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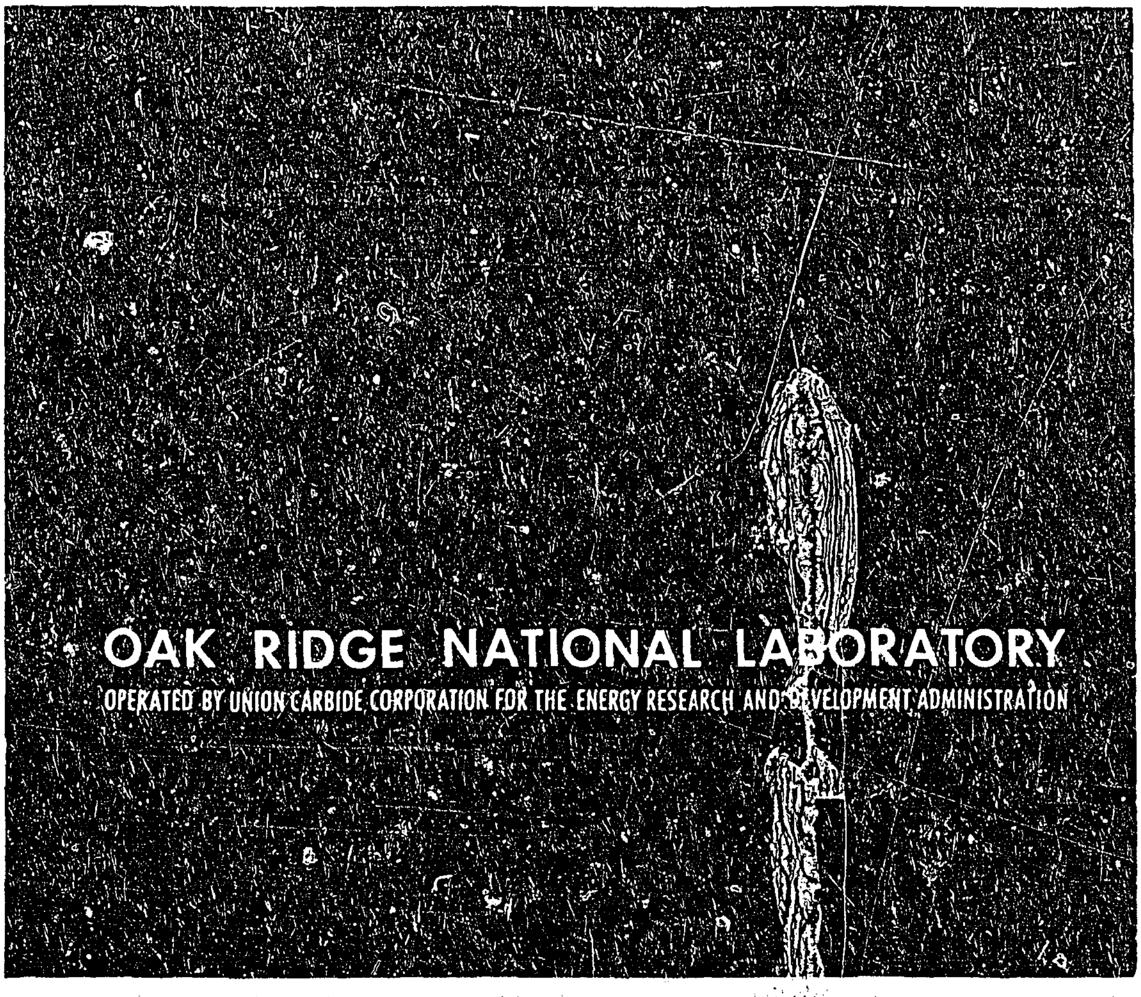
E. M. Oblow



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SENSITIVITY THEORY FOR GENERAL NON-LINEAR ALGEBRAIC EQUATIONS WITH CONSTRAINTS

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Abstract

The general development of a sensitivity theory for non-linear algebraic equations with constraints is presented. Adjoint equations suitable for evaluating derivatives of system response functions with respect to input parameters are derived. The role of the solution of the constrained problem in eliminating non-essential constraints is highlighted. Two sample problems, one linear and one non-linear, are solved to illustrate the theory.

I. Introduction

Sensitivity theory has been developed to a high state of sophistication for applications involving solutions of the linear Boltzmann equation or approximations to it.[1-5] The success of this theory in the field of radiation transport has prompted study of possible extensions of the method to more general systems of non-linear equations. Initial work in the U.S.[6,7] and in Europe[8] on the reactor fuel cycle show that the sensitivity methodology works equally well for those non-linear problems studied to date. In this paper the general non-linear theory for algebraic equations is summarized and applied to a class of problems whose solutions are characterized by constrained extrema. Such equations form the basis of much work on energy systems modelling and the econometrics of power production and distribution. It is valuable to have a sensitivity theory available for these problem areas since it is difficult to repeatedly solve complex non-linear equations to find out the effects of alternative input assumptions or the uncertainties associated with predictions of system behavior.

In Section II the sensitivity theory for a linear system of algebraic equations with constraints which can be solved using linear programming techniques is discussed. The role of the constraints in simplifying the problem so that sensitivity methodology can be applied is highlighted. In Sections II and IV the general non-linear method is summarized and applied to a non-linear programming problem in particular. Conclusions are drawn in Section V about the applicability of the method for practical problems.

II. A General Linear Programming Problem

A. Theory

To illustrate the sensitivity method for constrained systems, consider first a general linear problem. The aim in such a case is to find an extremum in a linear objective function $U(\bar{x})$ subject to a series of

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linear constraints $f_j(\bar{x})$ on the system state vector $\bar{x}$. Once the extremum is known, the extremum state vector $\bar{x}_e$ can be used to evaluate some particular system response function $R(\bar{x}_e)$. For this case (generally called the "forward problem"), the relevant functions can be written as follows:

$$U(\bar{x}) = \sum_{i=1}^{N} a_i x_i = \bar{a}\cdot\bar{x} \tag{1}$$

$$R(\bar{x}) = \sum_{i=1}^{N} b_i x_i = \bar{b}\cdot\bar{x} \tag{2}$$

with linearly independent constraints of the form

$$f_j(\bar{x}) = \sum_{i=1}^{N} c_{ji} x_i \geq d_j \qquad j=1,\ldots,J \geq N \tag{3}$$

A unique solution to the resulting equations for $\bar{x}_e$ is known from theory to be a vertex formed by the intersection of N of the J constraint functions.[9] This theorem is the basis for the linear programming (i.e., simplex) algorithms used to solve such problems.

Once the solution to the forward problem has been found and $R(\bar{x}_e)$ has been evaluated the following general sensitivity question can be posed: how will the system response change with changes in the system definition parameters (i.e., the a_i, b_i, d_i, and c_{ji}'s)? In terms of a system variable α, on which all the system parameters may depend, the sensitivity problem reduces to finding the sensitivity coefficient $dR(\bar{x}_e)/d\alpha$. Here, α can take on any number of different definitions depending on the particular sensitivity question being asked.

By virtue of the nature of the solution to the forward problem (i.e., $\bar{x}_e$ being a vertex) the set of equations which describe the behavior of $\bar{x}_e$ are simply the N independent constraint equations intersecting at $\bar{x}_e$. The extremum $\bar{x}_e$ is therefore a solution to the following system of N equations:

$$f_n(\bar{x}_e) = \sum_{i=1}^{N} c_{ni} x_{ei} = d_n \qquad n=1,\ldots,N \tag{4}$$

or alternately

$$\bar{f}(\bar{x}_e) = C\,\bar{x}_e = \bar{d} \tag{5}$$

where C is an NxN matrix with elements c_{ni}. To get Eq. (4) it has been assumed for notational simplicity that the N constraint surfaces essential to forming the vertex at $\bar{x}_e$ are the first N constraints of those given in Eq. (3). The linear programming solution of the forward equation eliminates the other non-essential constraints from consideration. Note also that the equations for $\bar{x}_e$ are independent of the parameters in the objective function, a general feature of all linear problems. Therefore, once the extremum vertex has been found, variations in the objective function have no effect on the response (i.e., $dR/da_i = 0$).

With the non-essential constraints eliminated from consideration, Eqs. (2) and (3) can be differentiated directly and conventional sensitivity methods applied to evaluate $dR/d\alpha$.[3] Direct differentiation yields:

$$\frac{\partial \bar{f}}{\partial \alpha} = \frac{\partial C}{\partial \alpha}\,\bar{x}_e + C\,\frac{\partial \bar{x}_e}{\partial \alpha} = \frac{\partial \bar{d}}{\partial \alpha} \tag{6}$$

$$\frac{dR}{d\alpha} = \frac{\partial \bar{b}}{\partial \alpha} \cdot \bar{x}_e + \bar{b} \cdot \frac{\partial \bar{x}_e}{\partial \alpha} \tag{7}$$

Letting $\bar{\phi} = [(\partial \bar{x}_e)/(\partial \alpha)]$, Eq. (6) can be rewritten as,

$$C\bar{\phi} = -\frac{\partial C}{\partial \alpha}\,\bar{x}_e + \frac{\partial \bar{d}}{\partial \alpha} \tag{8}$$

Equation (8) can be converted into an adjoint set of equations[3] to evaluate the second term on the right hand side of Eq. (7). Denoting the adjoint by $\bar{\phi}^*$, the final result is:

$$\frac{dR}{d\alpha} = \frac{\partial \bar{b}}{\partial \alpha} \cdot \bar{x}_e + \bar{\phi}^* \cdot \left(\frac{\partial \bar{d}}{\partial \alpha} - \frac{\partial C}{\partial \alpha}\,\bar{x}_e\right) \tag{9}$$

where $\bar{\phi}^*$ solves the following equation (generally called the "adjoint problem")

$$C^*\bar{\phi}^* = \bar{b} \qquad (10)$$

Here $\bar{\phi}^*$ is called the adjoint of $\bar{\phi}$, and the adjoint matrix C^*, is defined in conventional terms as:

$$\bar{\phi}^* \cdot C\bar{\phi} = \bar{\phi} \cdot C^*\bar{\phi}^* = \bar{\phi}^* \cdot \left(\frac{\partial \bar{d}}{\partial \alpha} - \frac{\partial C}{\partial \alpha}\,\bar{x}_e\right) = \bar{b}\cdot\bar{\phi} \qquad (11)$$

For this linear system of algebraic equations $C^* = C^{tr}$ (i.e., the transpose of C).

To find any sensitivity coefficient of interest, therefore, it is necessary to solve only a single adjoint equation [of the form of Eq. (10)] for each response. The solution is then used in Eq. (9) together with the explicit analytic derivaties $(\partial\bar{b}/\partial\alpha)$, $(\partial\bar{d}/\partial\alpha)$, and $(\partial C/\partial\alpha)$ evaluated at the extremum point solution of the forward problem $\bar{x}_e$. Note that the first term in Eq. (9) is the "direct effect" on R of variations in $\bar{b}$ with respect to α. The second term, called the "indirect effect," represents the variation of R as a result of variations in the extremum solution point $\bar{x}_e$ with respect to α. The indirect effect acts through variations in the matrix C (i.e., the constraints). Again no change in $U(\bar{x}_e)$ has any effect on R as was expected from the vertex nature of the solution for $\bar{x}_e$.

B. Example

A simple example which illustrates this method for a linearly constrained system is given by the following:

$$U(\bar{x}) = a_1x_1 + a_2x_2 \qquad (12)$$

$$R(\bar{x}) = b_1x_1 + b_2x_2 \qquad (13)$$

where $a_1 = 4$, $a_2 = 5$, $b_1 = 2$, $b_2 = -1$. The problem is to maximize $U(\overline{x})$ subject to the following constraints

$$f_1(\overline{x}) = c_{11}x_1 + c_{12}x_2 \leq d_1 \tag{14}$$

$$f_2(\overline{x}) = c_{21}x_1 + c_{22}x_2 \leq d_2 \tag{15}$$

$$f_3(\overline{x}) = x_1 \geq 0 \tag{16}$$

$$f_4(\overline{x}) = x_2 \geq 0 \tag{17}$$

with $c_{11} = 3$, $c_{12} = 7$, $d_1 = 10$, $c_{21} = 2$, $c_{22} = 1$, and $d_2 = 3$.

The solution to this forward problem is $x_1 = x_2 = 1$, $R(\overline{x}_e) = 1$, and $U(\overline{x}_e) = 9$. This identifies the vertex $\overline{x}_e = (1,1)$ as the extremum and the vertex intersection equations as

$$c_{11}x_1 + c_{12}x_2 = d_1 \tag{18}$$

$$c_{21}x_1 + c_{22}x_2 = d_2 \tag{19}$$

The sensitivity coefficient is therefore:

$$\frac{dR}{d\alpha} = \frac{\partial b_1}{\partial \alpha} x_1 + \frac{\partial b_2}{\partial \alpha} x_2 + x_1^* \left(\frac{\partial d_1}{\partial \alpha} - \frac{\partial c_{11}}{\partial \alpha} x_1 - \frac{\partial c_{12}}{\partial \alpha} x_2 \right) + x_2^* \left(\frac{\partial d_2}{\partial \alpha} - \frac{\partial c_{21}}{\partial \alpha} x_1 - \frac{\partial c_{22}}{\partial \alpha} x_2 \right) \tag{20}$$

where $x_1 = x_2 = 1$; $x_1^* = -4/11$ and $x_2^* = 17/11$; and $\overline{\varphi}^* = (x_1^*, x_2^*)$ is obtained by solving the following adjoint problem :

$$c_{11}x_1^* + c_{21}x_2^* = b_1 \tag{21}$$

$$c_{12}x_1^* + c_{22}x_2^* = b_2 \tag{22}$$

The results of this sample problem are summarized in Table I where the values of $dR/d\alpha$ are given as a function of α. The definition of α is assumed to be a different system parameter for each separate evaluation of Eq. (20).

For this problem, the response is most sensitive to variations in the value of c_{12}, with c_{11} and d_1 following close behind in importance. In terms of first order perturbations given approximately by:

$$\delta R/R = \frac{dR/R}{d\alpha/\alpha} \cdot \frac{\delta\alpha}{\alpha} \qquad (23)$$

a perturbation of 1% in c_{12} would result in a 4.6% change in R. If all the parameters had uncorrelated 1% uncertainties associated with them, an uncertainty of approximately 10.6% in R would be obtained from a sum of the squares of the individual uncertainties as follows:

$$\sigma_R = \sqrt{\sum_{i=1}^{I} \left(\frac{dR/R}{d\alpha_i/\alpha_i}\right)^2 \sigma^2_{\alpha_i}} \Bigg/ I = \frac{0.01}{8}\sqrt{\sum_{i=1}^{8} \left(\frac{dR/R}{d\alpha_i/\alpha_i}\right)^2} = 0.106 \qquad (24)$$

This response is clearly very sensitive to variations in the input parameters and is a good illustration of the value of performing a sensitivity analysis for the problem.

III. A General Non-Linear Problem

A. Forward Problem Solution Characteristics

The treatment required for a non-linear problem with general constraints varies according to type of solution found for the forward problem but the methods used follow closely the developments presented in the last section. Consider first the following problem, with non-linear objective and response functions $U(\overline{x})$ and $R(\overline{x})$ respectively, and J non-linear constraints $f_j(\overline{x})$ given as:

$$U(\bar{x}) = U(x_1,x_2,\ldots,x_N,a[\alpha]) \tag{25}$$

$$R(\bar{x}) = R(x_1,x_2,\ldots,x_N,b[\alpha]) \tag{26}$$

$$f_j(\bar{x}) = f_j(x_1,x_2,\ldots,x_N,c[\alpha]) \geq 0 \qquad j=1,\ldots,J \tag{27}$$

The forward problem is to find an extremum in the objective function [i.e., $U(\bar{x}_e)$] subject to the constraints and then to evaluate the response at the extremum point, $R(\bar{x}_e)$. Assuming that a solution to this problem exists and methods are available to solve for $\bar{x}_e$ (i.e., non-linear programming algorithms exist and work for this case), it is clear that the solution need not be a simple vertex. In general, the solution can be shown to be one of three possibilities:[9]

1) the objective function has an unconstrained extremum point which lies inside the region defined by the constraint surfaces, or
2) the objective function has an extremum point tangential to a surface defined by the constraints, or
3) the extremum is a simple vertex.

A separate sensitivity theory has to be developed for each of these eventualities with the forward solution being used again to identify the extremum point and constraint equations which are applicable.

B. Unconstrained Solutions

The case where the constraints play no role in determining the extremum [i.e., case (1)] is discussed first since the general non-linear sensitivity equations can be developed here for later use in constrained problems. For the unconstrained problem the solution to the forward problem reduces to finding an extremum in the objective function itself. That is, the following N simultaneous, non-linear equations must be solved:

$$g_n(\bar{x}_e) = \frac{\partial U(\bar{x})}{\partial x_n} = 0 \qquad n=1,\ldots,N$$

Assuming that a solution to this set of equations can be found, the sensitivity problem again entails finding the derivatives of R with respect to α evaluated at the extremum solution point $\bar{x}_e$. As in the linear case the first step is to differentiate R as a function of $\bar{x}_e$ directly to get:

$$\frac{dR}{d\alpha} = \frac{\partial R}{\partial b} \frac{\partial b}{\partial \alpha} + \frac{\partial R}{\partial \bar{x}_e} \cdot \frac{\partial \bar{x}_e}{\partial \alpha} \tag{29}$$

To get the needed values of $\partial \bar{x}_e / \partial \alpha$, Eq. (28) is differentiated to give:

$$\frac{\partial g_n}{\partial a} \frac{\partial a}{\partial \alpha} + \frac{\partial g_n}{\partial \bar{x}_e} \cdot \frac{\partial \bar{x}_e}{\partial \alpha} = 0 \qquad n=1,\ldots,N \tag{30}$$

or alternately in terms of $\bar{\phi} \equiv \partial \bar{x}_e / \partial \alpha$ and the matrix operator G with elements $\partial g_n / \partial x_{n'}$, we can write:

$$G\bar{\phi} = - \frac{\partial \bar{g}}{\partial a} \frac{\partial a}{\partial \alpha} \tag{31}$$

Here again it is assumed that R and U can be explicitly differentiated with respect to α and $\bar{x}_e$ to get the elements of the matrix G and the various derivatives of R.

Since many different representations of α will be used in a full sensitivity analysis, each new form of which would change only the nature of the source term in Eq. (31), an adjoint formulation of the equation is needed. It is possible to develop an equivalent adjoint problem since Eq. (30) is linear in $\bar{\phi}$. The matrix operator G is not a function of $\bar{\phi}$ and is in fact a matrix of constants, each element of which is a known derivative of some sort evaluated at the extremum point $\bar{x}_e$. These properties of the equation for $\bar{\phi}$ are quite general results of sensitivity theory, whereby non-linear forward equations give rise to linear sensitivity equations making the latter problems far easier to solve than the original problem.

Using an adjoint formulation of the problem results in a form for $dR/d\alpha$ which can be written as:

$$\frac{dR}{d\alpha} = \frac{\partial R}{\partial b}\ \frac{\partial b}{\partial \alpha} - \overline{\phi}^* \cdot \left(\frac{\partial \overline{g}}{\partial a}\ \frac{\partial a}{\partial \alpha} \right) \tag{32}$$

where $\overline{\phi}^*$ solves the following adjoint equation:

$$G^*\overline{\phi}^* = \frac{\partial R}{\partial \overline{x}_e} \tag{33}$$

and from the definition of the adjoint operator

$$G^*\overline{\phi}^*\cdot\overline{\phi} = \frac{\partial R}{\partial \overline{x}_e} \cdot \overline{\phi} = \overline{\phi}^*\cdot G\overline{\phi} = - \overline{\phi}^* \cdot \left(\frac{\partial \overline{g}}{\partial a}\ \frac{\partial a}{\partial \alpha} \right) \tag{34}$$

Note that G^* is an NxN square matrix with elements composed of various derivatives of the form $\partial g_{n'}/\partial x_n$.

Again it is clear that the solution of a single linear adjoint equation is all that is needed to evaluate the derivative of each response of interest with respect to all differential variations of the input parameters. The adjoint equation in fact is an algebraic set of coupled linear equations with constant coefficients in which only the source term depends on the response function R. If it were possible then to invert the G^* matrix explicitly, all response functions of interest could also be studied using the solution to only a single adjoint equation.

C. Constrained Solutions

For the constrained case, if the extremum point is a vertex the sensitivity problem is a simple extension of the developments in the previous section. That is, the equations for the vertex are given by the first N constraints, $f_n(\overline{x}_e) = 0$, $n=1,\ldots,N$ and the theory in Section III.B can be applied directly starting with Eq. (28) in which $g_n(\overline{x}_e) \equiv f_n(\overline{x}_e)$ for the present case. For the remaining case, in which the extremum is given by the tangential intersection of the objective function and a surface defined by the constraints, some additional developments are needed.

This problem is best posed by using Lagrange multipliers, k_i. The intersection of the extremum in $U(\bar{x})$ and constraint surfaces $f_i(\bar{x})$, $i=1,\ldots,I$ with $I < J$, is described by the extremum in a new objective function $H(\bar{x})$ given as:

$$H(\bar{x}) = U(\bar{x}) + \sum_{i=1}^{I} \kappa_i \, f_i(\bar{x}) \tag{35}$$

where from the definition of the constraint extremum vector and the constraint surfaces

$$H(\bar{x}_e) = U(\bar{x}_e) \tag{36}$$

since,

$$f_i(\bar{x}_e) = 0 \qquad i=1,\ldots,I \tag{37}$$

Again it has been assumed that the $i=1,\ldots,J$ constraints were ordered in such a fashion that those with $i > I$ are non-essential to the particular solution point $\bar{x}_e$.

It is clear then that the extremum point is a solution to the N+I set of simultaneous non-linear equations given by:

$$g_n(\bar{x}) = \frac{\partial H(\bar{x})}{\partial x_n} = 0 \qquad n=1,\ldots,N$$

$$g_{N+i}(\bar{x}) = \frac{\partial H(\bar{x})}{\partial k_i} = f_i(\bar{x}) = 0 \qquad i=1,\ldots,I \tag{38}$$

Solution of the general sensitivity problem now follows directly from the developments presented in the last section. In this case, however, there are N+I equations as opposed to just N equation before. The solution can therefore be written as:

$$\frac{dR}{d\alpha} = \frac{\partial R}{\partial b}\frac{\partial b}{\partial \alpha} - \bar{\phi}^* \cdot \left(\frac{\partial \bar{g}}{\partial a}\frac{\partial a}{\partial \alpha}\right) \tag{39}$$

where $\overline{\phi}^*$ now solves the linear adjoint equation

$$G^*\overline{\phi}^* = \frac{\partial R}{\partial \overline{x}_e} \tag{40}$$

Here the additional Lagrange variables k_i and the constraint surface equations $g_n(\overline{x})$, for $n > N$, imply a new notation of the following form:

$$\overline{x} = \overline{x}(x_1, x_2, \ldots, x_N, k_1, \ldots, k_I) \tag{41}$$

$$\overline{\phi} = \overline{\phi}\left(\frac{\partial x_1}{\partial \alpha}, \ldots\ldots, \frac{\partial x_N}{\partial \alpha}, \frac{\partial k_1}{\partial \alpha} \ldots \frac{\partial k_I}{\partial \alpha}\right) \tag{42}$$

so that the elements of G form an (N+I)x(N+I) matrix with $\overline{g}(\overline{x})$ defined in Eq. (38).

D. Example

As an illustration of the methods described in the last sections, consider the following non-linear problem with response $R(\overline{x})$ and objective function $U(\overline{x})$ (which is to be maximized):

$$U(\overline{x}) = a_1 x_1^{\alpha_1} x_2^{\alpha_2} \tag{43}$$

$$R(\overline{x}) = b_1 x_1 + b_2 x_2 \tag{44}$$

The system constraints are given by:

$$f_1(\overline{x}) = c_{11} x_1^{\beta_1} + c_{12} x_2^{\beta_2} + d_1 \leq 0 \tag{45}$$

$$f_2(\overline{x}) = x_1 \geq 0$$

$$f_3(\overline{x}) = x_2 \geq 0$$

where a_1, b_1, b_2, α_1, α_2, c_{11}, c_{12} = 1, β_1, β_2 = 2, and d_1 = -4. The constant response surface for this case is a simple straight line and the constant $f_1(\bar{x})$ constraint surface is a circle.

The specific values of the constants given, force the solution to lie on the constraint surface described by $f_1(\bar{x}) = 0$ and the Lagrangian formulation of the problem reduces to finding a maximum in $H(\bar{x})$ given by:

$$H(\bar{x}) = a_1 x_1^{\alpha_1} x_2^{\alpha_2} + k(c_{11} x_1^{\beta_1} + c_{12} x_2^{\beta_2} + d_1) \tag{46}$$

The three equations which the maximum point $\bar{x}_e$ is a solution to are therefore:

$$g_1(\bar{x}) = \frac{\partial H}{\partial x_1} = a_1 \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} + k c_{11} \beta_1 x_1^{\beta_1 - 1} = 0 \tag{47}$$

$$g_2(\bar{x}) = \frac{\partial H}{\partial x_2} = a_1 \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1} + k c_{12} \beta_2 x_2^{\beta_2 - 1} = 0 \tag{48}$$

$$g_3(\bar{x}) = \frac{\partial H}{\partial k} = c_{11} x_1^{\beta_1} + c_{12} x_2^{\beta_2} + d_1 = 0 \tag{49}$$

With the specific constant values given, the solution to this equation is $x_1 = x_2 = \sqrt{2}$, $R(\bar{x}_e) = 2\sqrt{2}$, $U(\bar{x}_e) = 2$, and $k = -1/2$.

Taking the derivative with respect to $\bar{x}$ of the functions and parameters in Eqs. (47)-(49) and using the definition of the matrix elements of G in terms of the derivatives $\partial g_n / \partial x_{n'}$, results in the following adjoint equation for $\bar{\phi}^*$.

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_3}{\partial x_1} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_3}{\partial x_2} \\ \frac{\partial g_1}{\partial k} & \frac{\partial g_2}{\partial k} & \frac{\partial g_3}{\partial k} \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \\ k^* \end{pmatrix} = \begin{pmatrix} \frac{\partial R}{\partial x_1} \\ \frac{\partial R}{\partial x_2} \\ \frac{\partial R}{\partial k} \end{pmatrix} \tag{50}$$

which in this case reduces to:

$$\begin{pmatrix} 2k & 1 & 2x_1 \\ 1 & 2k & 2x_2 \\ 2x_1 & 2x_2 & 0 \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \\ k^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{51}$$

A solution to this equation can be evaluated easily using the extremum values for x_1, x_2, and k found solving the forward problem. The result is $x_1^* = x_2^* = 0$, and $k^* = 1/(2\sqrt{2})$. These values allow the final expression for the general sensitivity coefficient to be written simply as:

$$\frac{dR}{d\alpha} = \sqrt{2}\ \frac{\partial b_1}{\partial \alpha} + \sqrt{2}\ \frac{\partial b_2}{\partial \alpha} - \frac{1}{2\sqrt{2}}\ \frac{\partial g_3}{\partial \alpha} \tag{52}$$

Results of sensitivity coefficient calculations for different definitions of α are given in Table II. The derivatives in Eq. (52) were evaluated analytically from the defining equations for $g_n(\overline{x})$ [i.e., Eqs. (47)-(49)]. These results make it clear that the response is sensitive to all the system input parameters except those which define $U(\overline{x})$. This is to be expected because of the symmetries imposed on the problem by the response and the circular constraint surface. The problem is also much less sensitive to the input parameters than the previous linear example.

IV. Summary and Conclusions

The developments presented make it clear that sensitivity theory can be extended successfully to cover a wide class of algebraic non-linear equations with and without constraints. A solution to the forward problem under investigation is the starting point for these developments. Once this solution is available all the derivatives needed to evaluate sensitivity coefficients can be reduced to a procedure for solving a single linear adjoint equation for each response of interest. This equation is easily solved since it is algebraic and has constant coefficients. In most cases (where the dimensionality of the resulting system of equations is not too large) the adjoint matrix operator containing the system constants can be inverted directly and all sensitivities for all responses can be evaluated from a single matrix inversion.

By analogy with sensitivity work on the radiation transport equation, the sensitivity coefficients made available by the methods developed here can be put to a number of important practical uses. For example, Taylor series expansions using the sensitivity coefficients (i.e., first derivatives) can be used as the basis for a second order accurate perturbation theory for the non-linear systems under investigation. In addition, a statistical uncertainty analysis of system responses can also be made if perturbation results are combined with assumptions about the nature of the uncertainties in the system input parameters. These results yield information about the level of confidence that can be placed in calculated system responses or projections of future behavior. Much work needs to be done in this area, especially for econometric and energy system modelling problems.

Table I. Sensitivity Coefficients of Linear Problem Response to Input Data

Input Parameter	$(dR/R)/(d\alpha/\alpha) \times 11$
$\alpha = a_1 = 4$	0
$= a_2 = 5$	0
$= b_1 = 2$	22
$= b_2 = -1$	-11
$= c_{11} = 3$	-40
$= c_{12} = 7$	51
$= c_{21} = 2$	12
$= c_{22} = 1$	28
$= d_1 = 10$	-34
$= d_2 = 3$	-17

Table II. Sensitivity Coefficients of Non-Linear Problem Response to Input Data

Input Parameter	$(dR/R)/(d\alpha/\alpha)$
$\alpha = a_1 = 1$	0
$= \alpha_1 = 1$	0
$= \alpha_2 = 1$	0
$= b_1 = 1$	0.5
$= b_2 = 1$	0.5
$= c_{11} = 1$	-0.25
$= c_{12} = 1$	-0.25
$= \beta_1 = 2$	-0.173
$= \beta_2 = 2$	-0.173
$= d_1 = -4$	0.5

References

1. A. Gandini, "A Generalized Perturbation Method for Bilinear Functionals of the Real and Adjoint Neutron Fluxes," J. Nucl. Energy 21, 755 (1967).

2. W. M. Stacy, Jr., Variational Methods in Nuclear Reactor Physics, Academic Press, N.Y. (1974).

3. D. E. Bartine, E. M. Oblow, and F. R. Mynatt, "Radiation Transport Cross-Section Sensitivity Analysis - A General Approach Illustrated for a Thermonuclear Source in Air," Nucl. Sci. Eng. 55, 147 (1974); see also E. M. Oblow, Nucl. Sci. Eng. 59, 187 (1976).

4. E. Greenspan, "Developments in Perturbation Theory," Advances in Nuclear Science and Technology, 9 (1977).

5. S. A. W. Gerstl, D. J. Dudziak, and D. W. Muir, "Cross-Section Sensitivity and Uncertainty Analysis with Application to a Fusion Reactor," Nucl. Sci. Eng. 62, 137 (1977).

6. D. R. Harris, "Sensitivity of Nuclear Fuel Cycle Costs to Uncertainties in Nuclear Data and Methods," Doctoral Thesis, Rennselaer Polytechnic Institute, Troy, N.Y. (October 1976).

7. M. L. Williams and C. R. Weisbin, "Sensitivity and Uncertainty Analysis for Linear Functionals of the Time-Dependent Nuclide Density Field," ORNL/TM-5799 (1977).

8. A. Gandini, "A Method of Correlation of Burnup Measurements for Physics Prediction of Fast Reactor Life," Nucl. Sci. Eng. 38, 1-7 (1969).

9. R. G. Geary and J. E. Spencer, Elements of Linear Programming, Griffin Press, London (1973).