2.77/$$

where $x = q^2/\lambda^2$. The solution of this equation for $\varepsilon > 0$ and small x is

$$K_d u_R(x) = K_d u_0^* + A\, x^{\omega/2} + \dots \qquad /2.78/$$

where

$$K_d u_0^* = \frac{6}{n+8}\,\varepsilon \left[1 + \varepsilon\, \frac{9n+42}{(n+8)^2}\right] + O(\varepsilon^3) \qquad /2.79/$$

and

$$\omega = \varepsilon - \frac{9n+42}{(n+8)^2}\,\varepsilon^2 + O(\varepsilon^3) \qquad /2.8o/$$

and A is a constant.

The critical behaviour of the system in the scaling regime is governed by the invariant coupling for small x . The critical exponents are determined by u_0^* which is called fixed point coupling. The term with exponent ω gives the first correction to the scaling behaviour. In order to determine e.g. η the Lie equation for the Green's function d has to be solved. Using the perturbational expression from eq. /2.74/ the Lie equation /2.67/ for the present case is

$$\frac{\partial \ln d(x,v,u)}{\partial x} = \frac{1}{x}\Big\{ \frac{n+2}{144}\, u_R^2(x) K_d^2 \left(1+\frac{9}{4}\varepsilon\right) - 5\,\frac{(n+2)(n+8)}{12^3}\, u_R^3(x) K_d^3 + \dots \Big\} \qquad /2.81/$$

where $x = q^2/\lambda^2$. The variable $v = \Lambda^2/\lambda^2$ does not appear. The solution of this equation is

$$d(x) \sim x^{\eta/2} \left(1 + B\, x^{\omega/2} + \dots \right) \qquad /2.82/$$

with

$$\eta = \frac{n+2}{2(n+8)^2}\, \varepsilon^2 \left[1 + \varepsilon \left(\frac{6(3n+14)}{(n+8)^2} - \frac{1}{4} \right) \right] + o(\varepsilon^4). \qquad /2.83/$$

The physical function $d(q^2)$ is obtained by taking $\lambda = \lambda_{phys} = \Lambda$,

$$d(q^2) \sim \left(\frac{q^2}{\Lambda^2}\right)^{\eta/2} \left(1 + B\left(\frac{q^2}{\Lambda^2}\right)^{\omega/2} + \dots \right). \qquad /2.84/$$

Since the total Green's function G differs from d by a factor q^{-2} , the small momentum behaviour of G is given by

$$G \sim \frac{1}{q^{2-\eta}} \left(1 + B' q^{\omega} + \dots \right), \qquad /2.85/$$

with the correct expression for η [17] and also the correction to scaling [26] is obtained correctly.

Other critical exponents can be obtained in a similar way, and all exponents are obtained correctly up to order ε^2 , at least. The fact that the variable Λ^2/λ^2 appears in $\tilde{\Gamma}$ gives no complication since this variable disappears from the Lie equations to this order. In higher orders this is probably not the case and other normalization condition has to be chosen.

A good candidate for such a normalization condition is the choice given by eqs. /2.35/, /2.36/ and /2.47/. The normalization factors in this case are

$$Z_1 = 1 - \frac{n+8}{12} u K_d\, f\left(\frac{\lambda^2}{\Lambda^2}\right) + \left(\frac{n+8}{12}\right)^2 u^2 K_d^2 \left[2 f^2\left(\frac{\lambda^2}{\Lambda^2}\right) - 2 f\left(\frac{\lambda^2}{\Lambda^2}\right) + \dots \right]$$

$$+ \frac{n^2+6n+20}{144} u^2 K_d^2 \left[\left(f\left(\frac{\lambda^2}{\Lambda^2}\right) - 1\right)^2 - 1 + \dots \right] \qquad /2.86/$$

$$+ \frac{5n+22}{36} u^2 K_d^2 \left[\frac{1}{2} \ln^2 \frac{\lambda^2}{\Lambda^2} - 2 \ln \frac{\lambda^2}{\Lambda^2} + \dots \right] + \dots$$

$$Z_3 = 1 + \frac{n+2}{144} u^2 K_d^2 \left[\left(1 - \frac{\varepsilon}{4}\right) \ln \frac{\lambda^2}{\Lambda^2} + \frac{\varepsilon}{2} \ln^2 \frac{\lambda^2}{\Lambda^2} + \dots \right]$$

$$+ \frac{(n+2)(n+8)}{12^3} u^3 K_d^3 \left[- \ln^2 \frac{\lambda^2}{\Lambda^2} + \dots \right] + \dots \qquad /2.87/$$

The transformed Green's function and vertex is obtained as

$$d = 1 + \frac{n+2}{144} u^2 K_d^2 \left(\frac{q^2}{\lambda^2}\right)^{\varepsilon} \left[\left(1 - \frac{\varepsilon}{4}\right) \ln \frac{q^2}{\lambda^2} - \frac{5}{2} + \frac{\varepsilon}{2} \ln^2 \frac{q^2}{\lambda^2} + \dots \right]$$

$$+ \frac{(n+2)(n+8)}{12^3} u^3 K_d^3 \left(\frac{q^2}{\lambda^2}\right)^{\varepsilon/2} \left[\ln^2 \frac{q^2}{\lambda^2} - 5 \ln \frac{q^2}{\lambda^2} + \frac{15}{2} + \dots \right] + \dots \qquad /2.88/$$

$$\tilde{\Gamma} = 1 + \frac{n+8}{12} u K_d \left(\frac{q^2}{\lambda^2}\right)^{\varepsilon/2} \left[\frac{2}{\varepsilon} \left(\left(\frac{q^2}{\lambda^2}\right)^{\varepsilon/2} - 1 \right) - 1 + \dots \right]$$

$$+ \frac{n^2+6n+20}{144} u^2 K_d^2 \left(\frac{q^2}{\lambda^2}\right)^{\varepsilon} \left[\frac{2}{\varepsilon} \left(\left(\frac{q^2}{\lambda^2}\right)^{\varepsilon/2} - 1 \right) - 1 + \dots \right]^2 \qquad /2.89/$$

$$+ \frac{5n+22}{36} u^2 K_d^2 \left[\frac{1}{2} \ln^2 \frac{q^2}{\lambda^2} - 2 \ln \frac{q^2}{\lambda^2} + 2 + \dots \right] + \dots$$

The perturbational expansion of the invariant coupling in this case is

$$u' = u \left(\frac{\lambda'^2}{\lambda^2}\right)^{-\varepsilon/2} \left\{ 1 + \frac{n+8}{12} u K_d \frac{2}{\varepsilon} \left[1 - \left(\frac{\lambda'^2}{\lambda^2}\right)^{-\varepsilon/2} \right] \right.$$

$$\left. + \left(\frac{n+8}{12}\right)^2 u^2 K_d^2 \ln^2 \frac{\lambda'^2}{\lambda^2} - \frac{9n+42}{72} u^2 K_d^2 \ln \frac{\lambda'^2}{\lambda^2} + \dots \right\} \qquad /2.90/$$

The Lie equation for the invariant coupling takes now the form

$$\frac{\partial u_R(x,u)}{\partial x} = \frac{u_R(x,u)}{x}\left\{ -\frac{\varepsilon}{2} + \frac{n+8}{12}\, u_R(x,u) K_d - \frac{3n+42}{72}\, u_R^2(x,u) K_d^2 + \dots \right\} \qquad /2.91/$$

where $x = q^2/\lambda^2$. Considering the region $x \ll 1$ the solution of this equation to order ε^2 is the same as that of eq. /2.77/, i.e. this equation leads to the same fixed point and the same exponent ω, although the equations are different. Using eq. /2.88/ for the reduced Green's function d, the same form is obtained for the Lie equation as in /2.81/ and again the exponent η is obtained correctly.

The advantage of this normalization condition is that the variable Λ^2/λ^2 drops out of the transformed functions d and $\tilde{\Gamma}$ and also from the invariant coupling. If this variable does not appear, the transformed functions have similar structure as the physical functions and the scaling relation expresses a non-trivial relation. This non-trivial relation is reflected in the Lie equations which allow the generation of higher order corrections from a low order perturbational expression. On the other hand, it is easy to check that the renormalization condition with u given by /2.47/ is the only one in the considered class, where the transformed functions may maintain the form of the physical functions.

Using a somewhat different normalization condition, as in our first example, the variable Λ^2/λ^2 does not drop out completely but the calculation of the critical exponents is still possible because it does not appear in the Lie equations. It should be emphasized that everywhere we have calculated the singular contributions only and neglected those terms which vanish when the cut-off goes to infinity. The appearance and disappearance of the variable Λ^2/λ^2 has been investigated in this approximation. Probably there is no such physical normalization condition which would lead to scaling relations where Λ^2/λ^2 does not appear at all. Scaling is obeyed asymptotically only where the smooth dependence on Λ^2/λ^2 can be neglected.

In the next section the renormalization procedure will be reformulated. Its relation to the usual renormalization treatment will be discussed in the last sections of this paper.

III. The new renormalization procedure

3.1 Formulation of the new procedure

It was shown in the preceding section that in the conventional Gell-Mann-Low renormalization scaling is achieved only after having introduced instead of the physical Green's function and vertex related functions with a new scaling variable .

In Wilson's theory the elimination of degrees of freedom and thereby scaling is performed on the physical system and there is no need to introduce auxiliary scaling variables. The physical system is mapped onto an equivalent one with the same number of variables eventually the number of coupling constants increases. In view of this fact it is tempting to suppose that the introduction of the scaling reference momentum is not a necessity in multiplicative renormalization and a scaling of the natural momentum cut-off can generate multiplicative renormalization of the Green's function and vertex.

This supposition is confirmed on the perturbational expressions for the Green's function and vertex as will be seen later on. The main idea of the new renormalization procedure is the assumption that a successive elimination of degrees of freedom or scaling of the cut-off can in fact generate a multiplicative renormalization. If this is true, the simple mathematics of multiplicative renormalization, the Lie differential equations can be applied using the cut-off as scaling variable and no extra scaling variable has to be introduced. We will formulate this now in a formal way.

The Hamiltonian of the system is written in the form

$$H = \int d^d x \left\{ \frac{r_0}{2} \varphi^2(x) + \frac{1}{2} \left[\nabla \varphi(x) \right]^2 + \frac{u_0}{4!} \Lambda^{\varepsilon} \left[\varphi^2(x) \right]^2 \right\} \qquad /3.1/$$

where we have introduced an explicit cut-off dependence of the coupling constant and u_0 is dimensionless.

Similarly as in the conventional multiplicative renormalization a mass renormalization is performed using eqs. /2.3/ and /2.16/. The new mass κ^2 is related to the coherence length ξ by the relation $\xi = \kappa^{-1}$ and will be a measure of the temperature. Introducing the reduced, dimensionless Green's function and vertex by the definitions $d = G / G^{(0)}$ and $\tilde{\Gamma} = \Gamma / u_0 \Lambda^{\varepsilon}$, these quantities depend on the dimensionless variables q^2/Λ^2 and κ^2/Λ^2 and on the dimensionless coupling

$$d = d\left(\frac{q^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0 \right) \qquad /3.2/$$

$$\tilde{\Gamma} = \tilde{\Gamma}\left(\frac{q_i^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0 \right) \qquad /3.3/$$

We can now formulate mathematically the basic assumption of this new method. It is assumed that when the large momentum degrees of freedom are eliminated by scaling the physical cut-off Λ to Λ', the dimensionless coupling constant u_0 can simultaneously be changed to u_0' in such a way that the dimensionless Green's function d and vertex $\tilde{\Gamma}$ in the original and transformed systems differ only by multiplicative factors Z_d and Z_Γ which depend only on Λ'/Λ and u_0, but are independent of the momentum and temperature variables. In addition, the new coupling constant u_0' must be related to the original coupling through the same factors Z_d and Z_Γ.

$$d\left(\frac{q^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0 \right) = Z_d \left(\frac{\Lambda'^2}{\Lambda^2}, u_0 \right) d\left(\frac{q^2}{\Lambda'^2}, \frac{\kappa^2}{\Lambda'^2}, u_0' \right) \qquad /3.4/$$

$$\tilde{\Gamma}\left(\frac{q_i^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0 \right) = Z_\Gamma^{-1} \left(\frac{\Lambda'^2}{\Lambda^2}, u_0 \right) \tilde{\Gamma}\left(\frac{q_i^2}{\Lambda'^2}, \frac{\kappa^2}{\Lambda'^2}, u_0' \right) \qquad /3.5/$$

$$u_0' \Lambda'^{\varepsilon} = u_0 \Lambda^{\varepsilon} Z_\Gamma^{-1} \left(\frac{\Lambda'^2}{\Lambda^2}, u_0 \right) Z_d^2 \left(\frac{\Lambda'^2}{\Lambda^2}, u_0 \right). \qquad /3.6/$$

These equations are formally similar to eqs. /2.52/ - /2.54/ with an essential difference, namely while eqs. /2.52/ - /2.54/ express an exact relation and relate the physical Green's function and vertex to a new function with an additional variable, eqs. /3.4/ - /3.6/ describe a supposed relationship between the physical functions but with modified parameters. Therefore the physical content of these equations is quite different. This will be discussed later on.

Contrary to eqs. /2.52/ - /2.54/, these equations are probably not exact and are valid for the singular parts of the functions d and $\tilde{\Gamma}$, for those contributions which determine the critical behaviour, but these relations will not hold out of the critical region. We will show in the next subsection that they can be verified in perturbation theory. It is very important in this respect that κ^2, which is the analog of the renormalized mass, must be properly chosen. Without mass renormalization, if the Green's function and vertex are functions of q^2/Λ^2 and t_0/Λ^2, it is not possible to find a new coupling u_0' which is independent of q^2 and t_0.

It is a great advantage of this method, that the new coupling u_0' does not depend on κ^2 but on Λ'^2/Λ^2 only. This allows us to calculate either the momentum or temperature dependence of the Green's function or other quantities with equal ease. It follows from the group property of this renormalization procedure that the transformation from Λ and u_0 to Λ'' and u_0'' can be done directly or through the intermediate state with Λ' and u_0'.

$$Z_d\left(\frac{\Lambda''^2}{\Lambda^2}, u_0\right) = Z_d\left(\frac{\Lambda'^2}{\Lambda^2}, u_0\right) Z_d\left(\frac{\Lambda''^2}{\Lambda'^2}, u_0'\right), \qquad /3.7/$$

$$Z_\Gamma\left(\frac{\Lambda''^2}{\Lambda^2}, u_0\right) = Z_\Gamma\left(\frac{\Lambda'^2}{\Lambda^2}, u_0\right) Z_\Gamma\left(\frac{\Lambda''^2}{\Lambda'^2}, u_0'\right). \qquad /3.8/$$

Introducing the function

$$u_R(x, u_0) = u_0 x^{\varepsilon/2} Z_d^2(x, u_0) Z_\Gamma^{-1}(x, u_0), \qquad /3.9/$$

it is easy to proof from eqs. /3.12/, /3.13/ and /3.6/ that

$$\begin{aligned} u_R\left(\frac{\Lambda''^2}{\Lambda^2}, u_0\right) &= u_0 \left(\frac{\Lambda''^2}{\Lambda^2}\right)^{\varepsilon/2} Z_d^2\left(\frac{\Lambda''^2}{\Lambda^2}, u_0\right) Z_\Gamma^{-1}\left(\frac{\Lambda''^2}{\Lambda^2}, u_0\right) \\ &= u_0' \left(\frac{\Lambda''^2}{\Lambda'^2}\right)^{\varepsilon/2} Z_d^2\left(\frac{\Lambda''^2}{\Lambda'^2}, u_0'\right) Z_\Gamma^{-1}\left(\frac{\Lambda''^2}{\Lambda'^2}, u_0'\right) \\ &= u_R\left(\frac{\Lambda''^2}{\Lambda'^2}, u_0'\right), \end{aligned} \qquad /3.1o/$$

and

$$u_0' = u_R\left(\frac{\Lambda'^2}{\Lambda^2}, u_0\right). \qquad /3.11/$$

As eq. /3.1o/ shows, the new coupling u_R is invariant under the scaling of the cut-off and a simultaneous tranformation of u_0 to u_0'. This quantity is therefore called invariant coupling or "invariant charge".

Similarly as in the conventional method a Lie differential equation can be derived for the Green's function, the vertex and the invariant coupling itself. For this latter this Lie equation is the differential form of eq. /3.1o/. Differentiating with respect to $s = \Lambda''^2/\Lambda^2$ and fixing Λ'^2 afterwards at $\Lambda'^2 = \Lambda''^2$ we get

$$\frac{d\, u_R(s, u_0)}{ds} = \frac{1}{s} \left. \frac{d\, u_R(\xi, u_R(s, u_0))}{d\xi} \right|_{\xi=1} \qquad /3.12/$$

For the Green's function and vertex two Lie equations can be derived, differentiating the logarithm of eqs. /3.4/ and /3.5/ with respect to either $x = q^2/\Lambda^2$ or $y = \kappa^2/\Lambda^2$ and fixing Λ'^2 by $q^2 = \Lambda'^2$ or $\kappa^2 = \Lambda'^2$ afterwards.

$$\frac{\partial \ln A(x,y,u_0)}{\partial x} = \frac{1}{x}\frac{\partial}{\partial \xi} \ln A\left(\xi, \frac{y}{x}, u_R(x,u_0)\right)\Big|_{\xi=1}, \qquad /3.13/$$

$$\frac{\partial \ln A(x,y,u_0)}{\partial y} = \frac{1}{y}\frac{\partial}{\partial \eta} \ln A\left(\frac{x}{y}, \eta, u_R(y,u_0)\right)\Big|_{\eta=1}, \qquad /3.14/$$

where A is any of d or $\tilde{\Gamma}$.

The physical picture is now very simple. Suppose we can calculate the Green's function for momenta near the cut-off and would like to know it for $q^2 \ll \Lambda^2$. By scaling the cut-off Λ to Λ' near to q we are again in the transformed system at a situation where the momentum is near the cut-off and the Green's function can be calculated, provided u_0' is known. Using the solution in the transformed system the Lie equation generates the solution for the original physical system. Once the Lie equation for the invariant coupling has been solved, the Lie equation for d and $\tilde{\Gamma}$ can be integrated to determine the momentum or temperature dependence of d and $\tilde{\Gamma}$.

It is not only the Green's function and vertex which obey multiplicative renormalization. There might be other quantities $A(q^2/\Lambda^2, \kappa^2/\Lambda^2, u_0)$ for which

$$A\left(\frac{q^2}{\Lambda'^2}, \frac{\kappa^2}{\Lambda'^2}, u_0'\right) = Z_A\left(\frac{\Lambda'^2}{\Lambda^2}, u_0\right) A\left(\frac{q^2}{\Lambda^2}, \frac{\kappa^2}{\Lambda^2}, u_0\right), \qquad /3.15/$$

i. e . under the same tranformation from Λ and u_o to Λ' and u_o' , the functions are transformed by a multiplicative factor z_Λ independent of q^2 or κ^2 . Example for such a quantity will be shown later on when calculating the specific heat. For these quantities as well a Lie equation can be derived, in the same form as eqs. /3.13/ and /3.14/.

3.2 Multiplicative factors and the fixed point

We will calculate the multiplicative factors and the invariant coupling in perturbation theory to show that eqs. /3.4/ - /3.6/ can be obeyed and the z factors are independent of q^2 and κ^2 . Since the calculation of d and $\tilde{\Gamma}$ as a function of two variables q^2/Λ^2 and κ^2/Λ^2 is increasingly difficult for higher order contributions, the special cases $\kappa^2=0$ and $q^2=0$ will be studied separately. The multiplicative factors and the renormalized coupling u_o' are determined in perturbation theory, to second and third order, respectively.

First we study the case when the temperature is fixed at the critical temperature, i.e. $\kappa^2=0$. The Green's function depends on one variables only. The vertex depends generally on three momenta. Similarly to our earlier calculation the momenta on the four legs will be chosen in a special way so that only one momentum variable is kept. This choice is shown in Fig. 5. The analytic expressions for d and $\tilde{\Gamma}$ have been given in eqs. /2.69/, /2.7o/ for $\kappa^2=0$. Using these analytic forms, the multiplicative factors and the new coupling can be determined in a self-consistent way from eqs. /3.4/ - /3.6/.

$$z_d = 1 + \frac{n+2}{144} u_0^2 K_d^2 \ln \frac{\Lambda'^2}{\Lambda^2} + \dots \qquad /3.16/$$

$$z_\Gamma^{-1} = 1 + \frac{n+8}{12} u_0 K_d \frac{2}{\varepsilon} \left[1 - \left(\frac{\Lambda'^2}{\Lambda^2} \right)^{-\varepsilon/2} \right]$$

$$+ \left(\frac{n+8}{12} \right)^2 u_0^2 K_d^2 \ln^2 \frac{\Lambda'^2}{\Lambda^2} - \frac{5n+22}{36} u_0^2 K_d^2 \ln \frac{\Lambda'^2}{\Lambda^2} + \dots \qquad /3.17/$$

and

$$u_0' = u_0 \left(\frac{\Lambda'^2}{\Lambda^2} \right)^{-\varepsilon/2} \left\{ 1 + \frac{n+8}{12} u_0 K_d \frac{2}{\varepsilon} \left[1 - \left(\frac{\Lambda'^2}{\Lambda^2} \right)^{-\varepsilon/2} \right] \right.$$

$$\left. + \left(\frac{n+8}{12} \right)^2 u_0^2 K_d^2 \ln^2 \frac{\Lambda'^2}{\Lambda^2} - \frac{9n+42}{72} u_0^2 K_d^2 \ln \frac{\Lambda'^2}{\Lambda^2} + \dots \right. \qquad /3.18/$$

On the other hand when all the momenta are fixed at q = 0 and the only variable is κ^2/Λ^2, the integration can again be performed /see the Appendix/ and we get

$$d = 1 + \frac{n+2}{144} u_0^2 K_d^2 \ln \frac{\kappa^2}{\Lambda^2} + \dots \qquad /3.19/$$

$$\tilde{\Gamma} = 1 + \frac{n+8}{12} u_0 K_d \left(\frac{\kappa^2}{\Lambda^2} \right)^{-\varepsilon/2} \left[\frac{2}{\varepsilon} \left(\left(\frac{\kappa^2}{\Lambda^2} \right)^{\varepsilon/2} - 1 \right) + 1 + \dots \right]$$

$$+ \frac{n^2+6n+20}{144} u_0^2 K_d^2 \left(\frac{\kappa^2}{\Lambda^2} \right)^{-\varepsilon} \left[\frac{2}{\varepsilon} \left(\left(\frac{\kappa^2}{\Lambda^2} \right)^{\varepsilon/2} - 1 \right] + 1 + \dots \right]^2$$

$$+ \frac{5n+22}{36} u_0^2 K_d^2 \left(\frac{\kappa^2}{\Lambda^2} \right)^{-\varepsilon} \left[\frac{1}{2} \ln^2 \frac{\kappa^2}{\Lambda^2} + \dots \right] + \dots \qquad /3.20/$$

Although eqs. /2.70/ and /3.20/ are not symmetric in q^2 and κ^2, the self-consistent solution of eqs. /3.4/ - /3.6/ with these analytic forms for d and $\tilde{\Gamma}$ produces the same expressions for z_d, z_Γ and u_0' as eqs. /3.16/ - /3.18/. This shows that at least in these two special cases when $q^2 = 0$ or $\kappa^2 = 0$, the z factors and the new coupling are the same. This finding is generalized in eqs. /3.4/ - /3.6/ as the basic equations of the new method.

The perturbational expansion of the invariant coupling as given in eq. /3.18/ is a good approximation for a small change of the cut-off, i.e. for $\Lambda'^2/\Lambda^2 \sim 1$. In studying the critical phenomena we have to determine the Green's function and other quantities for $q^2/\Lambda^2 \ll 1$ or $\kappa^2/\Lambda^2 \ll 1$ and therefore we need the invariant coupling for $\Lambda'^2/\Lambda^2 \ll 1$, i.e.

when almost all degrees of freedom are eliminated. Using eq. /3.18/ as the perturbational expansion of u_R, the Lie equation for the invariant coupling /3.12/ takes the form

$$\frac{\partial u_R(s,u_0)}{\partial s} = -\frac{u_R(s,u_0)}{s}\left\{-\frac{\varepsilon}{2} + \frac{n+8}{12}u_R(s,u_0)V_d - \frac{3n+14}{72}u_R^2(s,u_0)V_d^2 + \dots\right\}, \qquad /3.21/$$

where $s = \Lambda'^2/\Lambda^2$. The solution of this equation for $s \ll 1$ is

$$V_d u_R(s,u_0) = V_d u^* + A s^{\omega/2} + \dots, \qquad /3.22/$$

where

$$V_d u^* = \frac{6}{n+8}\varepsilon\left\{1 + 3\varepsilon\frac{3n+14}{(n+8)^2}\right\} + O(\varepsilon^3) \qquad /3.23/$$

and

$$\omega = \varepsilon - 3\frac{3n+14}{(n+8)^2}\varepsilon^2 + O(\varepsilon^3) \qquad /3.24/$$

The fixed point coupling u^* is of the order of ε and this confirms a posteriori that the perturbational calculation is adequate to calculate the right hand side of the Lie equations. It is also interesting to notice that this fixed point value coincides with the fixed point value given by eq. /2.79/ for that case of the conventional renormalization scheme where the Green's function and vertex are normalized to their values at the cut-off by assuming the normalization condition given by eqs. /2.35/-/2.36/. Since u^* is small /we are not going to extend this method for three dimensional systems/ the right hand side of the Lie equation for the Green's function can similarly be calculated in powers of the invariant coupling and the critical exponents can be obtained in the ε expansion.

IV. Calculation of the critical exponents η, ν and γ and the anomalous dimensions d_φ, d_{φ^2} and $d_{\varphi\varphi}$

4.1 Renormalization of the Green's function d

Knowing the behaviour of the invariant coupling as a function of Λ'^2/Λ^2, we can start now to evaluate the behaviour of the Green's function and vertex as a function of the momentum q or the inverse coherence length κ.

First we study the behaviour of the Green's function. We will restrict ourselves to the one-variable case and investigate the momentum dependence $(\kappa^2 = 0)$ and temperature dependence $(q^2 = 0)$ separately. In doing so the right hand sides of the Lie equations /3.13/ and /3.14/ simplify considerably, namely apart from the factors $1/x$ or $1/y$, respectively, the x or y dependence appears only through the invariant coupling.

$$\frac{\partial \ln d(x,u_0)}{\partial x} = \frac{1}{x}\Psi_1(u_R(x,u_0)), \qquad /4.1/$$

$$\frac{\partial \ln d(y,u_0)}{\partial y} = \frac{1}{y}\Psi_2(u_R(y,u_0)), \qquad /4.2/$$

where the generators Ψ_1, and Ψ_2 are given as

$$\Psi_1(u_R(x,u_0)) = \frac{\partial}{\partial \xi} \ln d(\xi, 0, u_R(x,u_0))\Big|_{\xi=1}, \qquad /4.3/$$

$$\Psi_2(u_R(x,u_0)) = \frac{\partial}{\partial \eta} \ln d(0, \eta, u_R(y,u_0))\Big|_{\eta=1}. \qquad /4.4/$$

Since the critical exponents η and γ are defined through the leading terms in the x and y dependence of d for $x \ll 1$ and $y \ll 1$ they can be obtained by replacing the invariant coupling on the right hand side of the Lie equation by its value at $x=0$ or $y=0$, i.e. by the fixed point coupling. Simple integration of these equations gives

$$d(x) \sim x^{\Psi_1(u_o^*)} \sim \left(\frac{q}{\Lambda}\right)^{2\Psi_1(u_o^*)} \quad \text{if} \quad x \ll 1 \qquad /4.5/$$

$$d(y) \sim y^{\Psi_2(u_o^*)} \sim \left(\frac{\kappa}{\Lambda}\right)^{2\Psi_2(u_o^*)} \quad \text{if} \quad y \ll 1 \qquad /4.6/$$

Since u_o^* is of the order of ε, the perturbational expansion of the generators in powers of the invariant coupling will automatically yield an expansion of the critical exponents in powers of ε. The function d is symmetric in the variables q^2 and κ^2, at least to second order in u_o as seen from eqs. /2.69/ and /3.19/ and we get from eqs. /4.3/ and /4.4/

$$\Psi_1(u_o^*) = \Psi_2(u_o^*) = \frac{n+2}{144} u_o^{*2} K_d^2 = \frac{n+2}{4(n+8)^2} \varepsilon^2 + O(\varepsilon^3) \qquad /4.7/$$

From the definition of the exponent η

$$G(q, \kappa=0) \sim \frac{1}{q^{2-\eta}} \qquad /4.8/$$

and therefore

$$d(q^2) = G(q^2)/G^{(o)}(q^2) \sim (q^2)^{\eta/2} \qquad /4.9/$$

Comparison with eq. /4.5/ and /4.7/ gives

$$2\Psi_1(u_o^*) = 2\Psi_2(u_o^*) = \eta . \qquad /4.10/$$

In order to determine the exponent η to order ε^3 the perturbational expansion of d to order u_o^3 has to be taken. This is given for $\kappa^2=0$ in eq. /2.69/. The Lie equation has the same from as eq. /2.81/ and the generator Ψ_1 is

$$\Psi_1(u_R) = \frac{n+2}{144} u_R^2 K_d^2 \left(1 + \frac{9}{4}\varepsilon\right) - \frac{5(n+2)(n+8)}{12^3} u_R^3 K_d^3 + \dots \qquad /4.11/$$

Inserting the fixed point coupling the exponent η is obtained correctly, as in eq. /2.83/

4.2 Temperature dependence of the coherence length and the susceptibility

Analogously to eqs. /4.8/ and /4.9/, the κ^2 dependence of the Green's function is

$$d(\kappa^2) \sim (\kappa^2)^{\eta/2} \qquad /4.12/$$

and

$$G(q^2=0, \kappa^2) \sim \frac{1}{\kappa^2}(\kappa^2)^{\eta/2} \sim \frac{1}{\kappa^{2-\eta}}, \qquad /4.13/$$

where eqs. /4.6/ and /4.1o/ have been used. The susceptibility χ is related to the Green's function by

$$\chi(T-T_c) \sim G(q^2=0, T-T_c), \qquad /4.14/$$

and the exponent γ is defined by

$$\chi(T-T_c) \sim (T-T_c)^{-\gamma} \qquad /4.15/$$

The inverse coherence length κ has been introduced as a measure of the temperature difference $T-T_c$ and

$$\kappa = \xi^{-1} \sim (T-T_c)^{\nu} \qquad /4.16/$$

and this is the definition of the exponent ν. Comparing these equations the scaling law

$$\gamma = (2-\eta)\nu \qquad /4.17/$$

follows immediately at least up to second order in ε.

We still have to determine the ε-expansion of ν or γ. The inverse coherence length κ has been introduced by relations /2.16/ or /2.18/ with

$$\delta m^2 = r_o - \kappa^2 \qquad /4.18/$$

where r_o is linear in the temperature. At the critical temperature /at $r_o = r_{oc}$ / $\kappa^2 = 0$ and the Green's function is divergent at $q^2 = 0$. Therefore r_{oc} can be defined by

$$\Sigma'(0,0) + r_{oc} = 0 . \qquad /4.19/$$

Combining eqs. /2.18/ and /4.19/ we get

$$\kappa^2 - \Sigma'(-\kappa^2, \kappa^2) + \Sigma'(0,0) = r_o - r_{oc} \sim T - T_c . \qquad /4.2o/$$

We have to keep in mind that the prime on Σ means that the appropriate subtraction procedure has to be performed step by step in the perturbational calculation of the self-energy.

Using eq. /4.16/ we write eq, /4.2o/ in the form

$$\kappa^2 - \Sigma'(-\kappa^2, \kappa^2) + \Sigma'(0,0) \sim (\kappa^2)^{\frac{1}{2\nu}} \qquad /4.21/$$

or

$$F(\kappa^2) = 1 - \frac{\Sigma'(-\kappa^2, \kappa^2) - \Sigma'(0,0)}{\kappa^2} \sim (\kappa^2)^{\frac{1}{2\nu} - 1} . \qquad /4.22/$$

This new function $F(\kappa^2)$ is multiplicatively renormalizable in the sense of eq. /3.2o/ and the Lie equation has the form

$$\frac{\partial \ln F(y, u_o)}{\partial y} = \frac{1}{y} \Psi_3 (u_R(y, u_o)), \qquad /4.23/$$

with $y = \kappa^2/\Lambda^2$ and

$$\Psi_3(u_R(y,u_0)) = \frac{\partial}{\partial\eta} \ln F(\eta, u_R(y,u_0))|_{\eta=1} . \qquad /4.24/$$

The asymptotic behaviour for $y \ll 1$ is again governed by the fixed point coupling and

$$F(\kappa^2) \sim (\kappa^2)^{\Psi_3(u_0^*)} \qquad \text{if} \qquad \kappa^2/\Lambda^2 \ll 1 \qquad /4.25/$$

The self-energy has been calculated to second order in u_0 for $q^2 = -\kappa^2$ with the result /see Appendix/

$$\Sigma'(-\kappa^2,\kappa^2) - \Sigma'(0,0) = \frac{n+2}{12} u_0 K_d \kappa^2 \ln\frac{\kappa^2}{\Lambda^2}$$
$$- \frac{n+2}{18} u_0^2 K_d^2 \kappa^2 \left(-\frac{3}{8}\ln^2\frac{\kappa^2}{\Lambda^2} + \frac{5}{8}\ln\frac{\kappa^2}{\Lambda^2} + \dots\right) + \dots \qquad /4.26/$$

Making use of this expression

$$\Psi_3(u_R(y,u_0)) = -\frac{n+2}{12} u_R(y,u_0) K_d + \frac{5(n+2)}{144} u_R^2(y,u_0) K_d^2 + \dots \qquad /4.27/$$

and

$$\Psi_3(u_0^*) = -\frac{n+2}{2(n+8)}\varepsilon - \frac{(n+2)(13n+44)}{4(n+8)^3}\varepsilon^2 + o(\varepsilon^3) \qquad /4.28/$$

Since ν is related to $\Psi_2(u_0^*)$ via eqs. /4.22/ and /4.25/

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)}\varepsilon + \frac{(n+2)(n^2+23n+60)}{8(n+8)^3}\varepsilon^2 + o(\varepsilon^3) \qquad /4.29/$$

Using the scaling law in eq. /4.17/ the ε-expansion of γ is obtained

$$\gamma = 1 + \frac{n+2}{2(n+8)}\varepsilon + \frac{(n+2)(n^2+22n+52)}{4(n+8)^3}\varepsilon^2 + o(\varepsilon^3) \qquad /4.30/$$

The integration of the Lie equation can be performed not only in those cases where the invariant coupling in the generators Ψ_i is approximated by the fixed point coupling. Using the form given in eq. /3.22/ which contains also the first correction when $\Lambda'^2/\Lambda^2 \ll 1$, simple integration gives

$$d(q^2) \sim q^{\eta} \left(1 + B q^{\omega} + \ldots\right) \qquad /4.31/$$

$$\kappa^2 = \xi^{-2} \sim t^{2\nu} \left(1 + C t^{\omega} + \ldots\right) \qquad /4.32/$$

$$\chi(\kappa^2) \sim \kappa^{-2+\eta} \left(1 + D \kappa^{\omega} + \ldots\right) \qquad /4.33/$$

or

$$\chi(t) \sim t^{-\gamma} \left(1 + E t^{\omega\nu} + \ldots\right) \qquad /4.34/$$

where $t = r_0 - r_{0c} \sim T - T_c$. These expressions show that the exponent ω characterizes the first corrections to the scaling behaviour.

4.3 Anomalous dimensions

Until now the behaviour of the Green's function has been considered. In a similar fashion the momentum dependence of the vertex can also be studied. The Lie equation for $\tilde{\Gamma}$ at $\kappa^2 = 0$ has the form

$$\frac{\partial \ln \tilde{\Gamma}(x, u_0)}{\partial x} = \frac{1}{x} \Psi_4\left(u_R(x, u_0)\right), \qquad /4.35/$$

with

$$\Psi_4\left(u_R(x, u_0)\right) = \frac{\partial}{\partial \xi} \ln \tilde{\Gamma}\left(\xi, 0, u_R(x, u_0)\right)\Big|_{\xi=1} \qquad /4.36/$$

The asymptotic behaviour of $\tilde{\Gamma}$ for $x \ll 1$ is

$$\tilde{\Gamma}(x) \sim x^{\Psi_4(u_0^*)} \sim (q^2)^{\Psi_4(u_0^*)} \qquad /4.37/$$

Using the analytic form of the perturbational expansion of the vertex given in eq. /2.7o/ we easily get

$$\Psi_4(u_R(x,u_0)) = \left\{ \frac{n+8}{12} u_R K_d \left(1+\frac{\varepsilon}{2}\right) - \frac{n^2+26n+108}{72} u_R^2 K_d^2 + \dots \right\} \qquad /4.38/$$

$$\cdot \left\{ 1 + \frac{n+8}{12} u_R K_d (-1-\varepsilon) + \dots \right\}^{-1}$$

and

$$\Psi_4(u_0^*) = \frac{\varepsilon}{2} - \frac{n+2}{2(n+8)^2}\varepsilon^2 + o(\varepsilon^3) \qquad /4.39/$$

This exponent is very closely related to the anomalous dimension d_φ , which can be defined /see Ref. [16]/through the relation

$$\tilde{\Gamma}(q^2) \sim q^{d-4d_\varphi} \qquad \text{if} \quad q^2/\Lambda^2 \ll 1 \qquad /4.4o/$$

From these relations we get $d - 4d_\varphi = 2\Psi_4(u_0^*)$ and

$$2d_\varphi = 2 - \varepsilon + \frac{n+2}{2(n+8)^2}\varepsilon^2 + o(\varepsilon^3) \qquad /4.41/$$

The scaling law $2d_\varphi = d - 2 + \eta$ is indeed satisfied.

Other, higher order vertices can also be studied and the anomalous dimension of higher order fields are easily calculated in this method. One of the authors [27] has applied multiplicative renormalization and the Lie equation to determine the anomalous dimension d_{φ^2} of the field $\varphi^2(x)$ to order ε . He has studied the function $\Gamma^{(2)}_{\varphi^2}(x,y) = \langle \varphi^2(x)\varphi^2(y)\rangle$ .Here we present another calculation to determine the anomalous dimension d_{φ^2} to order ε^2 from the behaviour of the one-particle irreducible part of

$$\Gamma^{(1,2)}_{\alpha\beta_1\beta_2}(x, x_1, x_2) = \langle \varphi^2_\alpha(x)\, \varphi_{\beta_1}(x_1)\, \varphi_{\beta_2}(x_2) \rangle_{\text{one particle irreducible}} \qquad /4.42/$$

The Fourier transform of this Green's function is

$$\Gamma^{(1,2)}_{\alpha\beta_1\beta_2}(q, p_1, p_2, u_0) = \int e^{i(qx + p_1x_1 + p_2x_2)} \langle \varphi^2_\alpha(x)\, \varphi_{\beta_1}(x_1)\, \varphi_{\beta_2}(x_2) \rangle\, d^dx\, d^dx_1\, d^dx_2$$

$$= \int \frac{d^dk}{(2\pi)^d} \langle \varphi_{k\alpha}\, \varphi_{q-k\,\alpha}\, \varphi_{p_1\beta_1}\, \varphi_{p_2\beta_2} \rangle \qquad /4.43/$$

This function is considered as an amputed Green's function, i.e. the external lines which are shown on the diagrammatic representation in Fig. 6. are not taken into account in the analytic expression. The diagrams are thus the same up to second order in the coupling as for the four-point vertex, only the geometrical factors are different.

From the structure of this function

$$\Gamma^{(1,2)}_{\alpha\beta_1\beta_2}(q, p_1, p_2, u_0) = \delta_{\beta_1\beta_2}\, \delta_{q, p_1+p_2}\, \Gamma^{(1,2)}(q, p_1, q-p_1, u_0) \qquad /4.44/$$

and for the special choice of $p_1 = 0$ the analytic expression is

$$\Gamma^{(1,2)}(q, 0, q, u_0) \doteq 1 + \frac{n+2}{12} u_0 K_d \left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon/2} \left[\frac{2}{\varepsilon}\left(\left(\frac{q^2}{\Lambda^2}\right)^{\varepsilon/2} - 1\right) - 1 + \dots\right]$$

$$+ \left(\frac{n+2}{12}\right)^2 u_0^2 K_d^2 \left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon} \left[\frac{2}{\varepsilon}\left(\left(\frac{q^2}{\Lambda^2}\right)^{\varepsilon/2} - 1\right) - 1 + \dots\right]^2 \qquad /4.45/$$

$$+ \frac{n+2}{24} u_0^2 K_d^2 \left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon} \left[\frac{1}{2}\ln^2\frac{q^2}{\Lambda^2} - 2\ln\frac{q^2}{\Lambda^2} + 2 + \dots\right] + \dots$$

It follows from the definition of the anomalous dimension d_{φ^2} [17], that for $q^2/\Lambda^2 \ll 1$

$$\Gamma^{(1,2)}(q, 0, q, u_0) \sim q^{-2d_\varphi + d_{\varphi^2}} \qquad /4.46/$$

This small momentum behaviour of $\Gamma^{(1,2)}$ can be obtained by using the Lie equation, since this function is multiplicatively renormalizable for which

$$\frac{\partial \ln \Gamma^{(1,2)}(x,u_0)}{\partial x} = \frac{1}{x}\Psi_s\left(u_R(x,u_0)\right), \qquad /4.47/$$

with

$$\Psi_s\left(u_R(x,u_0)\right) = \frac{\partial}{\partial \xi} \ln \Gamma^{(1,2)}\left(\xi,\, u_R(x,u_0)\right)_{\xi=1}, \qquad /4.48/$$

and the notation $\Gamma^{(1,2)}(x,u_0) = \Gamma^{(1,2)}(x,0,x,u_0)$ has been used with $x = q^2/\Lambda^2$. The asymptotic behaviour of $\Gamma^{(1,2)}(x,u_0)$ for $x \ll 1$ is easily obtained as

$$\Gamma^{(1,2)}(x,u_0) \sim x^{\Psi_s(u_0^*)} \sim (q^2)^{\Psi_s(u_0^*)} \qquad /4.49/$$

Using the perturbational expression in eq. /4.45/ the generator Ψ_s reads

$$\Psi_s(u_R) = \left\{\frac{n+2}{12}u_R K_d\left(1+\frac{\varepsilon}{2}\right) - 2\left(\frac{n+2}{12}\right)^2 u_R^2 K_d^2 - \frac{n+2}{12}u_R^2 K_d^2 + \dots\right\}$$
$$\times\left\{1 - \frac{n+2}{12}u_R K_d(1+\varepsilon) + \dots\right\}^{-1} \qquad /4.50/$$

Inserting the fixed point value of eq. /3.23/ we have

$$\Psi_s(u_0^*) = \frac{n+2}{2(n+8)}\varepsilon\left[1 + \frac{6(n+3)}{(n+8)^2}\varepsilon\right] + o(\varepsilon^3) \qquad /4.51/$$

The anomalous dimension d_{φ^2} can be calculated from the relation

$$d_{\varphi^2} = 2d_\varphi + 2\Psi_s(u_0^*) \qquad /4.52/$$

which is a consequence of eqs. /4.46/ and /4.49/.

Another exponent which is of interest and is related to an anomalous dimension is the cross-over exponent ϕ. Yamazaki and Suzuki [28] have shown that ϕ can be expressed in terms of the anomalous dimension $d_{\varphi\varphi}$ of the field $\varphi(x)\varphi(y)$ as

$$\phi = \nu(d - d_{\varphi\varphi}) \qquad /4.53/$$

$d_{\varphi\varphi}$ is in turn obtained from the momentum dependence of the Fourier transform of the one-particle irreducible part of the Green's function

$$\Gamma^{\shortparallel\,(n)}_{\alpha\beta\gamma_1\ldots\gamma_n}(x,y,x_1,\ldots x_n) = \langle \varphi_\alpha(x)\varphi_\beta(y)\varphi_{\gamma_1}(x_1)\ldots\varphi_{\gamma_n}(x_n)\rangle \quad \alpha\neq\beta \qquad /4.54/$$

The special case $x=y$, $n=2$ will be considered, in which case

$$\Gamma^{\shortparallel\,(2)}_{\alpha\beta\gamma_1\gamma_2}(q,p,u_0) = \int\frac{d^dk}{(2\pi)^d}\langle\varphi_{k\alpha}\;\varphi_{q-k\beta}\;\varphi_{p\gamma_1}\;\varphi_{q-p\,\gamma_2}\rangle \quad \alpha\neq\beta \qquad /4.55/$$

The essential difference with respect to $\Gamma^{(1,2)}$ is that here $\alpha\neq\beta$. Similarly as there, the amputed Green's function is studied. The diagrammatic representation is the same as for $\Gamma^{(1,2)}$ with different geometrical factors. The spin structure of the function is

$$\Gamma^{\shortparallel\,(2)}_{\alpha\beta\gamma_1\gamma_2}(q,p,u_0) = (\delta_{\alpha\gamma_1}\delta_{\beta\gamma_2}+\delta_{\alpha\gamma_2}\delta_{\beta\gamma_1})\,\Gamma^{\shortparallel\,(2)}(q,p,u_0), \qquad /4.56/$$

and again for $p=0$ we get

$$\begin{aligned}\Gamma^{\shortparallel\,(2)}(q,0,u_0) = 1 &+ \frac{1}{6}u_0K_d\left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon/2}\left[\frac{2}{\varepsilon}\left(\left(\frac{q^2}{\Lambda^2}\right)^{\varepsilon/2}-1\right)-1+\ldots\right]\\ &+\frac{1}{36}u_0^2K_d^2\left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon}\left[\frac{2}{\varepsilon}\left(\left(\frac{q^2}{\Lambda^2}\right)^{\varepsilon/2}-1\right)-1+\ldots\right]^2\\ &+\frac{n+4}{72}u_0^2K_d^2\left(\frac{q^2}{\Lambda^2}\right)^{-\varepsilon}\left[\frac{1}{2}\ln^2\frac{q^2}{\Lambda^2}-2\ln\frac{q^2}{\Lambda^2}+2+\ldots\right]+\ldots\end{aligned} \qquad /4.57/$$

This function is multiplicatively renormalizable and the Lie equation has the form

$$\frac{\partial\ln\Gamma^{\shortparallel\,(2)}(x,u_0)}{\partial x} = \frac{1}{x}\Psi_6(u_R(x,u_0)) \qquad /4.58/$$

where

$$\Psi_6(u_R) = \frac{\partial}{\partial \xi} \ln \Gamma''^{(2)}(\xi, u_R(x, u_0))\Big|_{\xi=1} \qquad /4.59/$$

and the notation $\Gamma''^{(2)}(x, u_0) = \Gamma''^{(2)}(x, 0, u_0)$ has been used with $x = q^2/\Lambda^2$.

The perturbational expression of $\Gamma''^{(2)}(x, u_0)$ in eq. /4.57/ leads to the following expression for the generator Ψ_6

$$\Psi_6(u_R) = \left\{ \frac{1}{6} u_R K_d \left(1 + \frac{\varepsilon}{2}\right) - \frac{1}{18} u_R^2 K_d^2 - \frac{n+4}{36} u_R^2 K_d^2 + \dots \right\} \cdot \left\{ 1 - \frac{1}{6} u_R K_d + \dots \right\}^{-1} \qquad /4.60/$$

or

$$\Psi_6(u_0^*) = \frac{1}{n+8} \varepsilon \left[1 - \frac{n^2 - 4n - 36}{2(n+8)^2} \varepsilon \right] + o(\varepsilon^3) \qquad /4.61/$$

The small momentum behaviour of $\Gamma''^{(2)}(x, u_0)$ is given as

$$\Gamma''^{(2)}(x) \sim x^{\Psi_6(u_0^*)} \sim (q^2)^{\Psi_6(u_0^*)} \qquad /4.62/$$

On the other hand the anomalous dimension $d_{\varphi\varphi}$ is defined [28] by the relation

$$\Gamma''^{(2)}(q, 0, u_0) \sim q^{-2d_\varphi + d_{\varphi\varphi}} \quad \text{if} \quad q^2/\Lambda^2 \ll 1 \qquad /4.63/$$

Comparison of these last two equations gives

$$d_{\varphi\varphi} = 2\Psi_6(u_0^*) + 2 d_\varphi \qquad /4.64/$$

Using our earlier result for d_φ /eq./4.41//, the anomalous dimension $d_{\varphi\varphi}$ is

$$d_{\varphi\varphi} = 2 - \frac{n+6}{n+8} \varepsilon - \frac{(n+4)(n-22)}{2(n+8)^3} \varepsilon^2 + o(\varepsilon^3), \qquad /4.65/$$

and the cross-over exponent is

$$\phi = \nu(d - d_{\varphi\varphi}) = 1 + \frac{n}{2(n+8)}\varepsilon + \frac{n(n^2+24n+68)}{4(n+8)^3}\varepsilon^2 + O(\varepsilon^3) \qquad /4.66/$$

in agreement with other calculations [19] .

V. Determination of the specific heat exponent α

The Gell-Mann-Low multiplicative renormalization is a very suitable tool to investigate the behaviour of the Green's function and vertices, because the renormalization transfo· ·ation is performed on these quantities. Its applicability to determine thermodynamical quantities, like the specific heat, is less straightforward. If the specific heat is expressed in terms of Green's functions we can eventually hope to get multiplicative renormalization for it and then the machinery of the Lie equations could be used to calculate the exponent α.

Larkin and Khmelnitskii [29] have shown that the leading singularity in the specific heat C_V can be obtained by studying the temperature dependence of the density-density correlation function. The thermodynamical relation defining C_V is

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V . \qquad /5.1/$$

Writing the free energy F in the usual form

$$F = -T \ln \mathrm{Tr}\left(e^{-H/T}\right) \qquad /5.2/$$

it is seen that the temperature dependence comes from two sources; an explicit dependence on T and the temperature dependence of the parameters of the Hamiltonian /3.1/. The leading singularity comes from this latter dependence. Since τ_0 in eq. /3.1/ is linear in T, the singular part of the heat capacity can be written as

$$C_v \sim \frac{\partial^2 F}{\partial T^2} \sim \frac{\partial^2 F}{\partial \tau_0^2} \sim \frac{\partial^2 F}{\partial (G^{(0)^{-1}})^2} \sim \int d^d x \, d^d y \, \langle \varphi^2(x) \varphi^2(y) \rangle , \qquad /5.3/$$

where only the most singular terms have been kept.

The free energy can be represented by connected, closed diagrams. Differentiation of the free energy with respect to the inverse of the Green's function means to introduce vertex points with one incoming and one outgoing lines without momentum transfer and therefore C_v can be obtained from the contribution of the diagrams in Fig. 7. There is no mass renormalization yet and that is the reason why in the second diagram there is a Hartree-loop on the bubble. We can perform the mass renormalization now, introducing $\varkappa^2$ instead of τ_0. This leads to the cancellation of the contribution of the second diagram /it is incorporated into $\varkappa^2$/.[x] Finally up to second order in u_0 the diagrams shown in Fig. 8. have to be taken into account.

The sum of these polarization diagram contributions is denoted by $\Pi(\varkappa^2, \Lambda^2, u_0)$. Since multiplicative renormali-

[x] To make the mass renormalization in the free energy diagrams before differentiation would be incorrect, because the free energy diagrams have a prefactor depending on the order of the diagram.

zability in the sense of eq. /3.2o/ does not follow from general arguments, it has to be checked whether $\Pi(\kappa^2, \Lambda^2, u_0)$ satisfies such scaling equation or not. The analytic contribution of the diagrams in Fig. 8. have been calculated with the result

$$
\begin{aligned}
\Pi(\kappa^2, \Lambda^2, u_0) = {} & n K_d (\kappa^2)^{-\varepsilon/2} \left(- f\left(\frac{\kappa^2}{\Lambda^2}\right) - 1 + \right) \\
& - \frac{n(n+2)}{12} u_0 K_d^2 (\kappa^2)^{-\varepsilon/2} \left(\frac{\kappa^2}{\Lambda^2}\right)^{-\varepsilon/2} \left(- f\left(\frac{\kappa^2}{\Lambda^2}\right) - 1 + \right)^2 \\
& + \frac{n(n+2)^2}{144} u_0^2 K_d^3 (\kappa^2)^{-\varepsilon/2} \left(\frac{\kappa^2}{\Lambda^2}\right)^{-\varepsilon} \left(- f\left(\frac{\kappa^2}{\Lambda^2}\right) - 1 + \right)^3 \\
& + \frac{n(n+2)}{144} u_0^2 K_d^3 (\kappa^2)^{-\varepsilon/2} \left(\frac{\kappa^2}{\Lambda^2}\right)^{-\varepsilon} \left(- \ln^2 \frac{\kappa^2}{\Lambda^2} + \dots \right) \\
& + \frac{n(n+2)}{48} u_0^2 K_d^3 (\kappa^2)^{-\varepsilon/2} \left(\frac{\kappa^2}{\Lambda^2}\right)^{-\varepsilon} \left(- \frac{2}{3} \ln^3 \frac{\kappa^2}{\Lambda^2} + \dots \right) + \dots
\end{aligned}
\qquad /5.4/
$$

where $\quad f(x) = \frac{2}{\varepsilon}\left(x^{\varepsilon/2} - 1\right)$.

The term linear in the logarithm have not been calculated for the last two terms since they are irrelevant if the specific heat exponent α is determined to order ε^2.

Making use of the perturbational expansion of u_0' as a function of Λ'^2/Λ^2 given in eq. /3.9/ a simple calculation shows that eq. /3.2o/ cannot be satisfied with a multiplicative factor z independent of κ^2. Similar situation has already been encountered in studying other physical problems like x-ray absorption [23] and one dimensional Fermi models [24]. These systems are multiplicatively renormalizable by scaling the cut-off similarly as the model studied here, the response functions, however are not multiplicatively renormalizable. We have learned on these examples that an auxiliary quantity can be introduced for these response functions which is already multiplicatively renormalizable and for which the Lie equation can be used. In these examples the zeroth order contribution was a simple logarithmic function and the auxiliary quantity was obtained by differentiating the response function with respect to this logarithm.

A straightforward generalization of this procedure suggests to introduce

$$\bar{\Pi}\left(\frac{\kappa^2}{\Lambda^2}, u_0\right) = \frac{\partial \Pi(\kappa^2, \Lambda^2, u_0)}{\partial \Pi^{(0)}(\kappa^2, \Lambda^2)} \qquad /5.5/$$

where

$$\Pi^{(0)}(\kappa^2, \Lambda^2) = n K_d (\kappa^2)^{-\varepsilon/2} \left(-f\left(\frac{\kappa^2}{\Lambda^2}\right) - 1 + \ldots\right)$$
$$= -n K_d (\kappa^2)^{-\varepsilon/2} \left(\ln\frac{\kappa^2}{\Lambda^2} + 1 + \ldots\right). \qquad /5.6/$$

The perturbational expansion of this auxiliary quantity is

$$\begin{aligned}\bar{\Pi}\left(\frac{\kappa^2}{\Lambda^2}, u_0\right) = 1 &+ \frac{n+2}{6} u_0 K_d \left(\frac{\kappa^2}{\Lambda^2}\right)^{-\varepsilon/2} \left(f\left(\frac{\kappa^2}{\Lambda^2}\right) + 1 + \ldots\right) \\ &+ \frac{(n+2)^2}{48} u_0^2 K_d^2 \left(\frac{\kappa^2}{\Lambda^2}\right)^{-\varepsilon} \left(f\left(\frac{\kappa^2}{\Lambda^2}\right) + 1 + \ldots\right)^2 \\ &+ \frac{n+2}{144} u_0^2 K_d^2 \left(\frac{\kappa^2}{\Lambda^2}\right)^{-\varepsilon} \left(2\ln\frac{\kappa^2}{\Lambda^2} + \ldots\right) \qquad /5.7/ \\ &+ \frac{n+2}{48} u_0^2 K_d^2 \left(\frac{\kappa^2}{\Lambda^2}\right)^{-\varepsilon} \left(2\ln^2\frac{\kappa^2}{\Lambda^2} + \ldots\right) + \ldots\end{aligned}$$

It is easily checked that this quantity up to this order obeys the relation

$$\bar{\Pi}\left(\frac{\kappa^2}{\Lambda'^2}, u_0'\right) = Z\left(\frac{\Lambda'^2}{\Lambda^2}, u_0\right) \bar{\Pi}\left(\frac{\kappa^2}{\Lambda^2}, u_0\right). \qquad /5.8/$$

Supposing that this equation is valid in higher orders as well, the Lie equation reads

$$\frac{\partial \ln \bar{\Pi}(y, u_0)}{\partial y} = \frac{1}{y} \Psi_7 (u_R(y, u_0)), \qquad /5.9/$$

where

$$\Psi_7\left(u_R(y,u_0)\right) = \frac{\partial}{\partial \eta} \ln \overline{\Pi}\left(\eta, u_R(y,u_0)\right)_{\eta=1} . \qquad /5.10/$$

The solution of this equation for small y is obtained by inserting eq. /3.22/ for u_R

$$\overline{\Pi}(y) \sim y^{\Psi_7(u_0^*)}\left(1 + F y^{\omega/2} + \dots\right). \qquad /5.11/$$

The exponent $\Psi_7(u_0^*)$ itself is obtained by using eq. /5.7/ to calculate $\Psi_7(u_R)$ from eq. /5.10/

$$\Psi_7(u_R) = \left[\frac{n+2}{6} u_R K_d \left(1-\frac{\varepsilon}{2}\right) + \frac{(n+2)^2}{24} u_R^2 K_d^2 + \frac{n+2}{72} u_R^2 K_d^2 + \dots\right] \cdot \left[1 + \frac{n+2}{6} u_R K_d + \dots\right]^{-1} . \qquad /5.12/$$

and

$$\Psi_7(u_0^*) = \frac{n+2}{n+8}\, \varepsilon \left(1 + \varepsilon \frac{13n+44}{2(n+8)^2}\right) + o(\varepsilon^3). \qquad /5.13/$$

Knowing the behaviour of $\overline{\Pi}(\kappa^2)$, the leading singularity in $\Pi(\kappa^2)$ is obtained by taking into account that $\Pi^{(0)}$ contains a factor $(\kappa^2)^{-\varepsilon/2}$ and therefore

$$\Pi(\kappa^2, \Lambda^2, u_0) \sim (\kappa^2)^{\Psi_7(u_0^*) - \varepsilon/2}\left(1 + F\left(\frac{\kappa^2}{\Lambda^2}\right)^{\frac{\omega}{2}} + \dots\right). \qquad /5.14/$$

Π is proportional to the specific heat but before getting the exponent α we have to reexpress κ^2 in term of $t \sim T - T_c$ by using eq. /4.32/. Finally we get

$$C_v(T - T_c) \sim t^{-\alpha}\left(1 + F' t^{\omega\nu} + \dots\right) \qquad /5.15/$$

where

$$\alpha = -2\nu\left(\Psi_7(u_0^*) - \frac{\varepsilon}{2}\right) = -\frac{n-4}{2(n+8)}\,\varepsilon - \frac{(n+2)^2(n+28)}{4(n+8)^3}\,\varepsilon^2 + o(\varepsilon^3) \qquad /5.16/$$

This is in agreement with earlier calculations for α' and obeys the scaling law

$$2 - \alpha = d\nu . \qquad /5.17/$$

The exponent α is usually calculated by making use of this relation. A direct calculation based on skeleton graph expansion has been carried out by Abrahams and Tsuneto [3o] . This result shows that this renormalization scheme works properly.

It is interesting to see that a similar calculation can be performed for the momentum dependence of the polarization operator. Considering the contribution of the same diagrams as in Fig. 8. as a function of an external momentum q at $\varkappa^2 = 0$ a straightforward calculation gives

$$\begin{aligned}
\Pi(q^2, \Lambda^2, u_0) = & -n K_d (q^2)^{-\varepsilon/2} \Big\{ \Big(f\big(\tfrac{q^2}{\Lambda^2}\big) - 1 + \dots \Big) \\
& + \frac{n+2}{12} u_0 K_d \Big(\frac{q^2}{\Lambda^2}\Big)^{-\varepsilon/2} \Big(f\big(\tfrac{q^2}{\Lambda^2}\big) - 1 + \dots \Big)^2 \\
& + \Big(\frac{n+2}{12}\Big)^2 u_0^2 K_d^2 \Big(\frac{q^2}{\Lambda^2}\Big)^{-\varepsilon} \Big(f\big(\tfrac{q^2}{\Lambda^2}\big) - 1 + \dots \Big)^3 \\
& + \frac{n+2}{144} u_0^2 K_d^2 \Big(\frac{q^2}{\Lambda^2}\Big)^{-\varepsilon} \Big(\ln^2 \frac{q^2}{\Lambda^2} + \dots \Big) \qquad /5.18/ \\
& + \frac{n+2}{48} u_0^2 K_d^2 \Big(\frac{q^2}{\Lambda^2}\Big)^{-\varepsilon} \Big(\frac{2}{3} \ln^3 \frac{q^2}{\Lambda^2} - 4 \ln^2 \frac{q^2}{\Lambda^2} + \dots \Big) + \dots
\end{aligned}$$

In the same way as above it is easily seen that this quantity is not multiplicatively renormalizable under a cut-off scaling, but the derivate of Π with respect to $\Pi^{(0)}$ is already a good quantity for which the Lie equation can be used

$$\begin{aligned}
\overline{\Pi}\Big(\frac{q^2}{\Lambda^2}, u_0\Big) = \frac{\partial \Pi(q^2, \Lambda^2, u_0)}{\partial \Pi^{(0)}(q^2, \Lambda^2)} = & \; 1 + \frac{n+2}{6} u_0 K_d \Big(\frac{q^2}{\Lambda^2}\Big)^{-\varepsilon/2} \Big(f\big(\tfrac{q^2}{\Lambda^2}\big) - 1 + \dots \Big) \\
& + \frac{(n+2)^2}{48} u_0^2 K_d^2 \Big(\frac{q^2}{\Lambda^2}\Big)^{-\varepsilon} \Big(f\big(\tfrac{q^2}{\Lambda^2}\big) - 1 + \dots \Big)^2 \\
& + \frac{n+2}{144} u_0^2 K_d^2 \Big(\frac{q^2}{\Lambda^2}\Big)^{-\varepsilon} \Big(2 \ln \frac{q^2}{\Lambda^2} + \dots \Big) \qquad /5.19/ \\
& + \frac{n+2}{48} u_0^2 K_d^2 \Big(\frac{q^2}{\Lambda^2}\Big)^{-\varepsilon} \Big(2 \ln^2 \frac{q^2}{\Lambda^2} - 8 \ln \frac{q^2}{\Lambda^2} + \dots \Big) + \dots
\end{aligned}$$

The Lie equation for $\overline{\Pi}(q^2)$ has the form

$$\frac{\partial \ln \overline{\Pi}(x)}{\partial x} = \frac{1}{x} \Psi_g(u_R(x, u_0)) \tag{5.20}$$

where

$$\Psi_g(u_R(x,u_0)) = \frac{\partial}{\partial \xi} \ln \overline{\Pi}(\xi, u_R(x, u_0))\Big|_{\xi=1} \tag{5.21}$$

Using the perturbational expansion of $\overline{\Pi}(x)$

$$\Psi_g(u_R) = \left\{ \frac{n+2}{6} u_R K_d \left(1+\frac{\varepsilon}{2}\right) - \frac{(n+2)^2}{24} u_R^2 K_d^2 + \right.$$
$$\left. + \frac{n+2}{72} u_R^2 K_d^2 - \frac{n+2}{6} u_R^2 K_d^2 + \ldots \right\} \left\{ 1 - \frac{n+2}{6} u_R K_d + \ldots \right\}^{-1} \tag{5.22}$$

and

$$\Psi_g(u_0^+) = \frac{n+2}{n+8} \varepsilon \left[1 + \frac{13n+44}{2(n+8)^2} \varepsilon \right] + o(\varepsilon^3) \tag{5.23}$$

This expression is the same as $\Psi_+(u_0^+)$ in eq. /5.12/ showing that in the asymptotic region $q^2/\Lambda^2 \ll 1$ and $\kappa^2/\Lambda^2 \ll 1$ the polarization operator is symmetric in q^2 and κ^2 although the perturbational expansion valid for $q^2/\Lambda^2 \sim 1$ or $\kappa^2/\Lambda^2 \sim 1$ does not show this symmetry.

Solving the Lie equation for $\overline{\Pi}(x)$

$$\overline{\Pi}(x) \sim x^{\Psi_g(u_0^+)} \left(1 + G x^{\omega/2} + \ldots\right). \tag{5.24}$$

$\Pi^{(0)}(q^2, \Lambda^2)$ contains a factor $(q^2)^{-\varepsilon/2}$. Taking this into account we get finally

$$\Pi(q^2, \Lambda^2, u_0) \sim q^{\lambda} \left(1 + G' q^{\omega} + \ldots\right) \tag{5.25}$$

with

$$\lambda = \frac{n-4}{n+8} \varepsilon + \frac{(n+2)(13n+44)}{(n+8)^3} \varepsilon^2 + o(\varepsilon^3). \tag{5.26}$$

This exponent has been calculated earlier by Ma [31] with the same results using the Feynman graph expansion and by one of the authors [27] using a method similar to the one presented here. The anomalous dimension d_{φ^2} is related to λ by the following relation $\lambda = -(d - 2d_{\varphi^2})$. The result of our previous calculation of d_{φ^2} given by eq. /4.52/ agrees with this recent calculation.

VI. Corrections to mean-field behaviour in four dimensional systems

In the preceding sections the critical behaviour of systems with dimensionality $d = 4 - \varepsilon$ $(\varepsilon > 0)$ was studied. The problem of phase transition in a four dimensional model is not of theoretical interest only since Larkin and Khmelnitskii [29] have pointed out that phase transition in uniaxial systems with dipolar interaction /uniaxial ferroelectrics/ is formally equivalent to a four dimensional problem. These systems are available for experiment [32] and therefore the theoretical study of four dimensional systems seems worthwhile.

The critical exponents are mean-field-like for $\varepsilon = 0$ and the real question is how corrections to this mean-field behaviour look like. The specific heat exponent α being zero, this correction will determine the actual behaviour of the specific heat near T_c . The specific heat itself was investigated by Larkin and Khmelnitskii [29]. Wegner and Riedel [33] obtained logarithmic corrections to the susceptibility and magnetization using Wilson's renormalization group approach.

First we show that these results can be obtained from our renormalization group approach as well and then further corrections will be considered.

The Lie equation for the invariant coupling /3.21/ has now the form

$$\frac{d u_R(s)}{ds} = \frac{u_R(s)}{s}\left\{\frac{n+8}{12}u_R(s)K_4 - \frac{3n+42}{72}u_R^2(s)K_4^2 + \dots\right\}. \qquad /6.1/$$

In the first step of an iterative solution the first term is kept only and we get

$$K_4 u_R(s) = -\frac{12}{n+8}\frac{1}{\ln s} + \dots \qquad /6.2/$$

wh ere s stands for x or y.

Inserting this expression into the Lie equations for the Green's function or polarization operator, only the lowest order term of the generators should be considered. Using eqs. /4.1/, /4.2/, /4.23/ and /5.9/ with the corres- ponding expressions for the respective generators

$$\frac{\partial \ln d(x)}{\partial x} = \frac{1}{x}\frac{n+2}{144}u_R^2(x)\cdot K_4^2 + \dots, \qquad /6.3/$$

$$\frac{\partial \ln d(y)}{\partial y} = \frac{1}{y}\frac{n+2}{144}u_R^2(y)K_4^2 + \dots, \qquad /6.4/$$

$$\frac{\partial \ln F(y)}{\partial y} = \frac{1}{y}\left(-\frac{n+2}{12}\right)u_R(y)K_4 + \dots, \qquad /6.5/$$

$$\frac{\partial \ln \overline{\Pi}(y)}{\partial y} = \frac{1}{y}\frac{n+2}{6}u_R(y)K_4 + \dots. \qquad /6.6/$$

These equations are solved in a straightforward way if the variable

$L = \ln x$ or $L = \ln y$ is introduced

$$d(x) \sim \exp\left\{ -\frac{n+2}{(n+8)^2}\frac{1}{\ln x} + \right\} = 1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln x} + \dots , \qquad /6.7/$$

$$d(y) \sim \exp\left\{ -\frac{n+2}{(n+8)^2}\frac{1}{\ln y} + \right\} = 1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln y} + \dots , \qquad /6.8/$$

$$F(y) \sim |\ln y|^{(n+2)/(n+8)} , \qquad /6.9/$$

$$\overline{\pi}(y) \sim |\ln y|^{-2(n+2)/(n+8)} . \qquad /6.1o/$$

The momentum and $\varkappa$ dependence of the Green's function and the temperature dependence of $\varkappa^2$ is obtained from /6.7/ - /6.9/ as

$$G(q^2) \sim \frac{1}{q^2}\left(1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln \frac{q^2}{\Lambda^2}} + \dots\right), \qquad /6.11/$$

$$G(\varkappa^2) \sim \frac{1}{\varkappa^2}\left(1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln \frac{\varkappa^2}{\Lambda^2}} + \dots\right), \qquad /6.12/$$

$$\varkappa^2 \left|\ln \frac{\varkappa^2}{\Lambda^2}\right|^{(n+2)/(n+8)} \sim t \qquad /6.13/$$

where $t \sim T - T_c$ and the relationship between the function $F(y)$ and $T - T_c$, given by eqs. /4.2o/ and /4.22/ has been used. By inverting this relation

$$\varkappa^2 \sim t\, |\ln t|^{-\frac{n+2}{n+8}} \qquad /6.14/$$

and

$$\chi(t) \sim G(t) \sim t^{-1} |\ln t|^{\frac{n+2}{n+8}} \left(1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln t} + \dots\right). \qquad /6.15/$$

For obtaining the specific heat $\overline{\pi}(y)$ has to be integrated

$$C(y) \sim \pi(y) \sim \int_1^y \frac{dy'}{y'}\, \overline{\pi}(y') \qquad /6.16/$$

$$C(y) \sim \begin{cases} \dfrac{1}{1 - 2\frac{n+2}{n+8}} |\ln y|^{1-2\frac{n+2}{n+8}} \sim \dfrac{n+8}{4-n} \left|\ln \dfrac{\varkappa^2}{\Lambda^2}\right|^{\frac{4-n}{n+8}} & \text{if } n \neq 4 \\ \ln|\ln y| \sim \ln\left|\ln \dfrac{\varkappa^2}{\Lambda^2}\right| & \text{if } n = 4 \end{cases} \qquad /6.17/$$

The temperature dependence of the specific heat is obtained by expressing $\varkappa^2$ in terms of t . The leading term is

$$C_v(t) \sim \begin{cases} \frac{n+8}{4-n} \left|\ln t\right|^{\frac{4-n}{n+8}} & \text{if } n \neq 4 \\ \ln\left|\ln t\right| & \text{if } n = 4 \end{cases} \qquad /6.18/$$

All these results agree with earlier works of Larkin and Khmelnitskii [29] and Wegner and Riedel [33] , and in some sense they are the counterparts of the expressions in ε expansion if only the linear terms are retained.
It is possible to get further corrections if in solving the Lie equation /6.1/ for the invariant coupling the second term is also considered, which then yields the counterparts of the ε^2 terms.

In the next step of the iterative solution the invariant coupling can be written in the form

$$K_4\, u_R(s) = -\frac{12}{n+8}\,\frac{1}{\ln s} + \Delta u_R(s) K_4 \qquad /6.19/$$

then eq. /6.1/ becomes

$$\frac{\partial \Delta u_R(s) K_4}{\partial s} = -\frac{2}{s}\,\frac{1}{\ln s}\,\Delta u_R(s) K_4 + \frac{72}{s}\,\frac{3n+14}{(n+8)^3}\,\frac{1}{\ln^3 s} + \dots \qquad /6.20/$$

This equation has the solution

$$\Delta u_R(s) K_4 = 72\,\frac{3n+14}{(n+8)^3}\,\frac{\ln\left|\ln s\right|}{(\ln s)^2} \qquad /6.21/$$

and therefore the invariant coupling is

$$K_4\, u_R(s) = -\frac{12}{n+8}\,\frac{1}{\ln s} + 72\,\frac{3n+14}{(n+8)^3}\,\frac{\ln\left|\ln s\right|}{\ln^2 s} + \dots \qquad /6.22/$$

The next correction is of order $1/\ln^2 s$. Apart from a small intermediate region around $s = e^{-1}$ both for $s \sim 1$ and for $s \ll 1$ the term $\ln|\ln s|/\ln^2 s$ is larger than the term with $1/\ln^2 s$ and will be neglected in this approximation. Due to this particular feature of the invariant coupling that it does not go simply in powers of $1/\ln s$, the next corrections to the Green's function, coherence length and specific heat are obtained by keeping the same Lie equations as /6.3/ - /6.6/, neglecting the higher powers of u_R, but inserting eq. /6.22/ for the invariant coupling. These equations can again be solved easily keeping the leading corrections only

$$d(x) = 1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln x} + 6\frac{(n+2)(3n+14)}{(n+8)^4}\frac{\ln|\ln x|}{\ln^2 x} + \dots, \qquad /6.23/$$

$$d(y) = 1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln y} + 6\frac{(n+2)(3n+14)}{(n+8)^4}\frac{\ln|\ln y|}{\ln^2 y} + \dots, \qquad /6.24/$$

$$F(y) = |\ln y|^{\frac{n+2}{n+8}}\left[1 + 6\frac{(n+2)(3n+14)}{(n+8)^3}\frac{\ln|\ln y|}{\ln y} + \dots\right] \qquad /6.25/$$

$$\Pi(y) = |\ln y|^{-2\frac{n+2}{n+8}}\left[1 - 12\frac{(n+2)(3n+14)}{(n+8)^3}\frac{\ln|\ln y|}{\ln y} + \dots\right] \qquad /6.26/$$

The Green's function is obtained from eqs. /6.23/ and /6.24/ as

$$G(q^2) = \frac{1}{q^2}\left(1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln\frac{q^2}{\Lambda^2}} + 6\frac{(n+2)(3n+14)}{(n+8)^4}\frac{\ln|\ln\frac{q^2}{\Lambda^2}|}{\ln^2\frac{q^2}{\Lambda^2}} + \dots\right) \qquad /6.27/$$

$$G(\varkappa^2) = \frac{1}{\varkappa^2}\left(1 - \frac{n+2}{(n+8)^2}\frac{1}{\ln\frac{\varkappa^2}{\Lambda^2}} + 6\frac{(n+2)(3n+14)}{(n+8)^4}\frac{\ln|\ln\frac{\varkappa^2}{\Lambda^2}|}{\ln^2\frac{\varkappa^2}{\Lambda^2}} + \dots\right) \qquad /6.28/$$

The temperature dependence of the coherence length is obtained implicitly from eq. /6.25/.

$$\varkappa^2 \left|\ln \frac{\varkappa^2}{\Lambda^2}\right|^{\frac{n+2}{n+8}} \left(1 + 6\,\frac{(n+2)(3n+14)}{(n+8)^3}\;\frac{\ln\left|\ln \frac{\varkappa^2}{\Lambda^2}\right|}{\ln \frac{\varkappa^2}{\Lambda^2}} + \ldots\right) \sim t \qquad /6.29/$$

Solving iteratively for $\varkappa^2$ we get

$$\varkappa^2 \sim t\,|\ln t|^{-\frac{n+2}{n+8}} \left(1 - 6\,\frac{(n+2)(3n+14)}{(n+8)^3}\;\frac{\ln|\ln t|}{\ln t} + \ldots\right) \qquad /6.3o/$$

and therefore the susceptibility is

$$\chi(t) \sim t^{-1}\,|\ln t|^{\frac{n+2}{n+8}} \left(1 + 6\,\frac{(n+2)(3n+14)}{(n+8)^3}\;\frac{\ln|\ln t|}{\ln t} + \ldots\right) \qquad /6.31/$$

The polarization operator as a function of $\varkappa^2$ is obtained by using eq. /6.16/

$$\Pi(y) \sim \frac{1}{2\frac{n+2}{n+8} - 1}\,|\ln y|^{1-2\frac{n+2}{n+8}} + 6\,\frac{3n+14}{(n+8)^2}\,|\ln y|^{-2\frac{n+2}{n+8}}\,\ln|\ln y| + \ldots \qquad /6.32/$$

or

$$\Pi\left(\frac{\varkappa^2}{\Lambda^2}\right) \sim \frac{n+8}{n-4}\left|\ln \frac{\varkappa^2}{\Lambda^2}\right|^{\frac{4-n}{n+8}} \left[1 - 6\,\frac{(n-4)(3n+14)}{(n+8)^3}\;\frac{\ln\left|\ln \frac{\varkappa^2}{\Lambda^2}\right|}{\ln \frac{\varkappa^2}{\Lambda^2}} + \ldots\right] \qquad /6.33/$$

Making use of eq. /6.3o/ and keeping in mind that both $\varkappa^2/\Lambda^2$ and t are less than unity and therefore

$$\begin{aligned} \left|\ln \frac{\varkappa^2}{\Lambda^2}\right| &= |\ln t| + \frac{n+2}{n+8}\,\ln|\ln t| + \ldots \\ &= |\ln t|\left(1 - \frac{n+2}{n+8}\;\frac{\ln|\ln t|}{\ln t} + \ldots\right) \end{aligned} \qquad /6.34/$$

the temperature dependence of the specific heat is

$$C_v(t) \sim \frac{n+8}{n-4}\,|\ln t|^{\frac{4-n}{n+8}} \left[1 - \frac{(n-4)(n^2-8n-68)}{(n+8)^3}\;\frac{\ln|\ln t|}{\ln t} + \ldots\right] \qquad /6.35/$$

In the special case of physical interest, $n=1$ this expression has a simpler form

$$C_v(t) \sim |\ln t|^{1/3} \left[1 - \frac{25}{81} \frac{\ln|\ln t|}{\ln t} + \dots \right]. \qquad /6.36/$$

This agrees with Brezin's result [34]. As it was shown by Brezin and Zinn-Justin [35] , the equivalence of d=4 problem with d=3 dipolar problem holds for the leading logarithmic corrections but not for the subleading divergences and therefore this correction term is not experimentally observable.

VII. Comparison of the conventional and new method

In the conventional formulation of the Gell-Mann-Low transformation the multiplicative renormalization of the Green's function and vertex is equivalent to the introduction of transformed functions with renormalized coupling and with an additional variable λ. This transformation can be performed on almost any system and in most of the cases it is not easier to study the transformed system than the original physical system. For a restricted class of problems, however, scaling is an inherent property of the system. In such cases the scaling transformation, which is characteristic to the physical system, can be expressed in the form of a multiplicative renormalization group transformation of a special kind, where the number of variables of the transformed functions is the same as that of the physical functions. In a renormalizable theory the new variable λ and the renormalized coupling can be introduced by an appropriate normalization condition in such a way that the transformed functions remain finite when the cut-off goes to infinity, i.e. the cut-off drops out from the transformed functions. We have seen examples for this in Sec. II. both with field theoretical and physical normalization. In this case the renormalization reflects a non-trivial scaling relation and can be effectively used to determine the critical behaviour by solving the Lie equation.

The specific heat is a good example to show that the simple multiplicative transformation cannot be applied is all of the cases to study scaling and the critical behaviour. Looking at the diagrams in Fig. 8. and performing the transformations /2.5/ - /2.8/ for these and higher order polarization diagrams, a relation analogous to eqs. /2.9/ and /2.1o/ can be obtained for $\Pi(\kappa^2, \Lambda^2, u_0)$

$$\Pi(\kappa^2, \Lambda^2, g_0, \delta m^2, \tilde{\Gamma}_0, G^{(0)}) = Z_3^2\, \Pi(\kappa^2, \Lambda^2, g, \delta m^2 Z_3, \tilde{\Gamma}_0 Z_1, G^{(0)} Z_3^{-1}) . \qquad /7.1/$$

This shows that the polarization operator is a multiplicatively renormalizable quantity. Fixing Z_1, Z_3 and δm^2 by an appropriate normalization condition as in Sec. II, scaling relations analogous to eqs. /2.26/ - /2.27/ or /2.58/ - /2.59/ can be derived. Unlike the Green's function and vertex, the variable Λ^2/λ^2 does not drop out from the Lie equations and therefore it cannot be used to obtain a summed up expression from the perturbational expansion, thus the scaling property of the specific heat is not apparent if a multiplicative renormalization transformation is performed on it.

In the conventional Gell-Mann-Low scheme the parameter λ in general has no physical meaning. The transformation maps the physical system onto another system, but the formal way of introducing λ usually does not allow a simple physical interpretation of the renormalization transformation.

In the new renormalization procedure, what we have proposed for other systems and for studying critical phenomena, the physical system is mapped onto an equivalent one by changing the physical cut-off Λ . The underlying physical picture is the same as in Kadanoff's cell construction or in Wilson's theory of eliminating degrees of freedom. The scaling relations as written in eqs. /3.4/ - /3.6/ are non-trivial relations in contrast to the trivial transformation relations of eqs. /2.26/ - /2.27/ or /2.58/ - /2.59/. As we emphasized these relations become non-trivial if the variable Λ drops out and then the scaling variable λ plays the same role in the conventional method as the cut-off in the new method. Thus one may expect that a formal relationship can be found.

Comparing eqs. /2.88/, /2.89/ for the transformed functions and eqs. /2.63/, /2.7o/ for the physical Green's function and vertex, it is seen that these functions with u and λ and with u_0 and Λ have exactly the same analytic form. This statement holds also if the κ^2 dependence is considered. Taking Z_1 and Z_3 as given in eqs. /2.86/ and /2.87/ and inserting then into the transformed Green's function and vertex at $q^2=0$, these functions have the form:

$$d = 1 + \frac{n+2}{144} u^2 K_d^2 \ln \frac{\varkappa^2}{\lambda^2} + \dots \qquad /7.2/$$

$$\tilde{\Gamma} = 1 + \frac{n+8}{12} u K_d \left(\frac{\varkappa^2}{\lambda^2}\right)^{-\varepsilon/2} \left[\frac{2}{\varepsilon}\left(\left(\frac{\varkappa^2}{\lambda^2}\right)^{\varepsilon/2} - 1\right) + 1 + \dots \right]$$

$$+ \frac{n^2+6n+20}{144} u^2 K_d^2 \left(\frac{\varkappa^2}{\lambda^2}\right)^{-\varepsilon} \left[\frac{2}{\varepsilon}\left(\left(\frac{\varkappa^2}{\lambda^2}\right)^{\varepsilon/2} - 1\right) + 1 + \dots \right]^2 \qquad /7.3/$$

$$+ \frac{5n+22}{36} u^2 K_d^2 \left(\frac{\varkappa^2}{\lambda^2}\right)^{-\varepsilon} \left[\frac{1}{2} \ln^2 \frac{\varkappa^2}{\lambda^2} + \dots \right] + \dots$$

These functions have the same analytic form as the physical functions in eqs. /3.1o/ and /2.11/ if u_0 and Λ is replaced by u and λ .

These similarities show that the conventional formulation of the Gell-Mann-Low renormalization using the normalization conditions in eqs. /2.43/, /2.44/ and /2.47/ and the new renormalization procedure are equivalent if only the singular parts of the Green's function and vertex are considered. This means automatically that the multiplicative factors Z_1 and Z_3 in eqs. /2.86/ and /2.87/ are the same as Z_Γ and Z_d in Sec. III. if λ and u is replaced by Λ' and u_0 .

VIII. Discussion and Conclusions

In this paper we have developed in detail a new method to study the critical phenomena for the Ginzburg-Landau-Wilson model. The physical picture which is the starting point of this method is very close to that of Wilson's renormalization theory [3], but it is formulated in a completely different mathematical framework. In fact the mathematical formulation is analogous to that of the Gell-Mann-Low multiplicative renormalization, actually, the new method is a simplified version of the Gell-Mann-Low method by assuming *a priori* the existence of scaling.

The essential new feature of Wilson's theory was to show that the number of degrees of freedom can be decreased by integrating out the large momenta and to make sure that the transformed system has the same thermodynamical behaviour as the original one, the parameters characterizing the system /coupling constants/ have to be changed simultaneously. Wilson emphasized the importance of the fixed point Hamiltonian which is obtained in the limit when all the degrees of freedom are eliminated. The parameters of the fixed point Hamiltonian determine the critical exponents. It was also realized [36] that the way the transformed Hamiltonian approaches the fixed point Hamiltonian determines the corrections to the scaling behaviour.

Adopting the same physical picture we have shown that this transformation leads at the same time to a multiplicative renormalization of the Green's function and vertex. In our opinion this is the central point in this new approach. In the same way as in Wilson's theory degrees of freedom are eliminated by recuding the momentum cut-off Λ to Λ' . Changing simultaneously and in an appropriate way the coupling constant u_0 to u_0', it may occur that not only the thermodynamical behaviour is the same before and after this transformation, but the Green's function and vertex differ only by multiplicative factors independent of the momentum and temperature variables.

We have shown this to be the case in perturbation theory for special choices of the momentum and temperature variables. Relying on this result we have supposed that the scaling of the momentum cut-off Λ generates a multiplicative renormalization. We could use then the simple mathematical structure of the Lie equations to get information on the critical behaviour.

The independence of the multiplicative factors of the choice of momentum variables is especially important. The scaling transformation is equivalent to transforming out the high momentum region in the internal lines of the Green's function and vertex. As the internal momentum variables are integrated over, the renormalization procedure can be worked out consistently only if the renormalization constants are independent of these variables.

In contrast to the usual Gell-Mann-Low renormalization no extra variable is introduced to scale the momentum because it is assumed, that the scaling does not change the form of the vertex and Green's functions. Furthermore, the scaling is parametrized by a single physical parameter, the momentum cut-off. Due to this fact it is very easy to interpret physically what the Lie equation means. Suppose we can calculate the Green's function or vertex for momenta near the cut-off in a perturbational way and for small coupling strength this is a good approximation. We want now to determine this quantity for smaller momenta where this perturbational expansion is not enough. We will transform our original system to a case where the new cut-off is at the momentum where the function is to be determined. In this transformed system the momentum is now at the new cut-off and therefore a perturbational calculation yields good approximation, provided the transformed coupling is small. Using the perturbational result for the transformed system the solution of the Lie equation provides us with a good non-perturbative result for the original problem.

The transformed coupling is small only if the dimensionality of the system is near to four, otherwise it might become of the order of unity and therefore it will not be allowe' 'r e the perturbational result any more.

The use ; momentum cut-off Λ as a scaling variable indicates c ırly that we study a statistical physical problem and not a field theoretical one. It should be stressed that the whole renormalization procedure is only a useful tool to get the low momentum behaviour starting from the $q^2 \sim \Lambda^2$ region by scaling down Λ to small values.

We have made an attempt to clarify the relationship of the simplified new method to the original Gell-Mann-Low technique. We have shown that in the Gell-Mann-Low technique it is not assumed that the physical functions obey a scaling relation. It is based on a completely general symmetry property and the symmetry transformation results in such new vertex and Green's functions that their functional forms are in general, different from the physical ones. In the general case, the transformation can be parametrized by at least one new parameter λ , which does not necessarily have any physical meaning. This new parameter means a new variable for the different functions. The transformation parametrized by λ usually leads to transformed functions with increased number of variables. By choosing different normalization conditions, different multiplicative factors and different transformed functions are obtained. It may happen that for some statistical physical system the transformed function is equivalent to the physical function if appropriate normalization condition is taken. In this case scaling is an inherent property of the system and all of its consequences can be explored. It has been shown that by applying slightly different parametrization of the transformation /i.e. slightly different normalization condition/, the transformed function may depend on the variable Λ^2/λ^2 but this dependence is unessential in the sense that this variable does not appear in the Lie equations and the critical behaviour is reproduced correctly. It should also be mentioned here that the multiplicative renormalization could be successfully applied to systems which do not obey exact scaling and show only an approximate scaling behaviour /see Ref. [25] /. In this case the extra variable has to be introduced and the renormalization procedure cannot be described by cut-off scaling.

Concerning the conventional Gell-Mann-Low technique, the most important result of this paper is that we have found the adequate normalization condition. Imposing this condition /see eqs. /2.43/, /2.44/ and /2.47// the transformed functions both in q^2 and $\varkappa^2$ dependence have the same analytic form in perturbation theory as the physical Green's function and vertex if those smoothly varying parts,which are irrelevant for the critical behaviour, are neglected. Thus the number of variables is not increased and the scaling parameter is in fact the cut-off. In this way the equivalence of the conventional Gell-Mann-Low renormalization and the new renormalization procedure proposed in Sec. III. is demonstrated. This consideration is not restricted to the problem of phase transition, thus that can be extended to any of those problems where cut-off scaling holds e.g. one-dimensional metallic system, Kondo problem, etc.

In our new renormalization procedure everything is reduced to calculate the renormalized, invariant coupling u_R and there is no need to introduce higher order couplings /three-, four-particle scattering/. In Wilson's formulation of the renormalization group transformation the introduction of higher order couplings is very essential and the renormalization procedure cannot be described consistently by keeping the two-particle scattering only. In first and second order in ε these higher order couplings are not relevant and our procedure is in agreement with that of Wilson. The relation of the conventional Gell-Mann-Low renormalization, our new procedure and Wilson's theory in higher orders is not clear and needs further studies. If the higher order couplings become relevant in higher order in ε, this may be reflected in the Gell-Mann-Low theory in the fact that the variable Λ^2/λ^2 does not drop out and it describes how far the system is scaled from the physical situation.

By extending the method discussed in the present paper more complicated systems than the simple isotropic φ^4 model can also be studied and it can be shown that in these cases as well cut-off scaling generates a multiplicative renormalization. This problem will be studied in a separate publication.

Finally we believe that this method can be applied to study the dynamics of critical behaviour after a rather straightforward extension of the formulation to include the energy variables. A calculation of this type has been carried out by Zawadowski and Grest [37] and they obtained similar results as Abrahams and Tsuneto [38] and De Dominicis et al [39] using very different methods.

Acknowledgements

One of the authors /A.Z./ is grafeful to E. Abrahams and M. Stephen to call his attention to the problem discussed in Sec. VI, and for the kind hospitality at the Departments of Physics of Rutger's University and of University of Illinois, Urbana. Another author /J.S./ is indebted to Ph. Nozieres for useful discussions and acknowledges the hospitality of the International Centre for Theoretical Physics, Trieste, where he was staying a shorter period working on this problem.

Appendix A.

In this appendix we give a comprehensive list of all the integrals which are relevant for the analytic contribution of the diagrams discussed in this paper. The integration over the momenta goes everywhere on a hypersphere of radius Λ.

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2+\varkappa^2)^2} = \frac{1}{2} K_d (\varkappa^2)^{-\varepsilon/2} \left\{ \frac{2}{\varepsilon}\left(1-\left(\frac{\varkappa^2}{\Lambda^2}\right)^{\varepsilon/2}\right) - 1 + \dots \right\} \qquad /A.1/$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} \frac{1}{p^2+q^2} = \frac{1}{2} K_d (q^2)^{-\varepsilon/2} \left\{ \frac{2}{\varepsilon}\left(1-\left(\frac{q^2}{\Lambda^2}\right)^{\varepsilon/2}\right) + 1 + \dots \right\} \qquad /A.2/$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \frac{1}{(p_1^2+\varkappa^2)^2} \frac{1}{p_2^2+\varkappa^2} \frac{1}{(p_1+p_2)^2+\varkappa^2} = \qquad /A.3/$$

$$= K_d^2 \left(\frac{1}{8} \ln^2 \frac{\varkappa^2}{\Lambda^2} + \mathrm{const} + \dots \right)$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \frac{1}{p_1^2} \frac{1}{(p_1+q)^2} \frac{1}{p_2^2} \frac{1}{(p_1+p_2)^2} =$$

$$= K_d^2 (q^2)^{-\varepsilon} \left[\frac{1}{8} \ln^2 \frac{q^2}{\Lambda^2} - \frac{1}{2} \ln \frac{q^2}{\Lambda^2} + \frac{1}{2} + \dots \right] \qquad /A.4/$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \left(\frac{1}{p_1^2+\varkappa^2} \frac{1}{p_2^2+\varkappa^2} \frac{1}{(p_1+p_2)^2+\varkappa^2} - \frac{1}{p_1^2} \frac{1}{p_2^2} \frac{1}{(p_1+p_2)^2} \right) =$$

$$= K_d^2 \varkappa^2 \left\{ -\frac{3}{8} \ln^2 \frac{\varkappa^2}{\Lambda^2} + \frac{3}{4} \ln \frac{\varkappa^2}{\Lambda^2} + \dots \right\} \qquad /A.5/$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \frac{1}{p_1^2} \frac{1}{(p_1+p_2)^2} \frac{1}{(p_2+q)^2} = K_d^2 q^{2-2\varepsilon} \left[\frac{1}{8} \ln \frac{q^2}{\Lambda^2} - \frac{5}{16} + \dots \right] \qquad /A.6/$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \left(\frac{1}{p_1^2+\kappa^2} \frac{1}{(p_1+p_2)^2+\kappa^2} \frac{1}{(p_2+q)^2+\kappa^2} - \frac{1}{p_1^2} \frac{1}{p_2^2} \frac{1}{(p_1+p_2)^2} \right)_{q^2=-\kappa^2} =$$

$$= K_d^2 \kappa^2 \left\{ -\frac{3}{8} \ln^2 \frac{\kappa^2}{\Lambda^2} + \frac{5}{8} \ln \frac{\kappa^2}{\Lambda^2} + \dots \right\} \qquad /A.7/$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \int \frac{d^d p_3}{(2\pi)^d} \frac{1}{p_1^2} \frac{1}{(p_1+p_2)^2} \frac{1}{(p_3+q)^2} \frac{1}{(p_2+p_3)^2} \frac{1}{p_2^2} =$$

$$= K_d^3 q^{2-3\varepsilon} \left[-\frac{1}{16} \ln^2 \frac{q^2}{\Lambda^2} + \frac{5}{16} \ln \frac{q^2}{\Lambda^2} - \frac{15}{32} + \dots \right] \qquad /A.8/$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \int \frac{d^d p_3}{(2\pi)^d} \frac{1}{(p_1^2+\kappa^2)^2} \frac{1}{(p_2^2+\kappa^2)^2} \frac{1}{(p_1+p_3)^2+\kappa^2} \frac{1}{(p_2+p_3)^2+\kappa^2} =$$

$$= K_d^3 \left[-\frac{1}{24} \ln^3 \frac{\kappa^2}{\Lambda^2} + o\left(\ln \frac{\kappa^2}{\Lambda^2} \right) + \dots \right] \qquad /A.9/$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \int \frac{d^d p_3}{(2\pi)^d} \frac{1}{p_1^2} \frac{1}{(p_1+q)^2} \frac{1}{p_2^2} \frac{1}{(p_2+q)^2} \frac{1}{(p_1+p_3)^2} \frac{1}{(p_2+p_3)^2} =$$

$$= K_d^3 \left[-\frac{1}{24} \ln^3 \frac{q^2}{\Lambda^2} + \frac{1}{4} \ln^2 \frac{q^2}{\Lambda^2} + o\left(\ln \frac{q^2}{\Lambda^2} \right) + \dots \right] \qquad /A.10/$$

$$\int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \int \frac{d^d p_3}{(2\pi)^d} \frac{1}{(p_1^2+\kappa^2)^3} \left[\frac{1}{p_2^2+\kappa^2} \frac{1}{(p_1+p_3)^2+\kappa^2} \frac{1}{(p_2+p_3)^2+\kappa^2} \right.$$

$$\left. - \frac{1}{p_2^2} \cdot \frac{1}{(p_1+p_3)^2} \frac{1}{(p_2+p_3)^2} \Big|_{p_1^2=-\kappa^2} \right] = \qquad /A.11/$$

$$= K_d^3 \left(-\frac{1}{32} \ln^2 \frac{\kappa^2}{\Lambda^2} + o\left(\ln \frac{\kappa^2}{\Lambda^2} \right) + \dots \right)$$

Appendix B

The critical exponents have been calculated in the paper by using the new renormalization procedure of Sec. III. The exponent η has also been calculated by the conventional method but using a physical normalization condition. Here we will calculate the same quantity with the field theoretical normalization to illustrate how this method works.

As we have emphasized earlier, normalizing the Green's function and vertex to unity, as in eqs. /2.14/ - /2.15/, has the consequence that the physical Green's function and vertex cannot be reproduced by a special choice of λ. Nevertheless if we are only interested in the critical exponents and do not want to determine the prefactors, this normalization condition is permitted.

Starting from the perturbational expansion calculated for finite cut-off Λ, the normalization conditions /2.14/ - /2.15/ lead to the following form for Z_1 and Z_3

$$Z_1^{-1} = 1 + \frac{n+8}{12} u K_d \left\{ \frac{2}{\varepsilon}\left(\left(\frac{\lambda^2}{\Lambda^2}\right)^{\varepsilon/2} - 1\right) - 1 - \varepsilon + \dots \right\} - \frac{5n+22}{36} u^2 K_d^2 \ln \frac{\lambda^2}{\Lambda^2} + \dots \qquad /B.1/$$

$$Z_3 = 1 + \frac{n+2}{144} u^2 K_d^2 \left[\left(1 - \frac{\varepsilon}{4}\right) \ln \frac{\lambda^2}{\Lambda^2} - \frac{5}{2} + \dots \right] + \frac{(n+2)(n+8)}{12^3} u^3 K_d^3 \left[- \ln^2 \frac{\lambda^2}{\Lambda^2} + 2 \ln \frac{\lambda^2}{\Lambda^2} + \frac{5}{2} + \dots \right] + \dots \qquad /B.2/$$

The transformed Green's function and vertex are obtained after eliminating Z_1 and Z_3,

$$d\left(\frac{q^2}{\lambda^2}, u\right) = 1 + \frac{n+2}{144} u^2 K_d^2 \left[\left(1 + \frac{9}{4}\varepsilon\right) \ln \frac{q^2}{\lambda^2} + \ldots\right] + \frac{(n+2)(n+8)}{12^3} u^3 K_d^3 \left[\ln^2 \frac{q^2}{\lambda^2} - 3 \ln \frac{q^2}{\lambda^2} + \ldots\right] + \ldots \qquad /B.3/$$

$$\tilde{\Gamma}\left(\frac{q^2}{\lambda^2}, u\right) = 1 + \frac{n+8}{12} u K_d \left\{\frac{2}{\varepsilon}\left[1 - \left(\frac{q^2}{\lambda^2}\right)^{-\varepsilon/2}\right] - (1+\varepsilon)\left[\left(\frac{q^2}{\lambda^2}\right)^{-\varepsilon/2} - 1\right] + \ldots\right\} + \frac{(n+8)^2}{12} u^2 K_d^2 \ln^2 \frac{q^2}{\lambda^2} - \frac{5n+22}{36} u^2 K_d^2 \ln \frac{q^2}{\lambda^2} + \ldots \qquad /B.4/$$

and the invariant coupling is

$$u_R\left(\frac{q^2}{\lambda^2}, u\right) = u\left(\frac{q^2}{\lambda^2}\right)^{-\varepsilon/2}\left\{1 + \frac{n+8}{12} u K_d \left[\left(\frac{2}{\varepsilon} + 1 + \varepsilon\right)\left(1 - \left(\frac{q^2}{\lambda^2}\right)^{-\varepsilon/2}\right) + \ldots\right] + \left(\frac{n+8}{12}\right)^2 u^2 K_d^2 \ln^2 \frac{q^2}{\lambda^2} - \frac{9n+42}{72} u^2 K_d^2 \ln \frac{q^2}{\lambda^2} + \ldots\right\} \qquad /B.5/$$

Writing now the Lie equation for the invariant coupling

$$\frac{\partial u_R(x,u)}{\partial x} = \frac{u_R(x,u)}{x}\left\{-\frac{\varepsilon}{2} + \frac{n+8}{12} u_R(x,u) K_d\left(1 + \frac{\varepsilon}{2}\right) - \frac{9n+42}{72} u_R^2(x,u) K_d^2 + \ldots\right\}, \qquad /B.6/$$

with $x = q^2/\lambda^2$. The solution of this equation can be written in the same form as eq. /2.78/, but with a different value for the fixed point.

$$K_d u_o^* = \frac{6}{n+8}\varepsilon\left[1 + \varepsilon\left(-\frac{1}{2} + \frac{9n+42}{(n+8)^2}\right)\right] + o(\varepsilon^3) \qquad /B.7/$$

The fixed point coupling being a non-measurable quantity, this difference has no consequence for physical quantities, like the critical exponents. Calculating η in this scheme, the Lie equation for d is

$$\frac{\partial \ln d(x,u)}{\partial x} = \frac{1}{x}\left\{ \frac{n+2}{144} u_R^2(x,u) K_d^2 \left(1+\frac{9}{4}\varepsilon\right) - 3\frac{(n+2)(n+8)}{12^3} u_R^3(x,u) K_d^3 + \dots \right\} \qquad /B.8/$$

This equation is different from eq. /2.81/ in the coefficient of the second term. Inserting the fixed point value from eq. /B.7/ we get exactly the same result as in eq. /2.28/ with the correct η and ω .

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Figure captions

Fig. 1. Typical low order diagrams for the Green's function.

Fig. 2. Typical low order diagrams for the vertex.

Fig. 3. Second and third order self-energy diagrams.

Fig. 4. First, second and third order vertex corrections.

Fig. 5. Third order vertex corrections with different orientations of the lines.

Fig. 6. First, second and third order corrections to the vertex $\Gamma^{(1,2)}_{\alpha\beta_1\beta_2}$

Fig. 7. Diagrammatic representation of the density-density correlation function.

Fig. 8. Zeroth, first and second order diagrams for the polarization π .

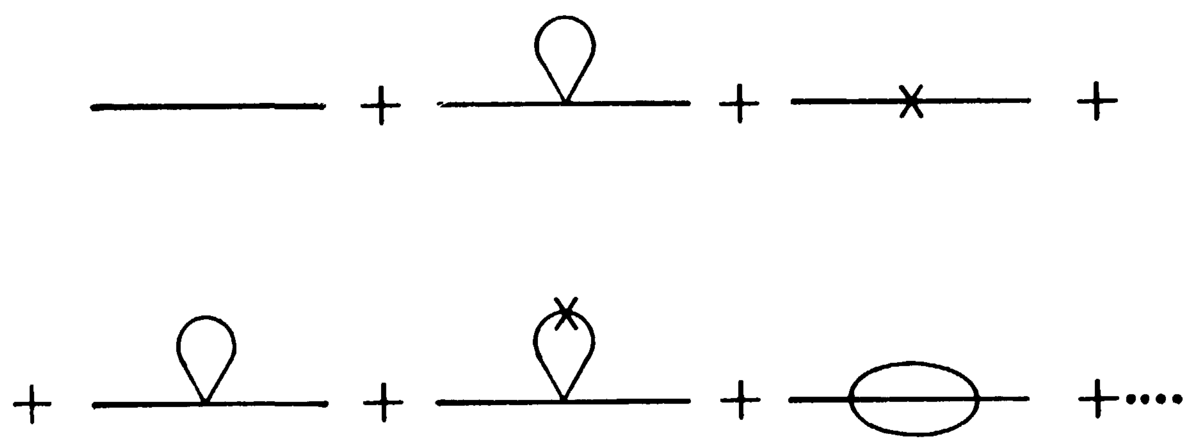


Fig. 1.

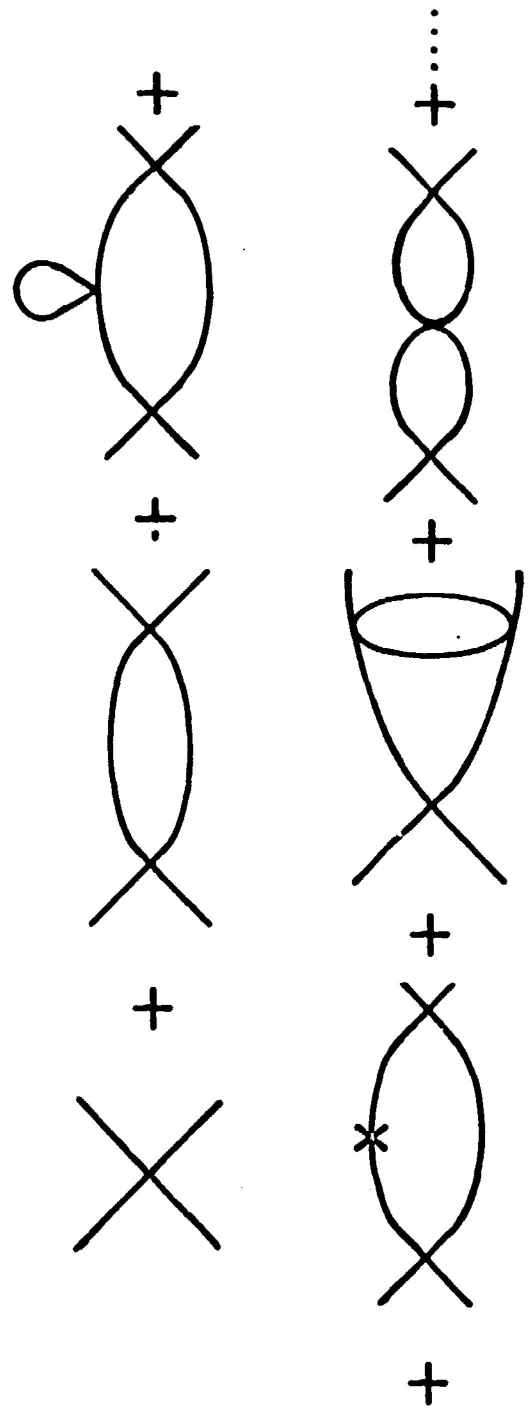


Fig. 2.

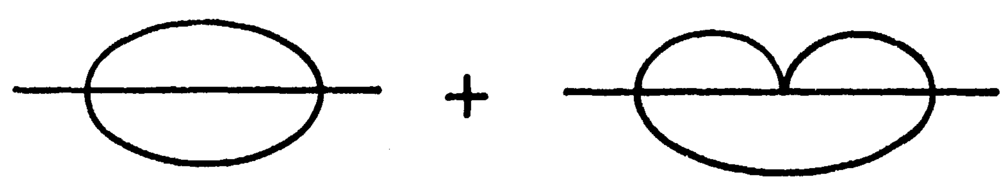
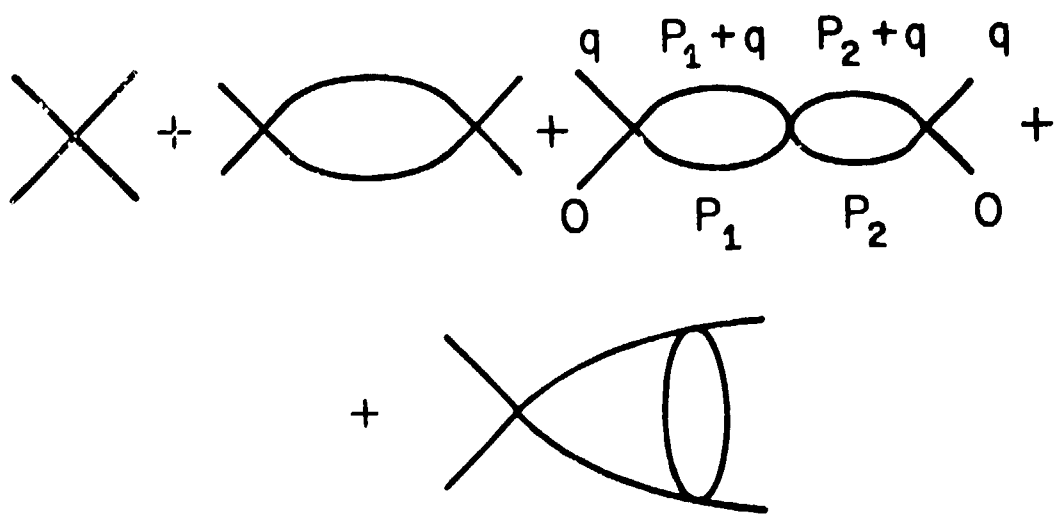


Fig. 3.



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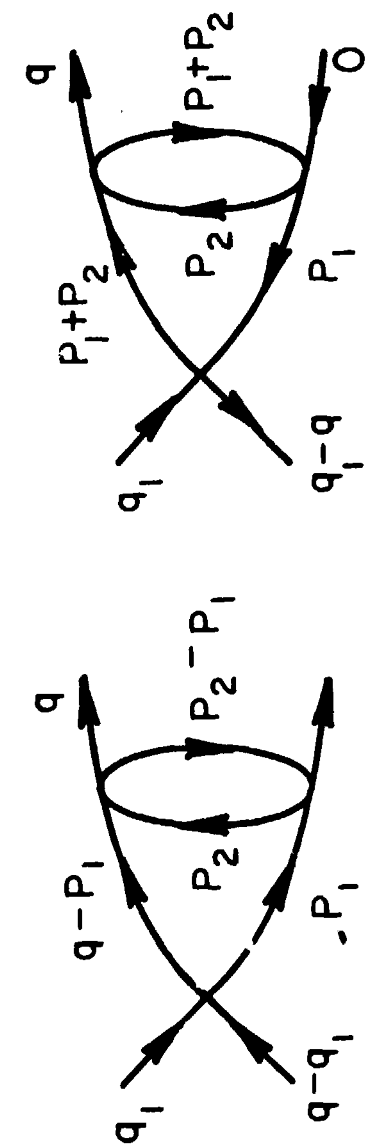
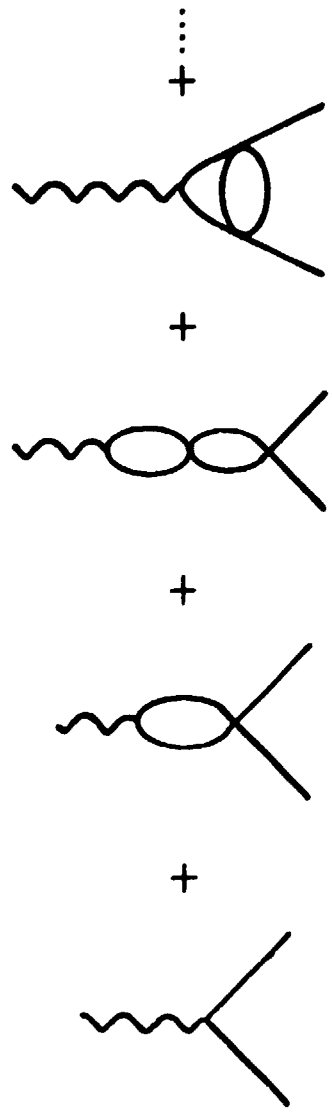


Fig. 5.



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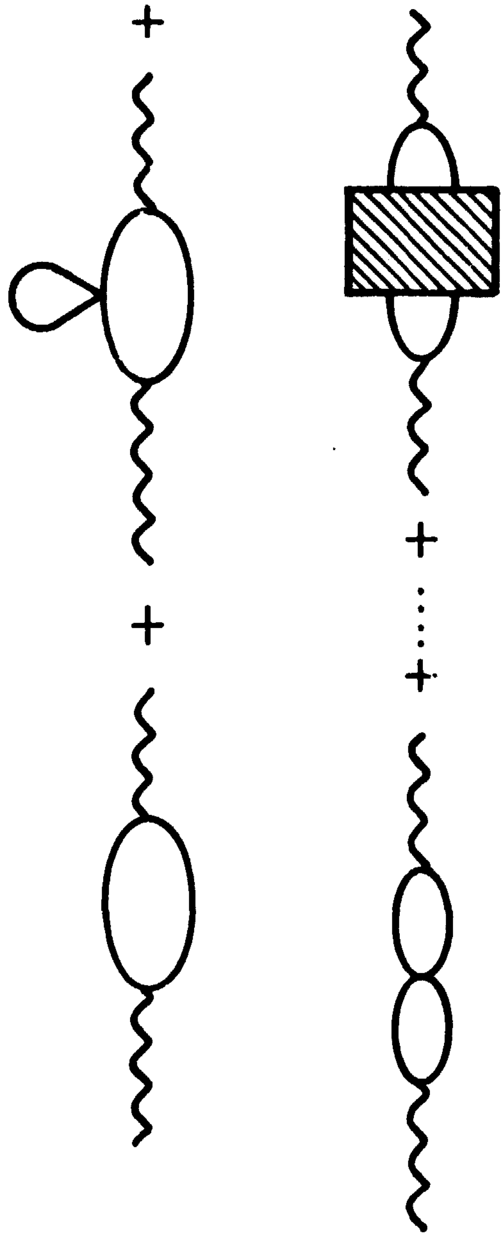

Fig. 7.

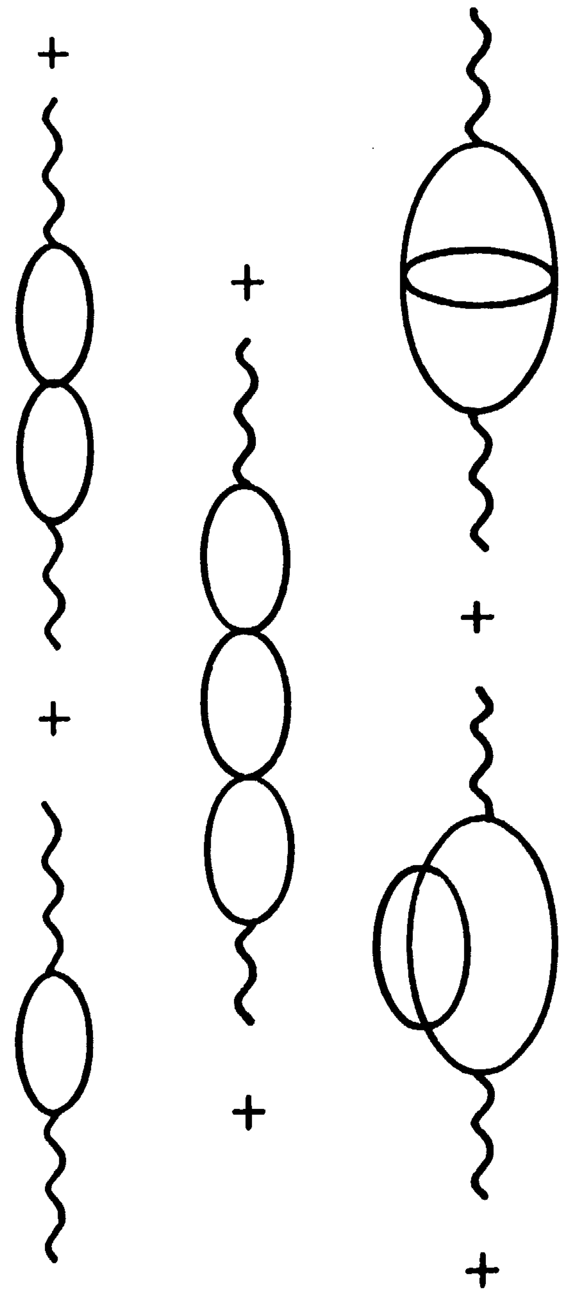

Fig. 8.

Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Siklós Tivadar, a KFKI Szilárdtestelméleti Tudományos Tanácsának szekcióelnöke
Szakmai lektor: Menyhárd Nóra
Nyelvi lektor: Menyhárd Nóra
Példányszám: 340 Törzsszám: 76-253
Készült a KFKI sokszorosító üzemében
Budapest, 1976. március hó