Journées mathématiques sur les perturbations singulières et la théorie de la couche limite. Lyon, France, 8-10 décembre 1976

CEA-CONF--3889

FR77C1777

COMPRESSIBLE COUNTERCURRENT FLOW IN A STRONGLY ROTATING CYLINDER

by Pierre LOUVET and Jean DURIVAULT
Division de la Chimie - CEN/Saclay
B.P. n° 2 - 91190 GIF-sur-YVETTE - FRANCE

Communication presented to the 'Symposium on singular perturbation problem and boundary layer theory'

LYON 8-9-10 Décembre 1976

COMPRESSIBLE COUNTERCURRENT FLOW IN A STRONGLY ROTATING CYLINDER

by Pierre LOUVET and Jean DURIVAULT
Division de la Chimie - CEN/Saclay
B.P. n° 2 - 91190 GIF-sur-YVETTE - FRANCE

ABSTRACT

The motion of a compressible viscous gas in a rapidly rotating cylinder closed at both ends is investigated by the linear theory. The rigid rotation state is perturbed slightly by source or sinks and by thermal gradients. The method of matched asymptotic expansions is used to find uniform solutions in powers of the Ekman number. In the Stewartson 1/3 layer along the side wall, the Ekman number ε at power 1/3 is taken of the same order of magnitude as the inverse of the square Mach number M ; this allows to take correctly into account the radial compressibility effect, contrary to previous works. This method is also applied to detached layers and to Stewartson 1/4 layers with $\varepsilon^{1/4} M^2 = O(1)$. The pattern of the flow in these layers is strongly altered as compared to incompressible case.

1. INTRODUCTION

The study of compressible fluid flow in a rapidly rotating cylinder is important in relation to gas centrifuges used for the enrichment of gaseous isotopic mixtures such as uranium hexafluoride UF_6.

Knowledge of velocity distribution in a gas centrifuge is of essential importance for the design and performances of the machines, as no attempt to measure velocity profiles have succeeded. Inside the rotating cylinder, the equilibrium state of rigid rotation is perturbed weakly so that the motion can be investigated by the linear theory, according to the classification of Greenspan [1]. The rigid rotation can be perturbed by prescribing a proper temperature distribution on the bounding walls : the top cover is maintained at a slightly higher temperature than the bottom cover.

For this thermally driven flow, Barcilon and Pedlowski [2], Hunter [3], Homsy and Hudson [4] did the pioneering work in the frame of Boussinesq approximation and used matched asymptotic expansions. However, due to the high value of the rotation speed, the strong radial stratification of pressure and density, associated with the predominance of the Coriolis force render this approximation invalid. Compressible linear analysis have been performed by Sakurai and Matsuda [5], Nakayama and Usui [6], Durivault and Louvet [7], but the compressibility is taken imperfectly into account in the Stewartson layers [8] along the side wall : the assumption [6] of a linear density variation does not remain valid for high speeds of rotation.

In this paper, as previously, we apply boundary layer analysis to seek the axisymmetric steady solution in thermally countercurrent centrifuge as well as countercurrent centrifuge mechanically driven by inlets and outlets. This last scheme has been already investigated in simpler cases by Matsuda and al. [9] for weak injections and by Nakayama and Usui [6] for strong injections. The emphasis is set here on the domain of validity of the linearization especially on transport terms in the flow and energy equations. We investigate successively all the parts of the flow. The motion in the inner core is degenerated and leads to a constant axial velocity. In the most general case, we recall the main results obtained for Ekman layers on top and bottom plates by various authors:Lotz [10] for variable temperature profiles and Matsuda [9] with injections and samples. The Stewartson layers, along the side wall or the injection layers (fig. 1), are of primarily importance because they remixed the already separated isotopes ; our analysis proposes a new method based on the choice of an appropriate order of magnitude between the compressibility and viscous forces. In the Stewartson 1/4 and detached layers, the authors have found a solution by the same method.

2. FUNDAMENTAL EQUATIONS AND SCALING ANALYSIS

2.1. Fundamental equations

We consider the steady motion of a compressible viscous gas in a cylinder of radius R and height 2h, rotating about its axis with a constant angular velocity ω (fig. 1). Our analysis is performed on the basis of the following assumptions :

- the flow is axisymmetric $\frac{\partial}{\partial \theta} = 0$.
- the gas is perfect and its transport coefficients are constant.
- the gravitational acceleration is negligibly small compared to the centrifugal acceleration.
- the shape factor $\beta = h/R$ is of the order unity.

The rigid unperturbed rotation state (subscript o) is given by :

$$V_{\theta 0} = \omega R, \quad V_{zo} = 0, \quad V_{ro} = 0$$

$$\frac{P_0(r)}{P_0(0)} = \frac{\rho_0(r)}{\rho_0(0)} = e^{S^2 r^2} \qquad (1)$$

where r is the dimensionless radial coordinate, p the pressure, ρ the density, V_θ, V_r and V_z the azimuthal, radial and axial components of velocity and S the speed ratio

$$S = \frac{\omega R \sqrt{M}}{\sqrt{2 \mathcal{R} T_o}} = \sqrt{\frac{\gamma}{2}}\, \mathcal{M} \qquad (2)$$

where $\mathcal{R}$ is the gas constant, T the temperature, M the molecular mass, γ the heat specific ratio and $\mathcal{M}$ the Mach number. In order to investigate the perturbed motion, we proceed to the change of variables :

$$v_z = \frac{Vz}{\alpha\ \omega R}\ ,\quad v_r = \frac{Vr}{\alpha \omega R}\ ,\quad v_\theta = \frac{V_\theta}{\alpha\ \omega R} - \frac{r}{\alpha}$$

$$p = \frac{P}{\alpha\ P_o(r)} - \frac{1}{\alpha}\ ,\quad \rho = \frac{\rho^*}{\alpha\ \rho_o(r)} - \frac{1}{\alpha} \tag{3}$$

$$T = \frac{T^*}{\alpha\ To} - \frac{1}{\alpha}$$

The dimensionless number α analoguous to a Rossby number denotes the importance of the perturbation. The set of Navier-Stokes equations including energy equation supplemented by an equation of state is the basic system of the problem. It is written here in the rotating frame

$$\nabla \cdot [\ \rho_o\ (1 + \alpha\ \rho)\ \vec{u}\ ] = 0 \tag{4}$$

$$\alpha\ \left[\frac{1}{2}\ \nabla(\vec{u}.\ \vec{u}) + (\ \nabla \times \vec{u})\ \times \vec{u}\right] + (\vec{2k}\ \times \vec{u})\ - T\ \vec{k} \times\ (\vec{k} \times \vec{r})$$
$$= \left[-\frac{1}{2S^2}\ \nabla p\ + 2\varepsilon\ \nabla .\ \bar{D}\ \right]\frac{1}{1 + \alpha\ \rho} \tag{5}$$

$$\frac{\gamma}{\gamma - 1}\ \alpha\ \left[\vec{u}.\nabla T\right](1 + \alpha\rho) = \alpha\ (\vec{u}.\nabla\ p\ ) + 2S^2 r v_r\ \ (1 + \alpha p)\ + \frac{2\gamma}{\gamma - 1}\ \frac{\varepsilon}{Pr}\ \nabla^2\ T$$
$$+ 4\ \alpha\varepsilon\ S^2\ \emptyset \tag{6}$$

where $\bar{D}$ is the rate of shear tensor, $\varepsilon = \mu\ /\ (2\ \dot{\rho}o(r)\ \omega\ R^2)$ the Ekman number, $Pr = \mu\ C\ p/K$ the Prandtl number, and $\vec{u}(\ v_\theta,\ v_r,\ v_z)$ the velocity in the rotating frame and where the viscous dissipation $\emptyset$ is given by :

$$\emptyset = 2\left[\ (\frac{\partial v_z}{\partial z})^2 + (\ \frac{\partial v_r}{\partial r})^2 + \frac{v_r^2}{r^2}\ \right] + (\ \frac{\partial v_\theta}{\partial z}\ )^2 + (\ \frac{\partial\ v_r}{\partial\ z} + \frac{\partial v_z}{\partial r})^2 \tag{7}$$
$$+\ (\frac{\partial v_\theta}{\partial r}\ - \ \frac{v_\theta}{r}\ )^2 - \frac{2}{3}(\frac{\partial v_z}{\partial z} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r})^2$$

Typical values are, at very high speed $\omega R = 800$ m/s

$$\varepsilon\ (R) = 1{,}6\ 10^{-9} << 1$$

$$\left.\begin{aligned} S^2 &= 40{,}7 \\ \gamma - 1 &= 0{,}065 \end{aligned}\right\}\ \text{so that}\ (\ \gamma - 1)\ \ S^2 = 0(1)$$

$$Pr\ = 1,\ \ \beta = 5,\ \alpha = 0\ \ \text{to}\ 5\ 10^{-2} << 1 \tag{8}$$

2.2. Linearized equations

When terms of order α are neglected in front of terms of order unity, the linearization leads to a much simpler system :

$$\frac{\partial v_z}{\partial z} + \frac{1}{r}\ (\frac{\partial r\ v_r}{\partial r}) + 2\ S^2\ r\ v_r\ = 0 \tag{9}$$

$$\frac{1}{2S^2}\frac{\partial p}{\partial z} = 2\varepsilon\left(\frac{4}{3}\frac{\partial^2 v_z}{\partial z^2} + \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{3}\frac{\partial^2 v_r}{\partial r\,\partial z} + \frac{1}{3r}\frac{\partial v_r}{\partial z}\right) \quad (10)$$

$$\frac{1}{2S^2}\frac{\partial p}{\partial r} - 2v_\theta + rT = 2\varepsilon\left(\frac{4}{3}\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{3}\frac{\partial^2 v_z}{\partial r\,\partial z} + \frac{\partial^2 v_r}{\partial z^2}\right) \quad (11)$$

$$v_r = \varepsilon\left(\frac{\partial^2 v_\theta}{\partial z^2} + \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right) \quad (12)$$

$$S^2 r\, v_r + \frac{\varepsilon}{Pr}\,\frac{\gamma}{\gamma-1}\left[\frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] = 0 \quad (13)$$

Perfect gas law $p = \rho + T$

where $z = \frac{Z}{R}$ is the dimensionless axial coordinate. (14)

The domain of validity of linearization is discussed in §2.5.

2.3. Boundary conditions

The usual zero velocity condition is taken at the wall of the centrifuge in absence of inlet or outlet. Otherwise, the axial velocity is equal to the inlet or outlet velocity ; these inlets and outlets can be approximated by sources and sinks in first approximation in order to calculate detached layers. The walls are assumed perfectly conducting. Their temperature is prescribed to be $Tw(z)$ for the side wall and $\pm$ $Tf(r)$ for the covers. The boundary conditions reads :

on the side wall : $r = 1$

$$T(1,z) = Tw(z), \quad v_\theta(1,z) = v_r(1,z) = v_z(1,z) = 0 \quad (15)$$

on the top cover : $z = \beta$

$$T(r,\beta) = Tf_{(r)}, \quad v_\theta(r,\beta) = v_r(r,\beta) = 0$$

$$v_z(r,\beta) = v_{zw}(r,\beta) = \begin{cases} 0 & \text{wall} \\ viz\ (r) & \text{for the } i^{th} \text{ inlet or outlet} \end{cases} \quad (16)$$

on the bottom cover: $z = -\beta$

$$T(r,-\beta) = -Tf(r), \quad v_\theta(r,-\beta) = v_r(r,-\beta) = 0$$

$$v_z(r,-\beta) = v_{zw}(r,-\beta) = \begin{cases} 0 & \text{at the wall} \\ viz & \text{for the } i^{th} \text{ port of injection} \end{cases} \quad (17)$$

2.4. Method of resolution

The Ekman number appears as a singular parameter in the Navier-Stokes equations. A uniform approximate solution is found by matched asymptotic expansion method. The outer expansion of a quantity g

$$g(r,z,\epsilon) = g^{(o)}(r,z) + \epsilon^{1/2} g^{(1)}(r,z) + \epsilon\, g^{(2)}(r,z) + \dots \quad (18)$$

corresponds to the main inner flow. Two inner expansions which corresponds to the boundary layers called the Ekman layers, of thickness $\epsilon^{1/2}$, along the top and bottom cover, read

$$g(r,z,\epsilon) = \bar{g}(r, \zeta, \epsilon) = \bar{g}^{(o)}(r, \zeta) + \epsilon^{1/2}\, \bar{g}^{(1)}(r, \zeta) + \dots \quad (19)$$

where ζ is the inner variable $\zeta = \frac{z \pm \beta}{\epsilon^{1/2}(r)}$ on the cover $z = \mp\beta$. Note that here $\epsilon^{1/2}$ is a function of r equal to $\epsilon^{1/2}(1)\, e^{\frac{S^2}{2}(r^2-1)}$, that must be small in all of the flow.

On the side wall (r=1), the existence of two layers one of thickness $O[\epsilon^{1/3}(1)]$ merged into another one of thickness $O[\epsilon^{1/4}(1)]$ have been proofed by Stewartson [8]. When the rate of injection or sampling located at radius r_1 is of order $O[\epsilon^{1/6}(r_1)]$ the existence of a detached shear layer parallel to the axis of the cylinder has been shown experimentally and theoretically [6]. These 1/3 layers allow the passage of the flow rate from one plate to the other one. The Stewartson 1/4 layer ensures the transition of the azimuthal velocity from the core to the wall ; this layer disappears in the case of an antisymmetric problem, using Stewartson's terminology [8] ($v_\theta(z,r) = -v_\theta(-z, r)$, $\Psi(z, r) = \Psi(-z,r)$, $T(z,r) = -T(-z,r)$, Ψ is the streamfunction). This is the case, when the wall temperature is symmetric relative to the midplane z = 0. The expansion read, in the Stewartson 1/3 layer located at radius r_1

$$g(r,z,\epsilon) = \tilde{g}(\xi, z, \epsilon) = \tilde{g}^{(0)}(\xi, z) + \epsilon_1^{1/12}\, \tilde{g}^{(1)}(\xi, z) + \epsilon_1^{1/6}\, \tilde{g}^{(2)}(\xi, z) + \dots \quad (20)$$

where $\xi = \frac{r - r_1}{\epsilon_1^{1/3}}$, $\epsilon_1 = \epsilon(r_1)$
and, in the Stewartson 1/4 layer $\eta = (r - r_1)/\epsilon_1^{1/4}$

$$g(r, z,\epsilon) = \hat{g}(\eta, z, \epsilon) = \hat{g}^{(o)}(\eta, z) + \epsilon_1^{1/12}\, \hat{g}^{(1)}(\eta, z) + \epsilon_1^{1/6}\, \hat{g}^{(2)}(\eta, z) + \dots \quad (21)$$

The corners of size $O(\epsilon_1^{1/4})$, $O(\epsilon_1^{1/2})$, called the Ekman extensions, by [4] are singular : there, the quasi-complete systems must be solved.

2.5. Validity of linearization

On the full set of equation, with $\alpha = O(1)$ which represents a more general case of flow in the rotating frame, a simple scale analysis shows that the boundary layers is of thickness $O(\epsilon^{1/2})$ (Classical boundary layer analysis). When the order of magnitude of α is smaller than unity, its order must be compared with ϵ in the various regions of the flow. When $\alpha = o(\epsilon^{1/3})$, the boundary layer analysis leads to usual results of linearized theory for the Stewartson layers. A more representative solution is found by linking α to ε [11]. Here the relation $\alpha = O(\varepsilon^{1/3})$ (resp. $\alpha = O(\varepsilon^{1/4})$) involves non-linear terms in the azimuthal equation.

For the Ekman layers, it is obvious that the limit of the solution of the non linear system as $\alpha \to 0$ is equal to the solution of the linear system.

In the core, the thermal convection term must be kept, if $\alpha\, \varepsilon^{-1/2}(r) = O(1)$ according to the analysis of Homsy and Hudson [4].

In this work, we restrict ourselves to the case of linear Stewartson layers $\alpha = o(\varepsilon_1^{1/3})$ and to the case where thermal convection can be neglected $\alpha = o(\varepsilon_1^{1/2})$.

3. EKMAN LAYERS

In this layer, all the quantities are of order unity except the first term of the expansion of the axial velocity of order $\varepsilon^{1/2}(r)$. Then, the substitution of expansion (19) into the linearized equations (10) to (14) leads to

$$\frac{\partial \bar{v}_z^{(o)}}{\partial \zeta} + \frac{1}{r}\frac{\partial (r\bar{v}_r^{(o)})}{\partial r} + 2S^2\, r\, \bar{v}_r^{(o)} = 0 \tag{22}$$

$$\frac{\partial \bar{p}^{(o)}}{\partial \zeta} = 0 \tag{23}$$

$$-2\bar{v}_\theta^{(o)} + r\,\bar{T}^{(o)} = 2\frac{\partial^2 \bar{v}_r^{(o)}}{\partial \zeta^2} - \frac{1}{2S^2}\frac{\partial \dot{\bar{p}}^{(o)}}{\partial r} \tag{24}$$

$$\bar{v}_r^{(o)} = \frac{\partial^2 \bar{v}_\theta^{(o)}}{\partial \zeta^2} \tag{25}$$

$$S^2 r\,\bar{v}_r^{(o)} + \frac{\gamma}{\gamma-1}\frac{1}{Pr}\frac{\partial^2 \bar{T}^{(o)}}{\partial \zeta^2} = 0 \tag{26}$$

The boundary conditions at the top and bottom plates $\zeta = 0$ are

$$\bar{v}_\theta^{(o)} = \bar{v}_r^{(o)} = 0$$

$$\bar{v}_z^{(o)} = \begin{cases} 0 & \text{at the wall} \\ v_{iz}(r) & \text{for the } i^{th} \text{ inlet or outlet} \end{cases}$$

$$\bar{T}^{(o)}(r,0) = Tf(r) \qquad \text{for } z = +\beta$$

$$\bar{T}^{(o)}(r,0) = -Tf(r) \qquad \text{for } z = -\beta \tag{27}$$

The standard elimination among (22) to (26) gives, with $4a^4 = 1 + Pr\,\frac{\gamma-1}{2\gamma}\,S^2 r^2$,

$$\frac{\partial^4 \bar{v}_r^{(o)}}{\partial \zeta^4} + 4\,a^4\,\bar{v}_r^{(o)} = 0 \tag{28}$$

The integration of equations (28), (25) and (26) is straightforward, using the fact that the solutions are finite as $|\zeta| \to \infty$

plate $z = \beta$

$$\bar{v}_r^{(o)} = 2\, a^2\, C_6^+ \; e^{a\zeta} \sin a\,\zeta$$

$$\bar{v}_\theta^{(o)} = C_6^+ \,(1 - e^{a\zeta} \cos a\zeta)$$

$$\bar{T}^{(o)} = \frac{2}{r}\left[\frac{rTf}{2} - C_6^+ \,(4\, a^4 - 1)(1 - e^{a\zeta} \cos a\zeta)\right] \qquad (29)$$

plate $z = -\beta$

$$\bar{v}_r^{(o)} = -\,2\, a^2\, C_6^- \; e^{-a\zeta} \sin a\,\zeta \qquad (30)$$

$$\bar{v}_\theta^{(o)} = C_6^- \left[1 - e^{-a\zeta} \cos a\,\zeta\right]$$

$$\bar{T}^{(o)} = \frac{2}{r}\left[-\frac{rTf}{2} - C_6^{\mp} \,(4a^4 - 1)\,(1 - e^{-a\zeta} \cos a\,\zeta)\right]$$

The relation between $C^+_6(r)$ and $C^-_6(r)$ is obtained by using the mass conservation in a cylinder of radius r bounded by the end plates. The algebraic sum of inlets and outlets mass flow rates $I = \sum_i \int_o^r \rho_o(r)\, v_{iz}(r, \pm\beta)\,dr$ is equal to the sum of the radial mass flow rate in the two Ekman layers of order $\varepsilon^{1/2}(r)$ and the radial mass flow rate in the core of order $\varepsilon(r)$; successively, we obtain

$$I = \int_o^\infty 2\pi\, r\bar{v}_r^{(o)} \;\rho_o(r)\, \varepsilon^{1/2}\, d\zeta + \int_{-\infty}^{o} 2\pi\, r\; \bar{v}_r^{(o)} \;\rho_o(r)\, \varepsilon^{1/2}\; d\zeta + O(\varepsilon) \qquad (31)$$

$$\frac{\partial \bar{p}^{(o)}}{\partial r} = \frac{8\, S^2 a^3\, e^{-\frac{S^2 r^2}{2}}}{r}\; I \qquad (32)$$

This radial pressure gradient is necessary to transfer the fluid from a port of injection at the radius r_i to another port at the radius r_j.
The final expressions of C_6^+ and C_6^- are :

$$C_6^+ = \frac{1}{4\,a^4}\left[\frac{rTf}{2} - \frac{1}{4S^2}\;\frac{\partial \bar{p}^{(o)}}{\partial r}\right] \qquad (33)$$

$$C_6^- = \frac{1}{4\,a^4}\left[-\frac{rTf}{2} - \frac{1}{4S^2}\;\frac{\partial \bar{p}^{(o)}}{\partial r}\right] \qquad (34)$$

The expression of the axial velocity $\bar{v}_z^{(o)}$ is obtained by integrating the continuity equation for the end plate $z = \beta$

End plate $z = \beta$

$$\bar{v}_z^{(o)} = \frac{1}{16a^3S^2}\left\{\left[\frac{\partial \bar{p}^{(o)}}{\partial r}\left(S^2r + \frac{3-4a^4}{8a^4r}\right) + \frac{\partial^2 \bar{p}^{(o)}}{\partial r^2}\right]\left[e^{a\zeta}(\cos a\zeta - \sin a\zeta) - 1\right]\right.$$

$$- \left(\frac{4a^4-1}{4a^4} + 2S^2r^2\right)a\zeta \frac{\sin a\zeta}{r} e^{a\zeta}\left.\frac{\partial \bar{p}^{(o)}}{\partial r}\right\} - \frac{Tf}{8a^3}\left\{\left[\left(\frac{4a^4-1}{4a^4} + 2S^2r^2\right)a\zeta \sin a\zeta\right.\right.$$

$$\left.- \left(S^2r^2 + \frac{1}{2} + \frac{3}{8a^4} + \frac{rdTf}{Tfdr}\right)(\cos a\zeta - \sin a\zeta)\right]e^{a\zeta} + S^2r^2 + \frac{1}{2} + \frac{3}{8a^4}$$

$$\left. + \frac{r}{Tf}\frac{dTf}{dr}\right\} + v_{iz} \qquad (35)$$

An analoguous solution is found for $z = -\beta$.

The main characteristics of the Ekman layer is the fact that the influence of injection leads to a symmetric problem : the pressure gradient which permits the passage of the flow from a port at radius r_i to a port at radius r_j is equal in the two layers and in the core at a given radius r so that the radial mass flow rate is the same in the two layers. On the contrary, an opposite temperature plate profile gives rise to opposite flow rates and to an antisymmetric problem in Stewartson's terminology [8]. Note that the solution is valid only if $1 \ll \frac{1}{S^2} \ll \varepsilon^{1/2}(r)$.

4. MAIN INNER FLOW

Substitution of expansion (18) into Navier-Stokes equations gives the appropriate scaling : $T^{(o)} = O(1)$, $v_\theta^{(o)} = O(1)$ $v_z^{(o)} = O(\varepsilon^{1/2})$, $v_r^{(o)} = O(\varepsilon)$ (36)

This yields to

$$\frac{\partial v_z^{(o)}}{\partial z} = 0 \qquad (37)$$

$$v_\theta^{(o)} = \frac{rT^{(o)}}{2} + \frac{1}{4S^2}\frac{\partial p^{(o)}}{\partial r} \qquad (38)$$

$$\frac{\partial p^{(o)}}{\partial z} = 0 \qquad (39)$$

$$v_r^{(o)} = \frac{\partial^2 v_\theta^{(o)}}{\partial z^2} + \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \qquad (40)$$

The solution of equation (37) and (39) reads

$$v_z(r, z) = \varepsilon^{1/2} v_z^{(o)} = \varepsilon^{1/2} f(r) \qquad (41)$$

$$p^{(o)}(r, z) = p(r) \qquad (42)$$

Thus, in the core, the flow is axial. The function f(r) is computed by matching the outer expansion with the Ekman layers (the axial velocity to the same value, as $\zeta \to \pm\infty$ for the two endplates

$$f(r) = \lim_{\zeta \to \pm\infty} \bar{v}_z^{(o)}(r,\zeta) \qquad (43)$$

The radial pressure p (r) is constant only in two cases :

a) if there is no injection

b) if there are injections at the same radius the mass flow entrance being equal to the outflow.

The energy equation reads in the core of the flow : $a(1) = a_1$

$$4 a_1^4 \left(\frac{\partial^2 T^{(o)}}{\partial z^2} + \frac{\partial^2 T^{(o)}}{\partial r^2} \right) + 3 \frac{1 + 4a^4(1) - 1}{r} \frac{\partial T^{(o)}}{\partial r} = \frac{4a_1^4 - 1}{2S^2} r \left(- \frac{\partial^3 p^{(o)}}{\partial r^3} - \frac{1}{r} \frac{\partial^2 p^{(o)}}{\partial r^2} + \frac{1}{r^2} \frac{\partial p^{(o)}}{\partial r} \right) \qquad (44)$$

This equation has been solved for small values of $4 a^4(1) - 1$ by Mastuda and coworkers [9].

An exact numerical solution of this equation has been found by Durivault and al. [12] as a particular case of energy equation including thermal convection.

5. STEWARTSON 1/3 - LAYER

5.1. Basic system

In this layer, the equilibrium density is expressed in function of the inner variable $\xi = \dfrac{r - r_1}{\varepsilon_1^{1/3}}$. Then

$$\frac{\rho_0(r)}{\rho_0(r_1)} = \exp\left[2S^2 r_1 \varepsilon_1^{1/3} \xi + S^2 r_1 \varepsilon_1^{2/3} \xi^2\right] \qquad (45)$$

As $1/S^2$ appears also as a singular parameter in the continuity equation [14], its order of magnitude have to be compared with $\varepsilon^{1/3}$ (Darrozes [13]). With the order of magnitude of § 2.1. and the help of Eq. (45), it appears that the parameter $\varepsilon^* = 2 S^2 r_1 \varepsilon_1^{1/3}$ must be taken of order the unity. Thus, the singularity is taken correctly into account. Notice that solutions like Nakayama's one [6] assuming $\varepsilon^* \ll 1$ are invalid if $1/S^2 \ll 1$. Thus $\rho_0(r)/\rho_0(r_1)$ can be approximated by $e^{\varepsilon^* \xi}$ to a term $\varepsilon_1^{1/3}$ smaller, as far as $\xi = O(\varepsilon^{1/6})$.

The local Ekman number ε_1 is linked to the layer Ekman number ε by

$$\varepsilon(r) = \varepsilon_1(r_1) \frac{\rho_0(r)}{\rho_0(r_1)} = \varepsilon_1 \left[e^{\varepsilon^* \xi} + O(\varepsilon_1^{1/3})\right] \qquad (46)$$

Thus, the equations of the flow reads [14] neglecting terms $\varepsilon_1^{1/3}$ smaller and using the expansion (21) (notice that ε^* has become regular)

$$\frac{\partial \tilde{v}_z^{(i)}}{\partial z} + \frac{\partial \tilde{v}_r^{(i)}}{\partial \xi} + \varepsilon^* \tilde{v}_r^{(i)} = 0 \tag{47}$$

$$r_1 \tilde{T}^{(i)} - 2 \tilde{v}_\theta^{(i)} = - \frac{r_1}{\varepsilon^*} \frac{\partial \tilde{p}^{(i)}}{\partial \xi} \tag{48}$$

$$2e^{-\varepsilon^* \xi} \frac{\partial^2 \tilde{v}_z^{(i)}}{\partial \xi^2} - \frac{r_1}{\varepsilon^*} \frac{\partial \tilde{p}^{(i)}}{\partial z} = 0 \tag{49}$$

$$\tilde{v}_r^{(i)} = e^{-\varepsilon^* \xi} \frac{\partial^2 \tilde{v}_\theta^{(i)}}{\partial \xi^2} \tag{50}$$

$$S^2 Pr \frac{\gamma - 1}{\gamma} r_1 \tilde{v}_r^{(i)} + e^{-\varepsilon^* \xi} \frac{\partial^2 \tilde{T}^{(i)}}{\partial \xi^2} = 0 \tag{51}$$

$i = 0, 1, 2$

In this layer, the first term in the expansion of v_r is of the order of the magnitude $\varepsilon^{1/3}$, the other quantities beeing of the order unity included the pressure, contrary to the incompressible case, for $i = 0$.

5.2. Wall layer ($r_1 = 1$)

In this case, the boundary conditions are at $\xi = 0$

$$\tilde{v}_r^{(i)} = \tilde{v}_\theta^{(i)} = \tilde{v}_z^{(i)} = 0, \quad \tilde{T}^{(o)} = T_W, \quad \tilde{T}^{(1)} = \tilde{T}^{(2)} = 0 \tag{52}$$

Let $\tilde{\psi}^{(i)}$ be the i^{th} order of the stream function in the Stewartson 1/3 layer defined as

$i = 0, 1, 2$

$$\tilde{v}_r^{(i)} = \frac{1}{e^{\varepsilon^* \xi}} \frac{\partial \tilde{\psi}^{(i)}}{\partial z}, \quad \tilde{v}_z^{(i)} = - \frac{1}{e^{\varepsilon^* \xi}} \frac{\partial \tilde{\psi}^{(i)}}{\partial \xi} \tag{53}$$

Standard elimination in the equation (47) to (51) gives :

$$4a^4(r_1) \frac{\partial^2 \tilde{\psi}^{(i)}}{\partial z^2} + e^{-2\varepsilon^* \xi} \left[\frac{\partial^6 \tilde{\psi}^{(i)}}{\partial \xi^6} - 8 \varepsilon^* \frac{\partial^5 \tilde{\psi}^{(i)}}{\partial \xi^5} + 25 \varepsilon^{*2} \frac{\partial^4 \tilde{\psi}^{(i)}}{\partial \xi^4} - 38 \varepsilon^{*3} \frac{\partial^3 \tilde{\psi}^{(i)}}{\partial \xi^3} + 28 \varepsilon^{*4} \frac{\partial^2 \tilde{\psi}^{(i)}}{\partial \xi^2} - 8 \varepsilon^{*5} \frac{\partial \tilde{\psi}^{(i)}}{\partial \xi} \right] = 0 \tag{54}$$

$i = 0, 1, 2$

To the zero order, solution of Eq. (54) can be written under the form :

$$\tilde{\psi}^{(o)}(\xi, z) = \sum_{n=1}^{\infty} A_n \, f_n(\xi) \, g_n(z) \tag{55}$$

where

$$g_n(z) = \sin \left[\frac{n\pi}{2\beta} (z - \beta) \right] \tag{56}$$

Letting $t = e^{\varepsilon^* \xi}$, the function $f_n(t)$ are the non-zero solution of the linear differential equation :

$$\mathcal{L}(fn) \equiv 4a_1^4 \left(\frac{n\pi}{2\beta}\right)^2 fn - \varepsilon^{*6}\left[2t\frac{d^3fn}{dt^3} + 10\,t^2\frac{d^4fn}{dt^4} + 7\,t^3\frac{d^5fn}{dt^5} + t^4\frac{d^6fn}{dt^6}\right] = 0 \qquad (57)$$

with two-point boundary conditions :

$$t = 1,\quad fn = \frac{dfn}{dt} = 0,\quad \frac{d^2fn}{dt^2} = 1 \qquad (58)$$

$t = 0$, fn and its derivatives are equal to zero. The point $t = 0$ beeing singular, the numerical integration starts from a point $t = \alpha$ close to $t = 0$ with the boundary conditions, which ensures that the regular solution is damped as $|\xi| \to \infty$

$$fn(\alpha) = \frac{dfn(\alpha)}{dt} = \frac{d^2fn\ (\alpha)}{dt} = 0 \qquad (59)$$

It is necessary to test that the solution is insensitive to the place of the starting point. A typical value of α is 10^{-6}. Our solution is checked on the incompressible solution [4] in the limit $\varepsilon^* \to 0$. The use of temperature boundary conditions allows the computation of coefficients An :

$$An = -\frac{r_1}{4}\frac{n\pi}{\beta}\left[\frac{d^4fn}{d\xi^4} - 4\varepsilon^*\frac{d^3fn}{d\xi^3} + 5\varepsilon^{*2}\frac{d^2fn}{d\xi^2} - 2\varepsilon^{*3}\frac{dfn}{d\xi}\right]^{-1}_{\xi=0} \cdot \int_{\beta}^{\beta} Tw\ \cos\left[\frac{n\pi}{2\beta}(z-\beta)\right] d\left(\frac{z}{\beta}\right) \qquad (60)$$

The integration of Eq. (50) leads to

$$\tilde{v}_\theta^{(o)}(\xi, z) = \sum_{n=1}^{\infty} An\ fn_\theta(\xi)\ \frac{dgn(z)}{dz}$$

$$fn_\theta(t) = \int_1^t \frac{1}{\varepsilon^{*2} s}\int_0^s \frac{fn(u)}{u}\ du ds \qquad (61)$$

and

$$\tilde{T}^{(o)} = -\Pr S^2\ \frac{\gamma-1}{\gamma}\ r_1\ \tilde{v}_\theta^{(o)} + T_W(z) \qquad (62)$$

The influence of the compressibility is shown on the figures 2 and 3 or different values of ε^*: the damping of the velocities is altered even for small values of ε^*.

<u>The next order solution</u> has the same form but the temperature boundary condition is changed.

$$\tilde{T}^{(1)}(0, z) = 0 \qquad (63)$$

Then, the solution for $\tilde{v}_r^{(1)}$, $\tilde{v}_z^{(1)}$ and $\tilde{v}_\theta^{(1)}$ reads

$$\tilde{v}_r^{(1)}(\xi, z) = \tilde{v}_z^{(1)}(\xi, z) = 0$$

$$\tilde{v}_\theta^{(1)} = c\xi + d \qquad (64)$$

where c and d can be obtained by matching with the Stewartson 1/4 layer. In the case of an antisymmetric problem, c = d = 0.

The second order solution allows the passage of the mass flow rate from one plate to the other one. This part is limited to the antisymmetric case. Althouth $\tilde{\Psi}^{(2)}$ does not vanish now at $z = \pm \beta$, $\tilde{\Psi}^{(2)} - F(\xi)$ does ; $F(\xi)$ is determined from the Ekman extansions § 7 . Thus, $\tilde{\Psi}^{(2)}$ is searched in the form

$$\tilde{\Psi}^{(2)}(\xi, z) = F(\xi) + \sum_{n=1}^{\infty} hn(\xi) \sin\left[\frac{2n+1}{2} \frac{\pi}{\beta} (z-\beta)\right] \tag{65}$$

The equation (54) for i = 2 requires that the functions $hn(\xi)$ satisfy the equation

$$L\left[hn(\xi)\right] = \frac{4}{(2n+1)\pi} \left\{ L[F(\xi)] - 4a_1^4 \left(\frac{2n+1}{2\beta}\pi\right)^2 F(\xi) \right\} \tag{66}$$

The boundary conditions for $\tilde{\Psi}^{(2)}$

$$\tilde{\Psi}^{(2)}(\xi, \pm\beta) = F(\xi)$$

$$\xi = 0, \quad \frac{\partial \tilde{\Psi}^{(2)}}{\partial z} = \frac{\partial \tilde{\Psi}^{(2)}}{\partial \xi} = 0$$

$$\tilde{\Psi}^{(2)} \text{bounded as } \xi \to -\infty \tag{67}$$

corresponds to the boundary conditions for $hn(\xi)$

$$h_n(\xi) \quad \text{bounded as } |\xi| \to \infty$$

$$h_n(0) = 0$$

$$\frac{dhn}{d\xi}(0) = \frac{4}{(2n+1)\pi} F'(0) \tag{68}$$

The last boundary condition is obtained after derivation of the axial and radial momentum equation

$$\left[\frac{d^4hn}{d\xi^4} - 4\varepsilon \frac{d^3hn}{d\xi^3} + 5\varepsilon^{*2} \frac{d^2hn}{d\xi^2}\right]_{\xi=0} = \frac{4}{(2n+1)\pi} \left[\frac{d^4F}{d\xi^4} - 4\varepsilon^* \frac{d^3F}{d\xi^3} + 5\varepsilon^{*2} \frac{d^2F}{d\xi^2}\right]_{\xi=0} \tag{69}$$

The numerical solution is shown on the figure 4 with the data previously used in [7]. It can be seen that the velocities to this order are small in front of those of the recirculation layers.

5.3. Detached layer

When the injection velocity is great enough of the order $\varepsilon_1^{1/6}$, the viscous stresses are predominant and, thus, a detached Stewartson layer exists [6], with the following orders of magnitude

$$\tilde{v}_z^{(o)} = O(\varepsilon_1^{1/6}),\ \tilde{v}_r^{(o)} = O(\varepsilon_1^{1/2}),\ \tilde{v}_\theta^{(o)} = O(\varepsilon_1^{1/6})$$
$$\tilde{T}^{(o)} = O(\varepsilon_1^{1/6}),\ \tilde{p}^{(o)} = O(\varepsilon_1^{1/6}),\ \tilde{\Psi}^{(o)} = O(\varepsilon_1^{1/2}) \quad (70)$$

The system to solve is the same as in §5.2 but the boundary conditions are different [15]

$$\lim_{|\xi| \to \infty} \tilde{v}_r^{(o)} = \tilde{v}_z^{(o)} = 0$$

$\tilde{v}_\theta^{(o)}$ and $\tilde{T}^{(o)}$ bounded.

$$\forall_z,\ \lim_{\xi \to \infty} \tilde{\Psi}^{(o)}(\xi, z) - \lim_{\xi \to -\infty} \tilde{\Psi}^{(o)}(\xi, z) = Q \quad (71)$$

$$z = \pm \beta,\ \tilde{\Psi}^{(o)}(z, \xi) = \begin{cases} -\frac{Q}{2} & \text{for } \xi < 0 \\ \frac{Q}{2} & \text{for } \xi > 0 \end{cases} \quad (72)$$

This last condition expresses that a mass flow rate Q of the order $\varepsilon_1^{1/2}$ at the radius r_1 goes through an annular space of thickness $\varepsilon_1^{1/3}$, so that the function of the radial coordinate r can be approximated by a Dirac distribution.

Like in 5.2, the integration is staighforward but the two cases $\xi < 0$ and > 0 due to the discontinuity of $\tilde{\Psi}^{(o)}$ at $\xi = 0$, are to be distinguished. For numerical reasons of precision, it is convenient to make the change of variables

$t = e^{\varepsilon^* \xi}$ for $\xi < 0$ and $u = e^{-\varepsilon^* \xi}$ for $\xi > 0$. So, we must solve the two differential equations : for $\xi < 0$, equation (54) and for $\xi > 0$, the sixth order equation

$$4a_1^4 \left|\frac{\pi n}{2\beta}\right|^2 f_n(u) - \varepsilon^{*6} u^3 \left[108 \frac{dfn}{du} + 468 \frac{u d^2 fn}{du^2} + 478 u^2 \frac{d^3 fn}{du^3} + 170 u^3 \frac{d^4 fn}{du^4} + 23 u^4 \frac{d^5 fn}{du^5} + u^5 \frac{d^6 fn}{du^6}\right] = 0 \quad (73)$$

For each n, the solutions of the two equations must satisfy the boundary conditions at $\xi \to \pm \infty$ $(u = t = 0)$ and match at $\xi = 0$, that is the derivatives of the functions $fn(\xi)$ equal at $\xi = 0$. In each half space, the solution of the 6^{th} order linear differential equation is a linear combination of three independant particular solutions bounded at $\xi \to \pm \infty$ (this is equivalent to three conditions).

$$\xi < 0 \qquad fn^-(\xi) = \sum \lambda_i f_{ni}(\xi) \qquad i = 1, 2, 3$$
$$\xi > 0 \qquad fn^+(\xi) = \sum \lambda_i f_{ni}(\xi) \qquad i = 4, 5, 6 \quad (74)$$

The coefficients λ_i are determined by the linear system

$$fn^+ - fn^- = 1$$

$$\frac{d^i fn^+(0)}{d\xi^i} = \frac{d^i fn^-(0)}{d\xi^i} \qquad i \in [1,5] \tag{75}$$

expressing the matching conditions. Since the mass flow rate Q is constant in the detached layer along z, An is derived from the expression

$$\sum_{n=1}^{\infty} An \left[fn^-(0) - fn^+(0) \right] gn\,(z) = Q \tag{76}$$

One obtain

$$An = -\frac{4}{n\pi} \qquad \text{for } n \text{ odd}$$

$$An = 0 \qquad \text{for } n \text{ even} \tag{77}$$

The expressions of the azimuthal velocity $\tilde{v}_\theta^{(0)}$ and of the temperature $\tilde{T}^{(0)}$ are obtained by the integration of equation (50) and (51)

$$\tilde{T}^{(o)} = \frac{\gamma - 1}{\gamma} PrS^2 \tilde{v}_\theta^{(o)} r_1 \tag{78}$$

$$\tilde{v}_\theta^{(o)} = \sum_{n=1}^{\infty} An \left(\frac{n\pi}{2\beta}\right) \cos\left[\frac{n\pi}{2\beta}(z-\beta)\right] f_{n\theta} \tag{79}$$

where fn_θ satisfy

$$\varepsilon^{*2}\, u \left(\frac{d}{du} + \frac{d^2}{du^2}\right) fn_\theta(u) = fn(u) \tag{80}$$

The detached layer have been computed with a peripherical speed of 600 m/s for inlets and outlets located at $r_1 = 0.895$ and $r_1 = 0.985$ (fig. 5). For the detached layers near the side wall, the compressibility effects are small since $\varepsilon^* = 0.12$. In this case, our computation has been compared with Nakayama's solution and is in very good agreement with it. Notice that, in both solutions, the maximum of <u>axial mass velocity</u> $\rho_o(r)\, \tilde{v}_z^{(o)}$ is slightly remove on the direction of the side wall, contrary to the incompressible case where the maximum is located at the radius of injection. For the inner layer, the value of ε^* is 0.64, our solution gives a very fast damping with increasing densities, and allows two separate detached layers, while the solution of Nakayama covers the outer radius and is out of its range of validity (Fig.5).

The convergence of the series is not fast : thirty terms are needed to obtain a precision of 5 %. The precision can be improved by a factor greater than ten by use of extrapolations [16] .

6. STEWARTSON 1/4 LAYER

6.1. Basic system

The same reasoning as for the 1/3 layer shows that a good approximation of the solution can be found if $2 S^2 r_1 \varepsilon^{1/4}_{(r_1)} = \alpha^*$, $\alpha^* = O(1)$. In this layer, the orders of magnitude of the first non-zero terms are

$$\hat{v}_\theta^{(o)} = O(1) \quad , \hat{T}^{(o)} = O(1) \quad , \quad \hat{\Psi}^{(o)} = O(\varepsilon_1^{1/2})$$
$$\hat{v}_z^{(o)} = O(\varepsilon_1^{1/4}), \quad \hat{v}_r^{(o)} = O(\varepsilon_1^{1/2}) \tag{81}$$

That implies

$$\frac{\partial v_z^{(o)}}{\partial z} + \frac{\partial \hat{v}_r^{(o)}}{\partial \eta} + \alpha^* \hat{v}_r^{(o)} = 0 \tag{82}$$

$$\frac{r_1}{\alpha^*} \quad \frac{\partial \hat{p}^{(o)}}{\partial z} = 0 \tag{83}$$

$$\frac{r_1}{\alpha^*} \quad \frac{\partial \hat{p}^{(o)}}{\partial \eta} - 2\hat{v}_\theta^{(0)} + r_1 \hat{T}^{(o)} = 0 \tag{84}$$

$$2\hat{v}_r^{(o)} = 2 e^{\alpha^* \eta} \frac{\partial^2 \hat{v}_\theta^{(o)}}{\partial \eta^2} \tag{85}$$

$$r_1 S^2 \hat{v}_r^{(o)} + \frac{\gamma}{\gamma - 1} \frac{1}{Pr} e^{-\alpha^* \eta} \frac{\partial^2 \hat{T}^{(0)}}{\partial \eta^2} = 0 \tag{86}$$

where $\eta = \dfrac{r - r_1}{\varepsilon_1^{1/4}}$

The solution of this system is obvious

$$\hat{v}_\theta^{(o)} = g(\eta) \tag{87}$$

$$\hat{v}_r^{(o)} = e^{-\alpha^* \eta} g''(\eta) \tag{88}$$

$$\hat{\Psi}^{(o)} = g'(\eta) z + d(\eta) \tag{89}$$

The functions $d(\eta)$ and $g(\eta)$ are determined by matching with Ekman layers, the Stewartson 1/3 layer and the core. The Stewartson 1/4 is created by a symmetric problem, $\hat{\Psi}(r, z) = - \hat{\Psi}(r, -z)$ when injections and samples occur at different radii with different mass flow rates.

6.2. Resolution

The free stream function and azimuthal velocity at the outlet of the Ekman layer must match with the Stewartson 1/4 layer, so that the solution can be read in the form of Ekman extensions :

$$v_\theta = F^+(\eta)\left[1 - e^{a_1\zeta}\cos a\zeta\right] \tag{90}$$

$$\Psi = 2\pi r_1 \rho_o(r_1)\,\varepsilon^{1/2}(r_1)\,G^+(\eta)\left[1 + (\sin a_1\zeta - \cos\zeta a)\, e^{a_1\zeta}\right] \tag{91}$$

in the corner $O(\varepsilon_1^{1/2})$, $O(\varepsilon_1^{1/4})$ along the top plate and

$$v_\theta = F^-(\eta)\left[1 - e^{-a_1\zeta}\cos a_1\zeta\right] \tag{92}$$

$$\Psi = 2\pi r_1 \rho_o(r_1)\,\varepsilon^{1/2}(r_1)\,G^-(\eta)\left[-1 + (\sin a_1\zeta + \cos a_1\zeta)e^{-a_1\zeta}\right] \tag{93}$$

along the bottom plate.

The matching of the azimuthal velocity involves

$$F+(\eta) = F-(\eta) = F(\eta) \tag{94}$$

with $F(\eta) \rightarrow -\frac{1}{2ar}\, e^{-\frac{S^2 r^2}{2}}\, I \quad , |\eta| \rightarrow \infty$

The functions $F(\eta)$ and $G(\eta)$ are linked by

$$\frac{1}{2\pi r\rho_o(r)\,\varepsilon^{1/2}}\frac{\partial^4\Psi}{\partial\zeta^4} + 4\,a^4(r)\,\frac{\partial v_\theta}{\partial\zeta} = 0 \tag{95}$$

in the extensions to match with Ekman layers, so that

$$G^+(\eta) = G^-(\eta) = a\frac{r}{r_1}\frac{\rho_o(r)}{\rho_o(r_1)}\frac{\varepsilon^{1/2}(r)}{\varepsilon^{1/2}(r_1)}\,F(\eta) \tag{96}$$

The matching of the extensions (Eq. (90) to (93)) with the 1/4 layer (Eq. (87) to (89)), leads to

$$d(\eta) = 0 \tag{97}$$

and requires to solve the homogeneous equation

$$g''(\eta) - \frac{a}{\beta}\, e^{\frac{a^* \eta}{2}}\, g(\eta) = 0 \tag{98}$$

with the following boundary conditions

$$g(0) = 1$$

$$\lim_{|\eta|\to\infty} g(\eta) = 0$$

where g is found in unit of radial mass flow rate.

The numerical integration of this differential equation with two-point boundary condition is performed by the same method as for the 1/3 layer and compared with the exact solution [17] for wall layers

$$g(\eta) = Io\left[\frac{4}{\alpha^*}\sqrt{\frac{a_1}{\beta}}\, e^{\frac{\alpha^*\eta}{4}}\right] / Io\left[\frac{4}{\alpha^*}\sqrt{\frac{a_1}{\beta}}\right] \qquad \eta \leq 0 \qquad (100)$$

$$g(\eta) = Ko\left[\frac{4}{\alpha^*}\sqrt{\frac{a_1}{\beta}}\, e^{\frac{\alpha^*\eta}{4}}\right] / K_0\left[\frac{4}{\alpha^*}\sqrt{\frac{a_1}{\beta}}\right] \qquad \eta \geq 0 \qquad (101)$$

where Io and Ko are the modified Bessel functions of the first and second kind. In Eq. (100), $g(\eta)$ (i.e the azimuthal velocity) does not go to 0 as $\eta \to -\infty$ (contrary to the solution (101) for $\eta > 0$). This is due to the too small damping of the density : even with the exact value of the density in the layer $\lim_{\eta\to-\infty} g(\eta) \neq 0$. (Fig. 6). For detached 1/4 layers, the solution is found in the form ($\phi = \frac{4}{\alpha^*}\sqrt{\frac{a_1}{\beta}}\, e^{\frac{\alpha^*\eta}{4}}$)

$$g(\eta) = \left[\frac{-K'_o}{Ko\, I'o - K'_o I_o}\right]_{\phi=\phi(o)} I_o(\phi) \qquad \eta \leq o$$

$$g(\eta) = \left[\frac{I'_o}{KoI'_o - K'_o\, I_o}\right]_{\phi=\phi(o)} K_o(\phi) + 1 \qquad \eta \geq o \qquad (102)$$

Though it gives the incompressible solution as $\alpha^* \to 0$, in the case of decreasing densities, one cannot obtain a solution which makes that the initially prescribed jump of velocity is cancelled.

The flow in the 1/4 layer induces a recirculation flow from the 1/4 layer in the 1/3 layer : its expression is obtained by matching the 1/4 and 1/3 layer. So, an antisymmetric stream function is found in the form

$$\hat{\Psi}^{(0)} = \sum_{n=1}^{\infty} An\; fn(\xi)\, \sin\left[\frac{n\pi}{\beta}(z-\beta)\right] \qquad (103)$$

with the boundary conditions

$$\xi \to -\infty , \quad fn(\xi), \quad \frac{d\, fn(\xi)}{d\,\xi}, \frac{d^2 fn(\xi)}{d\,\xi^2} \to 0$$

$$\xi = 0 , \quad \frac{d\, fn(\xi)}{d\,\xi} = 1; \quad \frac{d^2 fn(\xi)}{d\,\xi^2} = fn(\xi) = 0 \qquad (104)$$

7. EKMAN EXTENSIONS

In the corner of dimensions $O(\varepsilon_1^{1/2})$, $O(\varepsilon_1^{1/3})$ the solutions in the Stewartson and Ekman layers are non uniformly valid. So, it is convenient to seek an approached solution near the end plate $z = \beta$ in the form

$$\Psi_{EEK} = r_1\, \rho(r_1)\, \varepsilon^{1/2}(r_1)\, F(\xi)\left[1 - e^{a_1\zeta}(\cos a_1\zeta - \sin a_1\zeta)\right]$$

$$v_{\theta EEK} = \frac{E(\xi)}{a_1}\left[1 - e^{a_1\zeta}\cos a_1\zeta\right] \qquad (105)$$

When $\zeta \to -\infty$, the Ekman extensions must give the solutions in the Stewartson 1/3 layer

$$\lim \quad v_{\theta\,EEK} = \tilde{v}_\theta^{(o)}(\xi,\beta) + \varepsilon^{1/12}(r_1)\,\tilde{v}_\theta^{(1)}(\xi,\beta) + \varepsilon^{1/6}(r_1)\tilde{v}_\theta^{(2)}(\xi,\beta) + \ldots\ldots$$

$$\lim \quad \Psi_{EEK} = \varepsilon^{1/3}(r_1)\tilde{\Psi}^{(o)}(\xi,\beta) + \varepsilon^{5/12}(r_1)\,\tilde{\Psi}^{(1)}(\xi,\beta) + \varepsilon^{1/2}(r_1)\,\tilde{\Psi}^{(2)}(\xi,\beta) + \ldots\ldots \qquad (106)$$

This matching involves

$$\begin{aligned}
\tilde{\Psi}^{(o)}(\xi,\beta) &= 0\\
\tilde{\Psi}^{(1)}(\xi,\beta) &= 0\\
\tilde{\Psi}^{(2)}(\xi,\beta) &= e^{\frac{\varepsilon^*\xi}{2}}\,E(\xi) = F(\xi)\\
\tilde{V}_\theta^{(o)}(\xi,\beta) &= \frac{E(\xi)}{a_1} = \sum_{n=1}^{\infty}(-1)^n\frac{2\beta}{n\pi}\,An\, fn_\theta(\xi)
\end{aligned} \qquad (107)$$

8. CONCLUSION

The calculation of the flow in a rotating cylinder up to a peripherical speed equal to 1000 m/s have been performed by the use of matched asymptotic expansions and by setting the relative order of magnitude of the small parameters Ekman number,Rossby number and Mach number.
The flow in the Stewartson layers is deduced by matching the flow in the core and in the Ekman layers. The thickness of the detached and wall 1/3 layers and the amplitude of the oscillations increase (resp. decrease) with decreasing (resp. increasing)density.The pattern of the flow is strongly altered compared to incompressible case. For Stewartson 1/4 layer, a satisfactory solution can be found for increasing densities : on the contrary, no boundary layer solution can be found for decreasing densities.

In this type of flow dominated by Coriolis force and density stratification, the control of boundary conditions is of primarily importance for the suppression of recirculation flows disastrous in the case of isotope separation by centrifuges. These layers cannot be avoided in the case of detached layers 1/3 but in the case of side wall 1/3 layer, an isothermal wall or an insulated wall suppresses the recirculation. The 1/4 layer disappears either with thermal convection or if there are ports of injection so that there is no radial transport.

REFERENCES

[1] GREENSPAN, HP, Theory of rotating fluids, 1968, Cambridge University Press.

[2] BARCILON, V and PEDLOVSKY, J, On the steady motions produced by a stable stratification in a rapidly rotating fluid, J. Fluid. Mech. 1967, 29, 673-690.

[3] HUNTER, C, The axisymmetric flow in a rotating annulus due to horizontally applied temperature gradient. J. Fluid. Mech, 1968, 27, 753-778.

[4] HOMSY, G.M., and HUDSON, J.L., Centrifugally driven thermal convection in a rotating cylinder, J. Fluid Mechanics, 1969, 35, 33-52.

[5] SAKURAI, T and MATSUDA, T, Gas dynamics of a centrifugal machine, J. Fluid Mechanics, 1974, 62, 727-736.

[6] NAKAYAMA, W and USUI, S, Flow in rotating cylinder of a gas centrifuge, J. Nucl. Sci. Tech. 1974, 11, 242-262.

[7] DURIVAULT, J and LOUVET, P, Etude théorique de l'écoulement dans une centrifugeuse à contrecourant thermique, 1975, Rapport CEA-R-4714.

[8] STEWARTSON, K, On almost rigid rotations, J. Fluid Mechanics, 1957, 8, 17-26.

[9] MATSUDA, T, SAKURAI, T and TAKEDA, H, Source-sink flow in a gas centrifuge J. Fluid Mechanics, 1975, 69, 197-208.

[10] LOTZ, M. Kompressible Ekman-Grenzschichten in rotierenden Zylindern, Atomkernenergie, 1973, 22, 46-50.

[11] BENNETTS, D, A and HOCKING, L, M, On non linear Stewartson layers in a rotating fluid, Proc. R. Soc. Lond., 1973, A 333, 469-489.

[12] DURIVAULT, J, LOUVET, P, ROUVILLOIS, G, and SOUBBARAMAYER, Int. J. Heat and Mass Transfert, to be published (also communication to Euromech 80, Karlsruhe, 1976).

[13] DARROZES, J-S, The method of matched asymptotic expansions applied to problems involving two singular perturbation parameters, Fluid Dynamics transactions, 1972, 6, part. II, 119.

[14] DURIVAULT, J, LOUVET, P, Etude de la couche de Stewartson compressible dans une centrifugeuse à contrecourant thermique, C.R. Acad. Sc., séance du 10 mai 1976, série B, t 283, 79-82.

[15] LOUVET, P, and DURIVAULT, J, Compressible countercurrent flow in a centrifuge communication to Euromech 80, Karlsruhe (1976).

[16] BAKER, G, A, Essentials of Pade Approximants, Academic Press, New-York, 1975.

[17] BARK, FH, and BARK, TH, On vertical Boundary layers in a rapidly rotating gas, J of Fluid Mech, to be published.

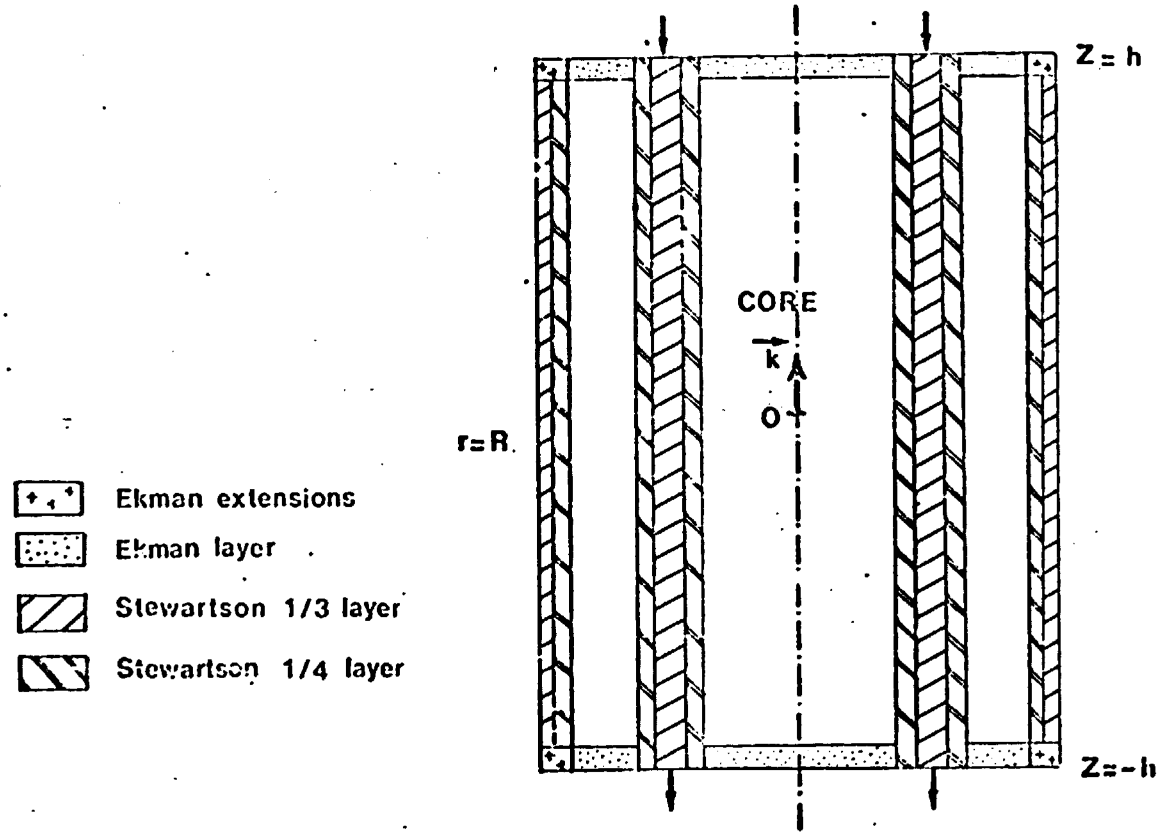

Fig. 1. Scheme of a centrifuge

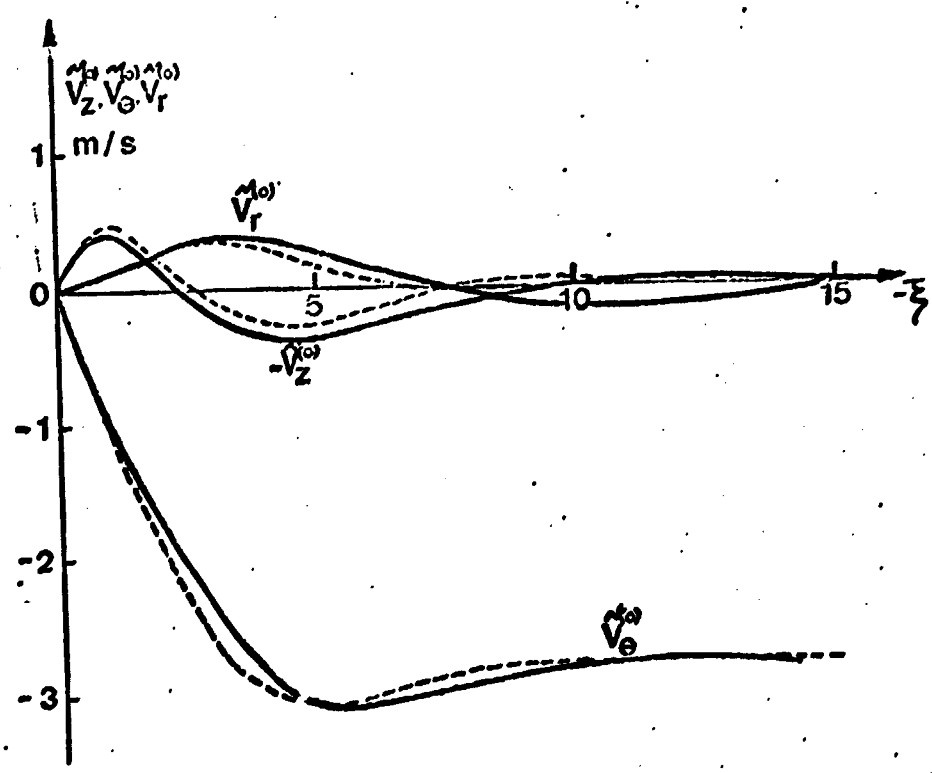

Fig. 2. Recirculating flow in the Stewartson $^{1/3}$ wall layer
$\varepsilon^* = 0.12$ — Present work --- incompressible solution

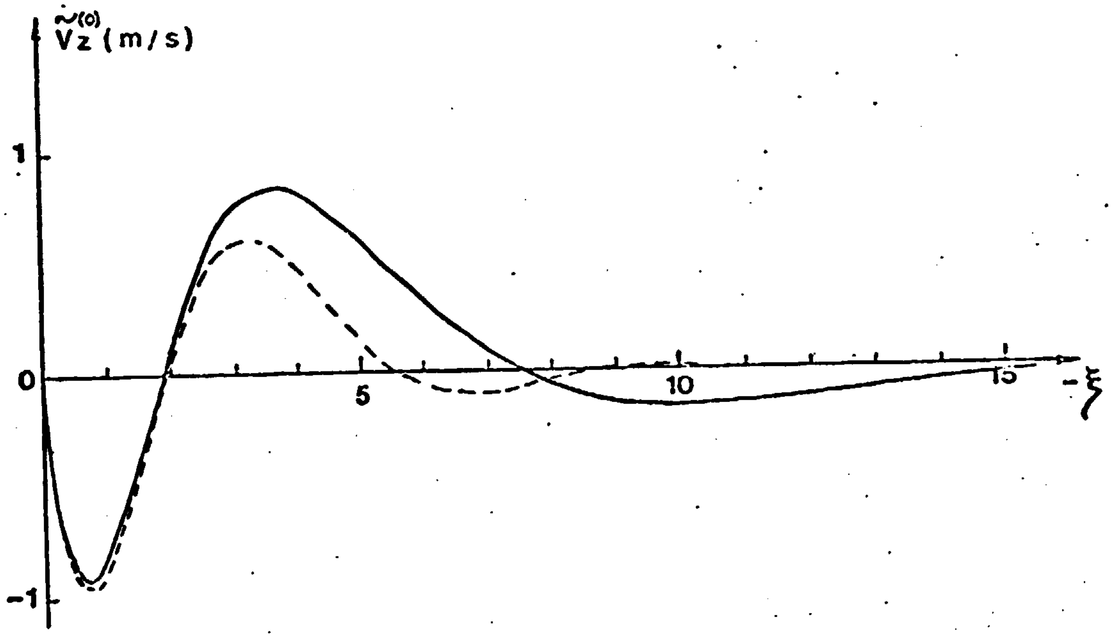

Fig. 3. Recirculating flow in the Stewartson 1/3 wall layer
Axial velocity profile $\varepsilon^* = 0.28$
— Present work - - - Incompressible solution

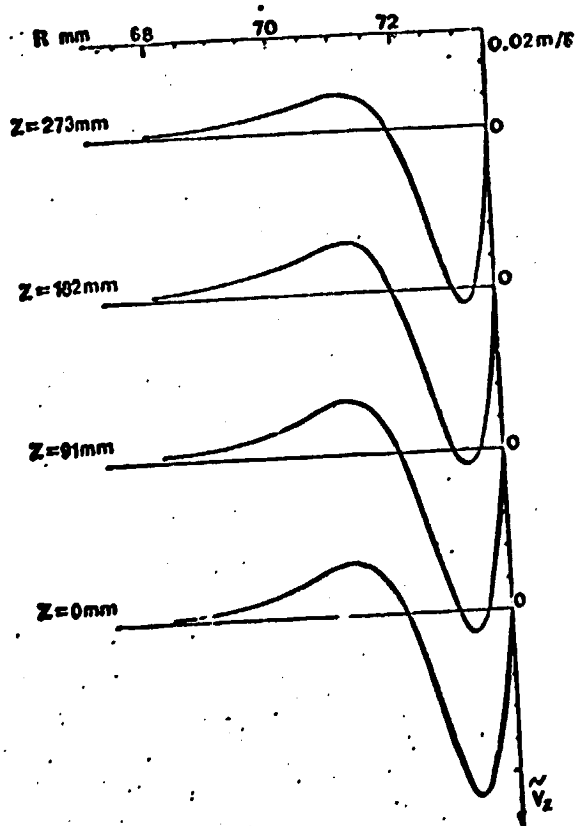

Fig. 4. Passage of the flow $O(\varepsilon^{1/2})$ in the Stewartson 1/3 wall layer
Axial velocity —— Present work

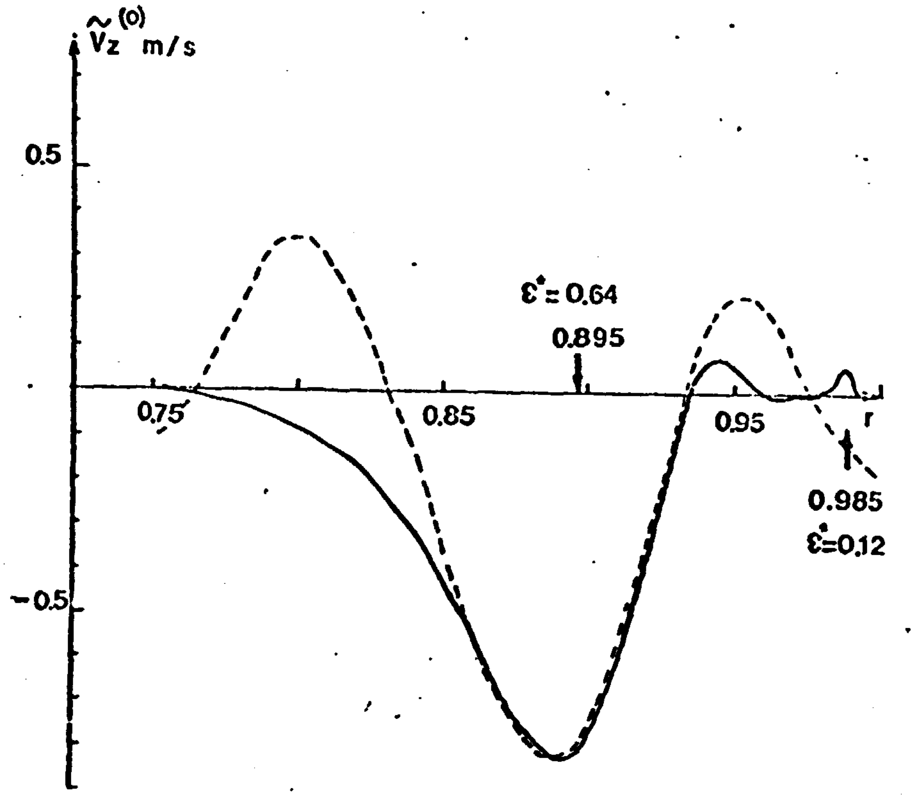

Fig. 5. Axial velocity in the detached 1/3 layer
— Present work – – – Nakayama's solution [6]
Inner injection $\varepsilon^* = 0.64$ Outer injection $\varepsilon^* = 0.12$

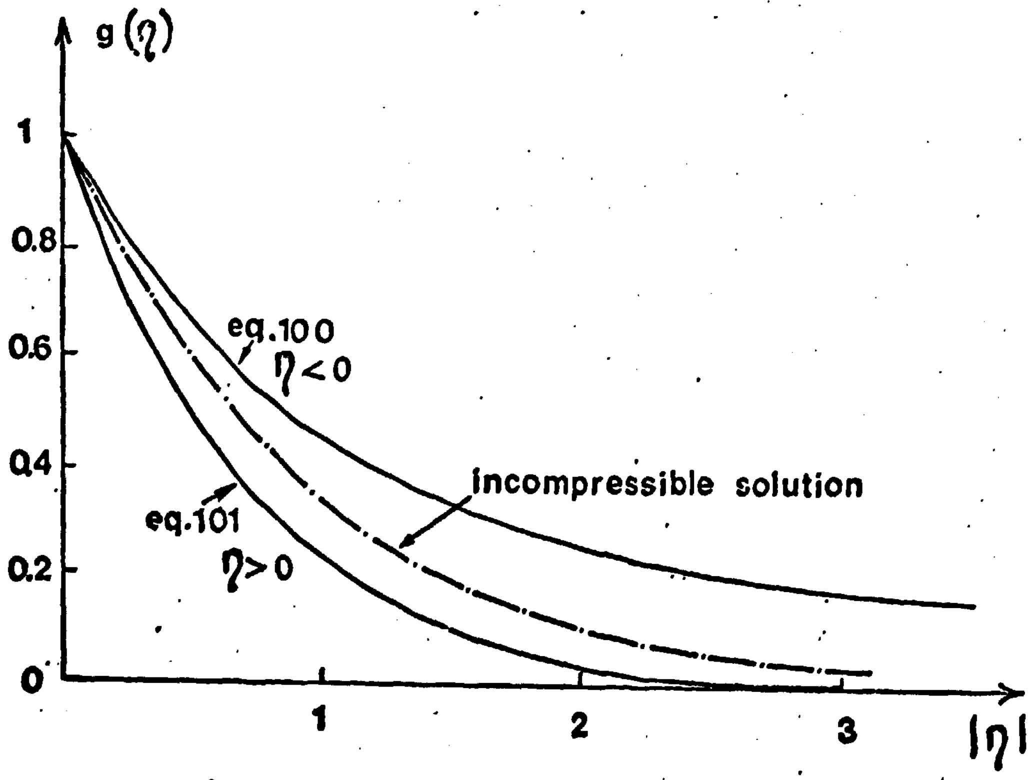

Fig. 6. Stewartson 1/4 wall layer
$\alpha^* = 1.21$