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Helical Equilibrium of

Current Carrying Plasma Column

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ABSTRACT

By means of the neighbouring equilibrium method the saturation level of the kink instability of a current carrying plasma column is investigated. We obtain the helical equilibrium which is the stationary saturated stage of kink instability. The analysis is performed for the rounded current profile case and shows that the shell has very strong suppression effect on the saturation level.

Kink instabilities, including the resistive one, still attract much attention in the study of the thermo-nuclear fusion research.[1] They are considered not only to be the cause of the whole plasma decontamination but also to give rise to the enhanced microscopic cross field transport.[1,2] Within the frame work of the linear theory, the analyses about the unstable region in the discharge condition[3,4] give good agreements with the experimental results.[5] Recently the studies about the interactions between these modes have been in progress in order to understand the observations in tokamaks.[6] The unstable modes in an actual plasma are usually observed in their nonlinearly saturated form because of their rapid growth rates in comparison with the discharge time. In this paper we obtain the saturation level of kink instability by using the neighbouring equilibrium method. In the nonlinearly stationary state, the fluctuation no longer grows (at least in the magnetohydrodynamic time scale), that is, the plasma satisfies the equilibrium equation $\vec{J} \times \vec{B} = \nabla p$. In other words, the plasma discharge moves from one cylindrical equilibrium to another neighbouring helical equilibrium. Thus by finding finite amplitude neighbouring helical equilibrium, we can obtain the saturation level of kink instabilities. In ref. [7] we have solved the fixed boundary case by this method. Here we consider the free boundary case.

In the large aspect ratio limit, a tokamak plasma is approximated as a current carrying plasma cylinder immersed in a strong longitudinal magnetic field. The toroidal symmetry is replaced by the longitudinal periodicity.

We study the finite amplitude helical equilibrium of a current carrying plasma column using helical coordinates, r, $\varphi = m\theta + k_z z$, $\chi = (mz - k_z r^2 \theta)/(m^2 + k_z^2 r^2)$ (r, θ, and z are the ordinary cylindrical coordinates with r = 0 corresponding to the axis of the cylinder). The considered system is helically symmetric, i.e., χ independent. We define B_φ and A_φ ($\vec{A}$ is vector potential) as

$$B_\varphi = k_z r B_\theta - m B_z \, , \tag{1}$$

$$A_\varphi = k_z r A_\theta - m A_z \equiv \psi \, . \tag{2}$$

The plasma equilibrium equation, $\vec{J} \times \vec{B} = \nabla p$, can be rewritten in the form as follows[8],

$$\left(\frac{m^2}{r^2} + k_z^2 \right) \frac{\partial^2 \psi}{\partial \varphi^2} + r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} \psi + \frac{2m^2}{r(m^2 + k_z^2 r^2)} \frac{\partial \psi}{\partial r} + \frac{2 m k_z B_z}{m^2 + k_z^2 r^2}$$

$$= - \frac{d}{d\psi} \left[\frac{1}{2} B_\varphi^2 + \mu_0 (m^2 + k_z^2 r^2) p \right] , \tag{3}$$

$$B = B (\psi), \quad p = p(\psi). \tag{4}$$

The longitudinal component of the current density is given by $J_z = \mu_0^{-1} B_z dB_\varphi / d\psi + m dp/d\psi$.

We consider that the plasma (radius a) is located in the conducting cylinder (radius b). The boundary condition is that B_r vanishes at the conducting wall, and the normal component of the magnetic field vanishes on the perturbed plasma boundary. We are interested in the neighbouring equilibrium, so that we expand ψ as

$$\psi(r,\varphi) = \psi_{p0}(r) + \psi_{p1}(r)\cos\varphi\cdot\alpha + \alpha^2\psi_{p2}(r) + \cdots \tag{5}$$

inside of the plasma,

$$\psi(r,\varphi) = \psi_{v0}(r) + \psi_{v1}(r)\cos\varphi\cdot\beta + \beta^2\psi_{v2}(r) + \cdots \tag{6}$$

in the vacuum,

where α and β are small expanding parameters which denote the amplitude of the helical perturbation. The boundary condition gives

$$\alpha^i\psi_{pi}(a) = \beta^i\psi_{vi}(a) \qquad (i = 0,1,2), \tag{7}$$

$$\alpha^i\psi_{pi}'(a) = \beta^i\psi_{vi}'(a) \qquad (i = 0,1,2), \tag{8}$$

and

$$\psi_{v1}(b) = 0, \tag{9}$$

where ' stands for d/dr.

Being given $B_\varphi(\psi)$ and $p(\psi)$, we can solve Eqs.(4) to (9). The linear solution (up to the 1st order of α) of Eqs.(4) to (9) have been obtained for various kind of current profiles numerically.[4] Now we consider the rounded current profile case where

$$B_\varphi = B_0 + \frac{k^2\psi^2}{2B_0} \tag{10}$$

holds with $k\psi \ll B_0$ [8]. This constraint is rather different from the ordinary flux conservation condition $D\psi/Dt = 0$, but says that the current profile does not change. The reason why we take this constraint is that from the experimental observations[5] we know that the current profile changes much more slowly than the evolution of MHD activities. The constant k is related to the safety factor q-value as shown later. We expand Eq.(4) using Eqs.(5) and (6) with $(ak_z)^2 \ll 1$ and the zero plasma pressure limit for simplicity. Retaining the terms up to the 2nd order of α, we obtain

$$\frac{\partial}{r\partial r} r \frac{\partial \psi_{p0}}{\partial r} + \frac{2k_z B_0}{m} + k^2 \psi_{p0} = 0, \tag{11}$$

$$\frac{\partial}{r\partial r} r \frac{\partial \psi_{p1}}{\partial r} - \frac{m^2}{r^2}\psi_{p1} + k^2 \psi_{p1} = 0, \tag{12}$$

$$\frac{\partial}{r\partial r} r \frac{\partial \psi_{p2}}{\partial r} + k^2 \psi_{p2} + \frac{k_z k^2}{2mB_0} \psi_{p1}^2 = 0. \tag{13}$$

We get ψ_0 which gives the diffused current profile (the current density smoothly goes to zero at the plasma boundary) , that is,

$$\psi_0 = -\frac{2k_z B_0}{k^2 m} [\ 1 - J_0(kr)/J_0(ka)\ ], \tag{14}$$

where $J_i(x)$ is the i-th order bessel function. It is convenient to use the normalized form Ψ instead of ψ as

$$\psi_i = \frac{2k_z B_0}{k^2 m} \Psi_i$$

The first order solutions are

$$\Psi_{p1}(r) = J_m(kr),$$

$$\Psi_{v1}(r) = (r/b)^m - (b/r)^m \quad . \tag{15}$$

The boundary condition (8) gives

$$\frac{J_m(ka)}{kaJ_{m-1}(ka)} = \frac{1-(b/a)^{2m}}{2m} \tag{16}$$

from which we determine k (i.e., q-value), that permits the existence of helical equilibrium of an infinitesimal amplitude.[4,8] This condition gives the linear stability criterion. We obtain the 2nd order solution noting the toroidal flux conservation condition $\int_{plasma} B_z r\, dr = \text{const.}$ as

$$\Psi_{p2}(r) = \left\{ \frac{J_0(ka)J_m^2(ka)}{4J_1^2(ka)} - \frac{J_m'(ka)}{2J_1(ka)} - \frac{3J_m^2(ka)}{4kaJ_1(ka)} \right\} J_0(kr). \tag{17}$$

By definition of the safety factor $q(r) = k_z r B_z / B_\theta$, we finally obtain

$$q(r) = m \left\{ 1 - \frac{2J_1(kr)}{krJ_0(ka)} + \alpha^2 \frac{d\Psi_{p2}}{dr} \right\}^{-1}. \tag{18}$$

If we consider the infinitesimal perturbation, taking α to be zero, Eq.(18) turns out to be

$$q(r) = m\,[\,1 - 2J_1(kr)/krJ_0(ka)\,]^{-1} , \tag{18'}$$

and Eqs.(16) and (18') determines the criterion of the linear stability q*. This result is the same as that in [8]. Proceeding to the nonlinear solution, from Eqs.(17) and (18) we solve the amplitude of the perturbation $\widetilde{B}_r/B_\theta\big|_a = 2J_m(ka)q^*(a)\alpha/k^2a^2$ as a function of q

$$\frac{\widetilde{B}_r}{B_\theta} = \frac{J_m(ka)}{ka}\sqrt{\frac{2q^*(a)}{a\Psi_{p2}'(a)}\left\{ q^*(a)/q(a) - 1 \right\}}\,, \tag{19}$$

where tilde ~ stands for the helical perturbation component.

Equation (19) gives the saturation amplitude of the kink instability. For instance let us consider the case of m = 1 mode. Equation (16) gives the well known criterion of the linear stability as

$$q^*(a) = (a/b)^2 \,. \tag{20}$$

This value does not depends on the current profile.[3,4] From Eq. (18) we solve $\widetilde{B}_r/B_\theta\big|_a$ which is shown in Fig.1 for various values of $(a/b)^2$. At a q-value which slightly exceeds q* for a given value of a/b, the fluctuation amplitude grows until it reaches the saturation according to the curve shown in Fig.1. The transition from a cylindrical equilibrium to a helical one takes place in the time scale of $\tau_A = B_\theta/a\sqrt{\mu_0\rho}$ which is usually much shorter than the time required for the q-value to change experimentally. So that the observed saturation level changes according to the change of q-value. If τ_A is comparable to the time of q-value change, we must use the constraint of the flux conservation $D\psi/Dt = 0$ as discussed before. If q-value becomes larger than unity, the perturbation connects with the tearing mode.

The nonlinear saturation level strongly depends on the finite a/b effect as in the case of the linear growth rate and the linear stability region. The shell effect indicated in Fig. 1 has a very large stabilizing and suppressing power to the kink instability of the tokamak plasma. It seems that the magnetic limiter[9] which naturally comes to give the smaller value of a/b has a disadvantage from the view point of MHD instabilities.

As shown above, using the method of neighbouring equilibrium we obtained the nonlinear saturation amplitude of the kink instability of the cylindrical tokamak. This analysis is performed for the special case of the current profile. The study about the fixed boundary mode has shown that the saturation amplitude strongly depends on the current profile.[7] More general cases have to be analyzed by means of numerical calculations.

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FIGURE CAPTION

Fig.1 The saturation amplitude $|\tilde{B}_r(a)/B_\theta(a)|$ is shown as a function of q(a) for various values of $(a/b)^2$. As a/b approaches to unity, the unstable region becomes narrower and the saturation level becomes lower.

Fig.1