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NEOCLASSICAL TRANSPORT IN AN ELLIPTIC TOKAMAK*

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ABSTRACT

Neoclassical transport for an elliptic tokamak in all collisional regimes is investigated by the technique of partitioning the velocity space. It is found that in a tokamak of moderate elongation, particle and ion heat confinement times are increased by a factor of σ^2, where σ is the ratio of vertical minor radius to horizontal minor radius. Ripple diffusion in an elliptic tokamak is also studied. Ion heat conductivity due to ripples is reduced by a factor of approximately σ^2.

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I. INTRODUCTION

It is now widely recognized that high beta (= plasma pressure/magnetic pressure) operation of any magnetically confined fusion device is a necessary ingredient in the design of an economical fusion system. High beta tokamak equilibria have characteristics very different from those usually assumed for low beta equilibria, e.g., the magnetic flux surfaces are no longer circular but are elongated in the vertical direction and triangular distortion is also introduced even if the outermost flux surface is kept circular. Usually, triangularity is not important except near the edge of the plasma, because it is a higher order flux surface distortion. For simplicity, we ignore triangularity and assume a system of elliptic flux surfaces as our model for the tokamak equilibrium. This model is fully described in Sec. 2.

The problem of neoclassical transport in an elliptic tokamak is a difficult one. So far, it has only been studied in various distinct collisional regimes.[1-3] Rigorous analysis based on the variational principle to calculate transport coefficients as smooth functions of collision frequency has not been carried out and is believed to be so complicated that it is probably not worth the effort. The calculation presented in Sec. 3 is an attempt to avoid this rigorous analysis by applying the technique of partitioning the velocity space[4] which was recently developed to reproduce standard neoclassical results. Using this method, the transport coefficients for all collision frequencies are obtained by solving the drift kinetic equation in different regions of velocity space and then combining the results. In principle, this technique can be applied with the exact collision operator. Again, for the sake of simplicity, we assume a Lorentz collision model which restricts our analysis to high Z (or $Z_{eff} >> 1$) plasma.

Although our present treatment is somewhat phenomenological, it should be emphasized that the purpose of this work is not to solve the neoclassical transport in high beta equilibria but to bring out the essential physics of neoclassical transport in elliptic tokamaks. The scaling of transport with respect to vertical elongation is obtained in Sec. 3. A physical explanation of this scaling is also given.

Besides the minimum loss neoclassical transport establishes for a tokamak plasma, another reason for studying neoclassical transport is that the ion heat conductivity it gives roughly agrees with experimental observation. However, at low collision frequency, ripple diffusion[5] dominates over neoclassical diffusion in a symmetric torus for ions. In Sec. 4, ripple diffusion in an elliptic tokamak is considered. Modification of Stringer's result[5] to account for ellipticity is evaluated.

II. THE EQUILIBRIUM MODEL

In (ψ, χ, ϕ) coordinates, where ψ is the poloidal flux and ϕ is the angle about the symmetry axis, the magnetic field can be written as $\vec{B} = I\vec{\nabla}\phi + \vec{\nabla}\phi \times \vec{\nabla}\psi$. I is a function of ψ only. In low beta situations, $I = RB_\phi$ is approximately constant. In a constant ϕ cross section, R represents the distance from the symmetry axis and z the vertical distance. An elliptic flux surface can be represented by

$$(R - R_c)^2 + z^2/\sigma^2 = \rho^2,$$

or by

$$R - R_o = \rho \cos \omega,$$

$$z = \sigma\rho \sin \omega,$$

where R_o designates the position of the magnetic axis, ρ is a flux surface variable, ω is a poloidal angle variable, and σ is the elongation of the ellipse in the z direction. In the most general case, both R_o and σ are functions of ρ, which corresponds to a system of nested but nonconcentric ellipses with varying ellipticity. For reasons of simplicity, we assume R_o and σ are independent of ρ, thus restricting ourselves to a system of concentric elliptic flux surfaces with a fixed ellipticity. Though this magnetic field model for finite beta tokamak equilibrium is not quite exact, it is nonetheless useful for our present purpose, since the effect of the shift of the center of the flux surface is small for low beta equilibria.

The safety factor q is then calculated by using $q = I(\oint dl/B_p R^2)/2\pi$. Neglecting terms of order ε^2, we have $q = \rho\sigma/R_o\Theta$, where $\Theta \equiv I^{-1}\, d\psi/d\rho$, and $\varepsilon = \rho/R_o$.

III. NEOCLASSICAL TRANSPORT

The gyrophase-averaged distribution function f satisfies

$$\frac{\partial f}{\partial t} + (V_{\parallel}\hat{n} + \vec{V}_D) \cdot \vec{\nabla} f = C(f).$$

Neglecting high beta effects, the drift velocity $\vec{V}_D$ is given by $\vec{V}_D = -V_{\parallel}\ \hat{n} \times \vec{\nabla}\ (V_{\parallel}/\Omega)$, where $\hat{n} = \vec{B}/B$ and C denotes the collision operator. In terms of coordinates (ρ, ω, ϕ), the drift kinetic equation can be written as[1]

$$\frac{\partial f}{\partial t} + \frac{V_{\parallel}\Theta}{\sigma\rho}\frac{B_\phi}{B}\frac{\partial f}{\partial \omega} + \frac{V_{\parallel}}{\sigma\rho}\frac{B_\phi}{B}\left(\frac{\partial}{\partial\omega}\frac{V_{\parallel}}{\Omega}\frac{\partial f}{\partial\rho} - \frac{\partial}{\partial\rho}\frac{V}{\Omega}\frac{\partial f}{\partial\omega}\right) = C(f), \tag{1}$$

where $B_\phi \equiv I/R$. If we assume $B \approx B_\phi$, then the integral operator that annihilates the bounce term in Eq. (1) has the same form as that in the circular case. Integrating this equation over velocity space and performing the flux surface average $\int \partial\omega\ h/2\pi$, we get the continuity equation,

$$\frac{\partial}{\partial t}\langle N\rangle = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\Gamma_\rho,$$

where

$$\Gamma_\rho = \langle \int d\vec{V}\ V_{D\rho}\ f\rangle,$$

$$V_{D\rho} = \frac{V_{\parallel}}{\sigma\rho}\frac{B_\varphi}{B}\frac{\partial}{\partial\omega}\frac{V_{\parallel}}{\Omega}, \tag{2}$$

$$h = 1 + \rho\cos\omega/R_o,$$

and $\langle A\rangle = \oint d\omega\ Ah/2\pi$.

Similarly, we have the heat conduction equation

$$\frac{\partial}{\partial t}\left(\frac{3}{2}NT\right) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\dot{Q}_\rho,$$

where $Q_\rho = \langle \int d\vec{V}\ V_{D\rho}\ \frac{1}{2}mv^2 f\rangle$. (3)

As in standard neoclassical theory, we assume that the lowest order solution of Eq. (1) is a Maxwellian distribution F constant on the flux surface,

$$F = N\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right),$$

where $N = N(\rho)$, $T = T(\rho)$.

Equation (1) then reduces to

$$\frac{V_\parallel \Theta}{\rho} \frac{\partial}{\partial \omega} + \frac{V_\parallel}{\rho} \frac{\partial}{\partial \omega} \frac{V_\parallel}{\Omega} \frac{\partial F}{\partial \rho} = \frac{B}{B_\phi} \sigma \, C(f). \tag{4}$$

Equation (4) has the same form as the corresponding equation for a circular tokamak except for the extra factor $\sigma B/B_\phi$ in front of the collision term. This equation can be solved asymptotically in three different regions in the velocity space.

In the electron equation, we ignore the electron-electron collision for convenience. This implies that the effective atomic number of the plasma is so large that electron-ion collisions dominate. The collision term is then:

$$C(f_e) = \nu_e(v) \frac{V_\parallel}{B} \frac{\partial}{\partial \mu} \mu V_\parallel \frac{\partial f_e}{\partial \mu}$$

where μ is the magnetic moment, $\nu_e(v) = \nu_e(V_e/V)^3$, and V_e is the electron thermal velocity. In the trapped region where $V \sim \varepsilon^{1/2} V$, the collision term can be neglected if

$$\varepsilon^{1/2} V\Theta/\rho >> \sigma\nu_e \, (V_e/V)^3/\varepsilon$$

or

$$V/V_e >> (Rq \, \nu_e/\varepsilon^{3/2} \, V_e)^{1/4} \equiv \nu_{*e}^{1/4}. \tag{5}$$

In the region where the inequality of Eq. (5) is satisfied, we can solve Eq. (4) as in Rosenbluth *et al.*[6] and obtain to lowest order

$$f_e = -\frac{F_e}{\Omega_e \Theta}\left\{\left[\frac{N'}{N} - \left(\frac{3}{2} - \frac{m_e v^2}{2T_e}\right)\frac{T'_e}{T_e}\right] v_\parallel + H(\lambda_c - \lambda)\left[\left(1 + \frac{T'_i}{T_e}\right)\frac{N'}{N}\right.\right.$$

$$\left.\left. - \left(\frac{3}{2} - \frac{m_e v^2}{2T_e}\right)\frac{T'_e}{T_e} - 0.172\,\frac{T'_i}{T_e}\right]\frac{v^2}{2}\int_{\lambda_c}^{\lambda}\frac{d\lambda}{\langle V_\parallel\rangle}\right\}, \tag{6}$$

where $\lambda = 2\mu B_o/v^2$ and $\lambda_c = 1 - \rho/R_o$. We notice that to this lowest order in collision frequency, ellipticity makes no difference except in $\langle V_\parallel\rangle$, because the collision term can be neglected and in the next order acts as a constraint to determine the flux surface constant part of f_e. When we invoke the assumption that the poloidal magnetic field can be neglected compared with the toroidal field, then even $\langle V_\parallel\rangle$ is the same for elliptic and circular tokamaks. This assumption is valid when the poloidal beta is less than or on the order of unity. However, even with the inclusion of the poloidal field in the magnitude of B, the result would not be changed by a factor of order more than unity.

When the effective collision frequency of a trapped electron is larger than its bounce frequency but the transit frequency of a circulating electron is still larger than its collision frequency, i.e.

$$\sigma\nu_e(v_e/v)^3 \ll v\Theta/\rho \ll \sigma\,\nu_e\,(v_e/v)^3\,\varepsilon^{-3/2},$$

or

$$\varepsilon^{3/8}\,\nu_{*e}^{\;1/4} \ll V/V_e \ll \nu_{*e}^{\;1/4}, \tag{7}$$

the electrons are in the so-called "plateau regime." In this region of the velocity space, the standard technique for solving Eq. (4) is to approximate C(f) by $(1/2)\nu v^2\partial^2 f/\partial v_\parallel^{\,2}$ and get[7]

$$\frac{V_{\parallel}\Theta}{\rho}\frac{\partial f_e}{\partial\omega} - \frac{\sin\omega}{\Omega_e R_o}\left(V_{\parallel}{}^2 + \frac{1}{2}V_{\perp}{}^2\right)\frac{\partial F_e}{\partial\rho} = 1/2\ \sigma\nu_e\ (V)V^2\frac{\partial^2 f_e}{\partial V_{\parallel}},$$

with a solution

$$f_e = \frac{\varepsilon V^2}{2\Theta\Omega_e}\frac{\partial F_e}{\partial\rho}\int_0^{\infty} dp\ \exp\ (-p^3\ \rho\sigma\nu_e V^2/6\Theta)\ \sin\ (\omega - V_{\parallel}p). \tag{8}$$

In the low energy region of the velocity space, when the collision frequency is larger than the transit frequency, i.e.

$$\sigma\nu_e\ (V_e/V)^3 >> V\Theta/\rho$$

or

$$V/V_e << \varepsilon^{3/8}\ \nu_{*e}{}^{1/4}, \tag{9}$$

we can solve Eq. (4) by ordering C(f) as the dominant term[6]. The solution in this region is[4]

$$f_e = \frac{N_1}{N}\ Fe + \frac{T_1}{T_e}\left(\frac{m_e v^2}{2T_e} - \frac{5}{2}\right)F_e \tag{10}$$

where

$$\frac{N_1}{N} = +\frac{3\pi}{32}\frac{\sigma\rho^2}{\Theta^2\tau_e}\left(\frac{25}{8}\frac{N'}{N} + \frac{5}{4}\frac{T'_e}{T_e}\right)\frac{\sin\omega}{R_o\Omega_e}$$

$$\frac{T_1}{T_e} = \frac{3}{32}\frac{\sigma\rho^2}{\Theta^2\tau_e}\left(\frac{3}{4}\frac{N'}{N} - \frac{1}{2}\frac{T'_e}{T_e}\right)\frac{\sin\omega}{R_o\Omega_e},$$

and

$$\tau_e = \frac{3m_e{}^{1/2}\ T_e{}^{3/2}}{4(2)^{1/2}\ e^4\ ZN\ \ell n\ \Lambda}.$$

Finally we have the generalization of the correction due to the boundary layer between trapped and untrapped electrons[7], which can be expressed as

$$Q_{11} = Q_{12} \approx 3.07 \left| 1 - 0.92 \, (\nu')^{1/2} \right| ,$$

where Q_{11} and Q_{12} are defined through

$$\Gamma = - \frac{4\varepsilon^{1/2}}{\sigma\rho\Omega_e^{\,2}\Theta} \int v dv \left(\frac{v^2}{2}\right)^2 F_e \nu Q_{11} \left(\frac{N'}{N} - \frac{3}{2}\frac{T'_e}{T_e}\right) - \frac{4^{1/2}}{\sigma\rho\Omega_e^{\,2}\Theta}\frac{T'_e}{T_e} \int v dv \left(\frac{v^2}{2}\right)^3 F_e \nu Q_{12} ,$$

$\nu' = \nu\varepsilon^{-3/2}/2$, and $\nu = 2\rho\sigma\nu_e(v)/\Theta v$.

After carrying out the velocity integration in Eq. (2) by using the appropriate f_e from Eqs. (6), (8), and (10) in the regions specified by inequalities (5), (7), and (9), we obtain an expression for the particle flux:

$$\Gamma = -N\varepsilon^{-3/2} \sigma^{-2} q^2 \rho_e^{\,2} \tau_e^{-1} \left| K_{11} \left(\frac{N'}{N} - \frac{3}{2}\frac{T'_e}{T_e}\right) + K_{12} \, T'_e/T_e \right| , \qquad (11)$$

where K_{11} and K_{12} have the same form as in Ref. 4,

$$K_{11} = 0.73e^{-Xc} \left| 1 - 0.90 \, \nu_{*e}^{\,1/2} e^{Xc} E_1 \, (Xc) \right| + \frac{(2\pi)^{1/2}}{8\nu_{*e}} \left| e^{-y_c}(2 + 2y_c + y_c^{\,2}) - e^{-Xc} (2 + 2Xc + Xc^2) \right| + \frac{\pi}{32} \varepsilon^{3/2} \left| \frac{75}{16} \mathrm{erf} \; y_c^{\,1/2} - \frac{e^{-y_c}}{\pi^{1/2}} y_c^{\,1/2} \left(\frac{75}{8} + \frac{25}{4} y_c - \frac{3}{2} y_c^{\,2}\right) \right| .$$

$$K_{12} = 0.73e^{-Xc} \left| 1 + Xc - 0.90 \, \nu_{*e}^{\,1/2} \right| + \frac{(2\Pi)^{1/2}}{8\nu_{*e}} \left| e^{-y_c}(6 + 6y_c + 3y_c + y_c^{\,3}) - e^{-Xc} (6 + 6Xc + 3Xc + Xc^3) \right| + \frac{\pi\varepsilon^{3/2}}{32} \left| \frac{285}{32} \mathrm{erf} \; y_c^{\,1/2} - \frac{e^{-y_c}}{\pi^{1/2}} y_c^{\,1/2} \left(\frac{285}{16} + \frac{95}{8} y_c - \frac{5}{4} y_c^{\,2}\right) \right| ,$$

$$x_c = 0.97\nu_{*e}^{1/2} ,$$

$$y_c = 0.97\varepsilon^{3/4} \nu_{*e}^{1/2} ,$$

$$E_1(x) \equiv \int_x^\infty e^{-t}\, dt/t .$$

Equation (11) states that if q is held constant before and after elongation, the particle loss will decrease as σ^{-2}. There is a factor of $\pi/2E\left[(\sigma^2-1)/\sigma^2\right]$ difference between Eq. (4) and Refs. 2 and 3 because of the difference of the flux surface variables used and the definitions of q. But this presents only a factor of 1 to 1.57 difference as σ varies from one to infinity. Similar calculation shows that the neo-classical ion heat confinement time also increases by a factor of σ^2 in an elliptic tokamak as compared to the circular case if q is kept constant.

The $1/\sigma^2$ reduction of neoclassical particle and heat fluxes due to ellipticity has a simple physical explanation. From Eqs. (2) and (3), these fluxes are proportional to the normal component of magnetic drift velocity to the flux surface, v_{Dp}, and the distortion of the distribution function due to this drift which is also proportional to v_{Dp}. When the flux surface changes from a circle to an ellipse with its major axis in the vertical direction, V_{Dp} is on the average reduced by $1/\sigma$ because the magnetic drift is also in the vertical direction. Thus, there is a total $1/\sigma^2$ reduction of the neoclassical particle and heat fluxes if the flux surface changes from a circle to an ellipse keeping the same ρ, R_o, and q.

IV. RIPPLE DIFFUSION

In investigating the effect of ripples in an elliptic tokamak, poloidal field cannot be ignored compared with toroidal field because the size of ripples is even smaller. Therefore, we write

$$B \simeq B_o \left[1-\varepsilon \cos \omega + \frac{1}{2}\left(\frac{\varepsilon}{q}\right)^2 + \frac{\varepsilon A}{2} \cos^2 \omega - \delta(\rho,\omega) \cos n\phi\right],$$

where $A \equiv (\sigma^2 - 1 + 2q^2)/q^2$, $\delta(\rho,\omega)$ is the size of the ripples, and n is the number of toroidal field coils. Following Stringer,[5] we find the effective ripple well depth along the field line to be

$$\Delta(\omega) = \frac{B_{max} - B_{min}}{B_o}$$

$$= 2\delta \left[\sqrt{1-\alpha^2 g^2} - \alpha\{\frac{\pi}{2} - \sin^{-1} (\alpha|g|)\}|g|\right], \qquad (12)$$

where $g = \sin \omega\ (1-A \cos \omega)h$ and $\alpha = \varepsilon/nq\delta$.

The drift kinetic equation can be written as

$$\frac{V}{R}\frac{\partial f}{\partial \phi} + V_{D\rho}\frac{\partial F}{\partial \rho} = \nu \frac{V}{B_\phi}\frac{\partial}{\partial \mu}\mu V \frac{\partial f}{\partial \mu},$$

where $V_{D\rho} = -\dfrac{B_o \varepsilon \cos \omega\ (1-A \cos \omega)}{\sigma\rho\Omega B}(V^2 + \mu B)$.

This equation can be solved in the same way as the previous calculations to obtain

$$f = \frac{1}{\nu}\frac{\partial F}{\partial \rho}\frac{mc}{e\sigma R_o}\left(\mu - \frac{E}{B_{max}}\right)\frac{B_\phi B_o}{B^2} \sin \omega\ (1-A \cos \omega).$$

Substituting this result in Eq. (2), we get

$$\Gamma_{i\rho} - \frac{64}{9}\frac{\delta o^{3/2}\ I(\alpha o)}{(2\pi)^{3/2}}\left(\frac{cT_i}{eB_o R\sigma}\right)^2 \frac{27.42}{\nu_{ii}}\left[\frac{N'}{N} + \frac{e\phi'}{T_i} + 3.37\frac{T'_i}{T_i}\right],$$

$$\Gamma_{e\rho} = -\frac{64}{9}\frac{\delta_o^{3/2}\, I(\alpha_o)}{(2\pi)^{3/2}}\left(\frac{cT_e}{eB_oR\sigma}\right)^2 \frac{12.78}{\nu_{ei}}\left[\frac{N'}{N} - \frac{e\phi'}{T_e} + 3.45\frac{T'_e}{T_e}\right].$$

After the radial potential is determined by quasi-neutrality, Eq. (3) leads to

$$Q_{i\rho} = -46.5\,\frac{\delta_o^{3/2}\, I(\alpha_o)}{\nu_{ii}}\left(\frac{cT_i}{eB_oR\sigma}\right)^2 T'_i\ . \tag{13}$$

In the above expressions, $I(\alpha_o)$ is the generalization of Stringer's $I(\alpha)$,[5]

$$I(\alpha_o) = \frac{1}{\pi}\int_0^{2\pi} d\omega \left(\frac{B_o}{B}\right)^{13/2}\left(\frac{\Delta}{2\delta_o}\right)^{3/2} \sin^2\omega\,(1-A\cos\omega)^2\ , \tag{14}$$

where $\delta(\rho,\omega) \equiv \delta_o(\rho)\,\xi(\omega)$ and $\alpha_o = \varepsilon/nq\delta_o$.

In Eq. (14) the poloidal variation of δ is also kept. The factor of $(B_o/B)^{13/2}$ in the integrand of Eq. (14) is neglected in most calculation because $B_o \sim B$ is assumed.

When the ω dependence of δ is neglected Eq. (14) has been evaluated numerically for different values of σ and $\varepsilon = 1/3$ (corresponding to the edge of a tokamak reactor plasma where the effect of ripples is largest). The result is shown in Fig. 1, from which we notice that $I(\alpha)$ is not very sensitive to σ. Thus, the effect of ellipticity of ion heat conduction due to ripples is mainly contained in the factor σ^{-2} in Eq. (9). Stringer's result,[5] which corresponds to $\sigma = 1$ and $B_o \simeq B$, is shown by the dashed line in Fig. 1.

V. CONCLUSIONS

Summarizing the results of this paper, we conclude that (i) neoclassical transport in an elliptic tokamak will be smaller than in a circular tokamak by a factor of σ^{-2}, where σ is the elongation of the ellipse in the vertical direction if the safety factor q is kept constant, and (ii) ion heat conductivity due to toroidal field ripples is also reduced by a factor of σ^{-2} provided that the ripple size is constant.

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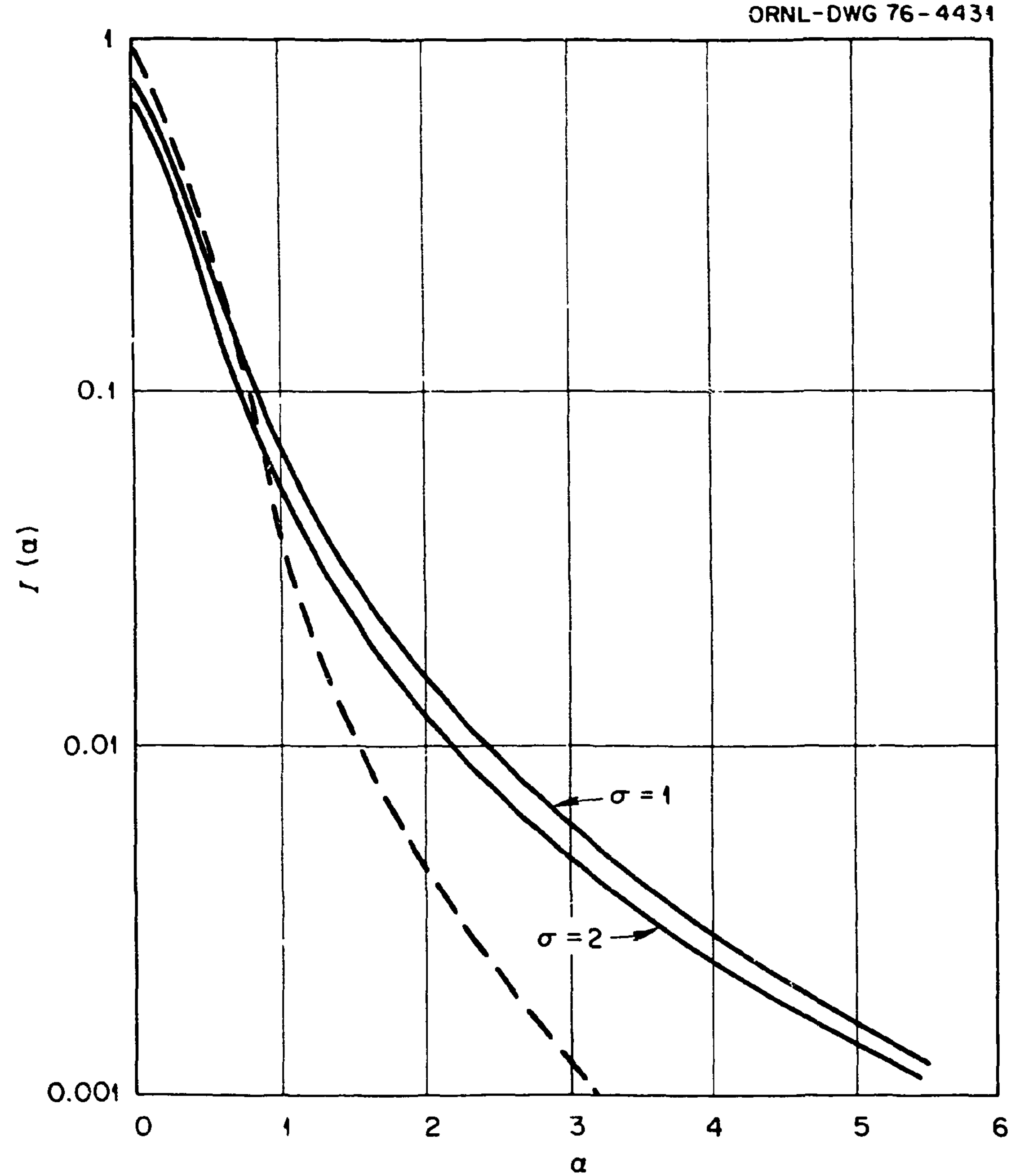

Fig. 1. I(α) factor resulting from reduction of ripple well depth by effects of toroidal variation and elliptic flux surfaces. For comparison, Stringer's result (from Ref. 5) is shown by the dashed line.