YANG MILLS INSTANTONS, GEOMETRICAL ASPECTS [x]

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Centre de Physique Théorique, CNRS , Marseille

[x] Lectures given at the International School of Mathematical Physics, Erice, 27 June - 9 July, 1977.

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- FOREWORD -

These notes are based on seminar notes prepared during the year 1976-1977 at the Centre de Physique Théorique du CNRS, Marseille, by : W. Franklin, C.P. Korthals-Altes, J. Madore, J.L. Richard, R. Stora, and private lectures by I.M. Singer to the author, to whom, however all incorrections should be attributed.

R. Stora

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I - INTRODUCTION

The word instanton [1] has been coined by analogy with the word soliton. They both refer to solutions of elliptic non linear field equations with boundary conditions at infinity (of euclidean space time in the first case, euclidean space in the second case) lying on the set of classical vacua in such a way that stable topological properties emerge, susceptible to survive quantum effects, if those are small. Under this assumption, instantons are believed to be relevant to the description of tunnelling effects between classical vacua [2] and signal some characteristics of the vacuum at the quantum level, whereas solitons should be associated with particles, i.e. discrete points in the mass spectrum : In one case the euclidean action is finite, in the other case, the energy is finite. From the mathematical point of view, the geometrical phenomena associated with the existence of solitons have forced physicists to learn rudiments of algebraic topology [3] . The study of euclidean classical Yang Mills fields involves naturally mathematical items falling under the headings :

- differential geometry (fibre bundles, connections)
- differential topology (characteristic classes, index theory)

and, more recently

- algebraic geometry.

Most of the machinery is old enough so that it can be learnt from mathematical books or sets of lecture notes where complete bibliographies can be found. It is out of question to give here a complete review of the mathematical apparatus. We shall rather pick out some of the results and show how they apply to the specific case at hand.

These notes are divided as follows :

Section II is devoted to a description of the physicist's views
Section III is devoted to the mathematician's views.

These notes are sketchy in the sense that very few technical details are fully described. Displaying them all would have required reproducing large portions of mathematical books. Emphasis has been put on some details of the 19th century geometry which is not easily accessible anymore, and not currently known to physicists The more accessible mathematical items are referred to as accurately as possible, including chapters, paragraphs, page numbers. It is thus hoped that these notes can be used as a guide through the recent literature.

II - THE PHYSICIST'S VIEWS

The problem to be solved is the following : find euclidean Yang Mills fields $A_\mu^\alpha(x)$ which minimize locally the euclidean action

$$S = \frac{1}{4g^2} \sum_{\substack{\alpha,\beta \\ \mu,\nu}} \int d^4x \; F_{\mu\nu}^\alpha \, g_{\alpha\beta} \, g^{\mu\rho} g^{\nu\sigma} \, F_{\rho\sigma}^\beta \tag{1}$$

The notations are as follows

$x \in E^4$ four dimensional euclidean space.

α : labels an orthonormal basis of the Lie algebra $\mathcal{G}$ of a simple compact Lie group G, unless otherwise specified $G = SU2$ for which there is the largest available information.

$g_{\alpha\beta}$: Killing form of

$g^{\mu\nu}$: flat riemannian metric in E^4

d^4x : volume element in E^4 corresponding to $g^{\mu\nu}$

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + f_{\beta\gamma}^\alpha A_\mu^\beta A_\nu^\gamma \tag{2}$$

$f_{\beta\gamma}^\alpha$: structure constants of $\mathcal{G}$.

The first class of instantons found by Belavin et al [4] is by now well known. It has the following characteristics :

$$S < \infty \qquad F_{\mu\nu}(x) \xrightarrow[|x|\to\infty]{} 0$$

$$A_\mu(x) \underset{|x|\to\infty}{\sim} g^{-1}(\hat{x}) \, \partial_\mu g(\hat{x})$$

where the homotopy class of $S_\infty^3 \ni x \xrightarrow{g} G$ corresponds to the integer $n = \pm 1$. Both cases, $n = \pm 1$ are treated together, by considering a Yang Mills field $A_\mu^{\alpha\beta}$ with value in the Lie algebra of $SO4$ which is the direct sum of two copies of the $SU2$ Lie algebra. The topological number n is related to a Chern number (the integral of a Chern characteristic class) :

$$2n = \frac{1}{4\pi^2} \int F_{\mu\nu}^\alpha \, \epsilon^{\mu\nu\rho\sigma} g_{\alpha\beta} \, F_{\rho\sigma}^\beta \; d^4x \tag{3}$$

For given n , absolute minima of S are reached for

$$F_{\mu\nu}^\alpha = \pm \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} \, F_{\rho\sigma}^\alpha = \pm (*F)_{\rho\sigma}^\alpha \tag{4}$$

which in particular imply the usual field equations

$$\nabla^{\mu} F^{\alpha}_{\mu\nu} = 0 \tag{5a}$$

$$\nabla^{\mu} (*F)^{\alpha}_{\mu\nu} = 0 \tag{5b}$$

However, all solutions which have been so far constructed saturate the absolute bound

$$S = \frac{2\pi^2}{g^2} |n| \tag{6}$$

deduced from the identity

$$\int \left(F^{\alpha}_{\mu\nu} \pm *F^{\alpha}_{\mu\nu} \right) g_{\alpha\beta}\, g^{\mu\rho}\, g^{\nu\sigma} \left(F^{\beta}_{\rho\sigma} \pm *F^{\beta}_{\rho\sigma} \right) d^4x \geqslant 0 \tag{7}$$

The $n=1$ solutions assume several equivalent forms [4],[5],[1]

$$\begin{aligned} A^{\alpha}_{\mu} &= \frac{x^2}{x^2+\lambda^2} \left[g^{-1}(x)\, \partial_{\mu}\, g(x) \right]^{\alpha} \\ &= -\eta^{\alpha}_{\mu\nu}\, \frac{2x^{\nu}}{x^2+\lambda^2} \end{aligned} \tag{8}$$

where

$$\begin{aligned} g(x) &= \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{\sqrt{x^2}} = \frac{\tilde{x}}{\sqrt{x^2}} \\ \eta^{\alpha}_{\mu\nu} &= \frac{1}{2}\, \mathrm{tr}\, \tilde{\sigma}_{\mu}\, \sigma_{\nu}\, \sigma^{\alpha} \end{aligned} \tag{9}$$

Through a conformal transformation which leaves both the euclidean action and the topological invariant unchanged, or a gauge transformation one gets the following equivalent form [1] :

$$A^{\alpha}_{\mu} = \eta^{\alpha}_{\mu\nu}\, \partial^{\nu} \log \left(1 + \frac{\lambda^2}{x^2} \right) \tag{10}$$

later generalized by 't Hooft [6] for higher n-values :

$$\begin{aligned} A^{\alpha}_{\mu} &= \eta^{\alpha}_{\mu\nu}\, \partial^{\nu} \log \rho \\ \frac{\Box \rho}{\rho} = 0 \qquad \rho &= 1 + \sum_{1}^{n} \frac{\lambda_i^2}{(x-x_i)^2} \end{aligned} \tag{11}$$

The $SO4$ version which puts together solutions pertaining to opposite n's reads .

$$A_\mu^{\alpha\beta} = \Sigma_{\mu\nu}^{\alpha\beta} \partial^\nu \log \rho \tag{12}$$

where the $\Sigma_{\mu\nu}^{\alpha\beta}$'s are the matrix elements of the SO_4 Lie algebra :

$$\Sigma_{\mu\nu}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta - \delta_\mu^\beta \delta_\nu^\alpha \tag{13}$$

This collection of solutions has been enlarged by Jackiw, Nohl and Rebbi [7] into a $5n+4$ parameter family with

$$\rho = \frac{1}{(x-x_0)^2} + \sum_1^n \frac{\lambda_i^2}{(x-x_i)^2} \tag{14}$$

It was also argued by these authors that there ought to be solutions depending on $5n + 3(n-1) = 8n - 3$ parameters corresponding to $n-1$ relative orientations of isospin axis, for instanton number n and this was checked in the neighbourhood of the known solutions, in the linear approximation [8]. This situation has been further analyzed by Brown, Carlitz, Lee [9] who relate the dimensionality of instanton fluctuations to that of minimally coupled massless fermions belonging to the adjoint representation. The latter is connected to the Adler anomaly, through an argument of S. Coleman [10], and hence to the instanton number.

Although the fermion problem is interesting in itself [1] and can be handled for an arbitrary compactification of E^4, [11], it is only directly related to the instanton problem in the case where the metric is flat. The argument can then be summarized as follows :

Let

$$A_\mu^\alpha = \mathring{A}_\mu^\alpha + a_\mu^\alpha \tag{15}$$

and let us impose the Landau gauge [11] condition in the background field $\mathring{A}$ which we assume to correspond to a self dual solution :

$$\mathring{\nabla}^\mu a_\mu = 0 \tag{16}$$

The linearized system then reads

$$\tilde{\nabla} \underset{\sim}{a} = 0 \tag{17}$$

where

$$\begin{aligned} \tilde{\nabla} &= \tilde{\sigma}_\mu \nabla^\mu \\ \underset{\sim}{a} &= \sigma_\mu a^\mu \end{aligned} \tag{18}$$

Since every quaternion $\underset{\sim}{a}$ is determined by its first column $\underline{a}$, one has

$$\tilde{\nabla}\,\underline{a} = 0 \qquad (19)$$

Conversely, for each solution $\underline{a}$ of this spinor equation, there corresponds a two dimensional real manifold of solutions of the initial equation, corresponding to the one dimensional complex manifold of solutions $\lambda\,\underline{a}$, λ complex. This in turn is equivalent to the massless Dirac equation

$$\overset{\circ}{\not{\nabla}}\,\psi = 0 \qquad (20)$$

together with the chirality condition

$$\psi = \gamma_5\,\psi \qquad (21)$$

(in the Weyl representation).

The rest of the argument which fits very well within the methods to be described in the next section involves several steps :

i) for a given self or anti-self duality property of the gauge field, the Dirac equation possesses only chiral or anti-chiral solutions

ii) the difference between the number of chiral and anti-chiral solutions can be evaluated in terms of the Adler anomaly, i.e. the instanton number.

This developping subject owes much to physicists who have first made a number of remarkable guesses. It seems however that mathematicians have taken over with powerful - and rigorous - techniques. It is to be noticed that one of the first contributors, A.S. Schwarz [4],[11] left a name in the theory of characteristic classes and was the first to have used the powerful index theory as early as April 1976 [12]. Later, M.F. Atiyah and I.M. Singer, the main contributors in this ten year old theory, and collaborators [13], have both reproduced A.S. Schwarz's work and gone beyond with the help of the hitherto unused techniques of algebraic geometry [14].

Some mathematical aspects dug out by physicists have not been exploited so far, namely, those related to the general conformal invariance of the problem : the function ρ involved in the 't Hooft ansatz can be identified with the conformity factor [15] occurring in the line element of a non compact manifold conformal to E^4 (flat for self or anti-self dual $F^{\alpha}_{\mu\nu}$, with constant curvature in the case of general solutions of the field equations). These remarks have not been fully exploited y.., because much of the mathematics used so far relies on the compactness of the manifolds that are used.

III - THE MATHEMATICIAN'S VIEWS

It is a matter of philosophy whether in principle a Yang Mills fields ought to be associated with a connection on a principal fibre bundle [16]. It is a fact that Yang Mills fields considered in the previous section are of this type and that the corresponding mathematical apparatus can be used either to streamline previously obtained results or to obtain new results.

We shall now review the various items enumerated in the previous section from a more mathematical point of view.

1. The n=1 instantons, a geometrical description [5], [15], [18].

Let us first map $E^4 \cup \infty$ into $S^4 \subset E^5$ through a stereographic projection. Call $\mathcal{A}$, $\mathcal{F}$, the differential forms

$$\mathcal{A} = A^{\alpha}_{\mu}\, dx^{\mu}\, e_{\alpha} \tag{22}$$

$$\mathcal{F} = \frac{1}{2}\, F^{\alpha}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}\, e_{\alpha} \tag{23}$$

where e_{α} is a basis of $\mathcal{G}$. We shall not distinguish the forms on $E^4 \cup \infty$ and their inverse images on S^4. Since the stereographic projection is conformal, it preserves

$$S = \frac{1}{4} \int (\mathcal{F}, *\mathcal{F}) \tag{24}$$

$$2n = \frac{1}{4\pi^2} \int (\mathcal{F}, \mathcal{F}) \tag{25}$$

where $*$ denotes the dual for whatever Riemannian metric is involved, and $(\ ,\ )$ is the Killing form of $\mathcal{G}$.

There are concrete examples of fibre bundle with structure group either $SO4$ or $SU2 \times SU2$ pertinent to the instanton antiinstanton doubling or $SU2$, pertinent to single instanton or antiinstanton description :
the $SO4$ principal bundles with basis S^4 are known to depend on two integers [17]. The simplest non trivial one is $SO5$ $(SO5 \backslash SO4 = S^4)$. The Maurer Cartan form on $SO5$, $\omega = g^{-1}dg$ with value in the Lie algebra of $SO5$ can be restricted to the Lie algebra of $SO4$ [18] and one can check by choosing coordinates that it is the n = 1 instanton in its initial version, which is $SO5$ invariant. The conformal transforms of this solution are obtained by restricting the Maurer Cartan form on $SO(5,1)$ to a right coset modulo $SO5$, and then to the Lie algebra of $SO4$. We thus obtain a five parameter family of solutions indexed by a point of $SO(5,1) \backslash SO5$, each solution being invariant under left translation by a subgroup of $SO(5,1)$ conjugate

to $SO5$. In this version one has to go from S^4 to E^4 by a stereographic projection, and it is actually much more direct to work with the covering groups $USp2$ of $SO5$ – the 2 x 2 unitary group with quaternion elements – and $SL(2,H)$ – the 2 x 2 unimodular group with quaternion elements – of $SO5,1$. The quotient $USp2 \backslash SU2 \times SU2$ is the projective quaternionic line $P_1(H)$, i.e. the set of pairs of quaternions (x,y) under the equivalence relation $(x,y) \sim (qx,qy)$ where q is an arbitrary non vanishing quaternion. $P_1(H)$ can be used naturally as a model of compactified E^4 and the formulae given by Jackiw and Rebbi [5] are directly recovered by the constructions indicated above. In particular, it is easy to verify that the corresponding curvature fulfills the self-duality condition [4], by using its expression in terms of the Maurer Cartan form on the one hand, and local coordinates on the other hand [18].

One can similarly deal with the SU2 version by considering $S^7 = USp2 \backslash SU2$ and appropriately restricting the Maurer Cartan form [18].

2. The 't Hooft instantons as connections on principal bundles [6],[19].

On S^4 (resp. $P_1(H)$), a has n singularities at x_i, $i=1\dots n$ and, in the neighbourhood of such a singularity

$$a \underset{x \to x_i}{\sim} g_i^{-1}\, d g_i \tag{26}$$

where g_i is the translated by x_i of g given by formula (9). Cover S^4 (resp. $P_1(H)$) by $n+1$ open sets : n ball-neighbourhoods Ω_i of x_i ($\Omega_i \cap \Omega_j = \emptyset$), $\Omega_0 = \complement \bigcup_i \Omega_i/2$

Define

$$\begin{aligned} a_0 &= a_{|\Omega_0} \\ a_i &= \mathrm{ad}\, g_i^{-1}\, a + g_i\, d g_i^{-1} \end{aligned} \tag{27}$$

Then, by the conventional construction of principal bundles, there is a bundle with transition functions

$$g_{i0} = g_i^{-1} \quad \text{in } \Omega_i \cap \Omega_0 \tag{28}$$

and a connection defined by $a_0, a_i, i=1\dots n$, on it (cf. Kobayashi Nomizu [18], p. 66).

Although there are canonical examples of $SO4$ fibre bundles over S^4 for arbitrary allowed topologies [20] – indexed by two integers – they have not suggested so far any geometrical characterization of the connections which minimize the euclidean action.

3. The manifold of connections minimizing the Euclidean action, local aspects.

There are two essentially equivalent versions of [11] [13] of the study of the manifold of solutions of the self or antiself duality condition Eq. 4 .

One is based on the linearized system

$$\tilde{\overset{\circ}{\nabla}} \underset{\sim}{a} = 0 \tag{29}$$

in the neighbourhood of a solution $\overset{\circ}{a}$, a connection on some principal bundle $\mathcal{B}$ over S^4 . $\tilde{a}$ is a section of the $SO4\times G$ bundle $T^{*1}(S^4)\underset{\mathcal{B}}{\times} \mathcal{G}$ with basis S^4 associated with $\mathcal{B}$ and the riemannian structure on S^4. $\tilde{\overset{\circ}{\nabla}}$ is a first order elliptic operator i.e. its first order symbol $\tilde{\xi}$, (obtained by replacing $\frac{\partial}{\partial x}$ by $i\xi$ in the higher degree terms) is invertible for $\xi \neq 0$. It maps sections of $T^{*1}(S^4) \underset{\mathcal{B}}{\times} \mathcal{G}$ into sections of $(T^{*0} \oplus T_+^{*2})(S^4) \underset{\mathcal{B}}{\times} \mathcal{G}$ where $T_+^{*2}(S^4)$ denotes the space of self dual 2-forms. Since $\tilde{\overset{\circ}{\nabla}}$ is elliptic, the dimensionality of the space of solutions ker $\tilde{\overset{\circ}{\nabla}}$ is finite [23] since S^4 is compact. The index theorem can be applied : [22] [23] [24] [25]

$$\mathrm{Ind}\, \tilde{\overset{\circ}{\nabla}} = \dim \ker \tilde{\overset{\circ}{\nabla}} - \dim \ker \tilde{\overset{\circ}{\nabla}}^T \tag{30}$$

can be computed in terms of topological data ($\tilde{\overset{\circ}{\nabla}}^T$ now maps $(T^{*0}\oplus T_+^{*2})(S^4)\underset{\mathcal{B}}{\times} \mathcal{G}$ into $T^{*1}(S^4) \underset{\mathcal{B}}{\times} \mathcal{G}$, and is the usual adjoint) because of its "universality". The calculation proceeds through a formal algebra of characteristic classes whose terms factor out into factors involving the $\mathcal{G}$ fibration, expressible by means of the character of the adjoint representation, and factors involving the basis, expressible in terms of the character of the $SO4$ representation in $T^{*1}(S^4)$ and $(T^{*0}\oplus T_+^{*2})(S^4)$ respectively. For $G = SUN$ the formula reads

$$\mathrm{Ind}\, \tilde{\overset{\circ}{\nabla}} = 4\, N\, n(\mathcal{B}) - \Big(\underbrace{\frac{\chi(S^4)}{2}}_{=1} + \underbrace{\frac{p^1(T(S^4))}{6}}_{=0} \Big)(N^2-1) \tag{31}$$

χ = Euler Poincaré characteristic ; $\chi(S^4)=2$; $p^1(T(S^4))$ = Pontrjagin number of S^4 ; $p^1(T(S^4))=0$; $n(\mathcal{B})$ = Chern number of $\mathcal{B}$.

Next one shows that

$$\dim \ker \tilde{\overset{\circ}{\nabla}}^T = 0 \tag{32}$$

The calculation is a bit lengthy and repeatedly makes use of two arguments which are schematized below : a positivity argument classical in the Hodge theory of harmonic forms [23] ,[29] , and an irreducibility argument concerning $\overset{\circ}{a}$;

From

$$\overset{\circ}{\nabla}^{T} H = 0 \qquad \begin{array}{l} H = (h, f) \\ h \in T^{*0} \underset{\mathcal{B}}{\times} \mathcal{G} \\ f \in T_{+}^{*2} \underset{\mathcal{B}}{\times} \mathcal{G} \end{array} \tag{33}$$

we deduce

$$\int_{S^4} (\overset{\circ}{\nabla}^{T} H, \overset{\circ}{\nabla}^{T} H) = 0 \tag{34}$$

hence, from positivity

$$\overset{\circ}{\nabla} h = 0 \tag{35}$$

which in turn implies

$$\overset{\circ}{\nabla}^{2} h = [\overset{\circ}{F}, h] = 0 \tag{36}$$

Actually Eq. 36 is equivalent to Eq. 35 :

$$0 = \int_{S^4} (h, \overset{\circ}{\nabla}^{2} h) = \int (\overset{\circ}{\nabla} h, \overset{\circ}{\nabla} h) \tag{37}$$

hence, Eq. 35 follows from positivity.

If $\overset{\circ}{F}$ spans the $SU2$-Lie algebra everywhere, (the irreducibility property, which is true here, it follows that h vanishes. This argument incidentally shows why the background Landau gauge does not leave any gauge freedom (Eq. 36 has no non-vanishing solution).

The method used by Atiyah, Hitchin and Singer [13] is essentially equivalent : they consider the elliptic complex [23][24][25]

$$0 \to T^{*0}(S^4) \underset{\mathcal{B}}{\times} \mathcal{G} \xrightarrow{\overset{\circ}{\nabla}} T^{*1}(S^4) \underset{\mathcal{B}}{\times} \mathcal{G} \xrightarrow{\overset{\circ}{\nabla}_{+}} T_{+}^{*2}(S^4) \underset{\mathcal{B}}{\times} \mathcal{G} \to 0 \tag{38}$$

($\overset{\circ}{\nabla}_{+} \nabla_{+} = 0$ by the self-duality of $\overset{\circ}{F}$).
Under the irreducibility hypothesis for $\overset{\circ}{F}$, the topological index of this complex is identical with the one previously computed and so is its analytical index (sum of dimensions of various kernels). The formulation is however slightly different : instead of removing the gauge freedom by fixing the background Landau gauge condition, the gauge freedom is eliminated by subtracting $\dim \operatorname{Im} \overset{\circ}{\nabla}$, which is

correct in all cases.

Once the linearized system has been analyzed, there remains to prove that each solution of the linearized system gives rise to a true solution, in a neighbourhood of $\overset{\circ}{a}$. This requires the use of the infinite dimensional implicit function theorem on top of

$$\dim \ker \overset{\circ}{\nabla}{}^{T} = 0$$

in the first version,

$$\dim \ker \overset{\circ}{\nabla}{}_{+}^{T} = 0$$

in the second version [25] where the question is to study the deformation of the complex (38).

The last question [11] connected with this has to do with the solutions of the massless Dirac equation Eqs. (20, 21). $\overset{\circ}{\not\nabla}$ is an elliptic operator from positive chirality fields to negative chirality fields, and coincides with its adjoint, the two spaces being interchanged. The index of $\overset{\circ}{\not\nabla}$, $n_+ - n_-$, ($n_\pm$ = # positive/negative chirality of fermion zero modes) can be computed again using topological data [22], [11], [13], [9] . On the other hand, it is related to the Adler Bardeen anomaly, as remarked by S. Coleman [10] : it can be computed by investigating the corresponding Laplacian and diffusion operator [25] [27] [30] :

$$\mathrm{Ind}\, \overset{\circ}{\not\nabla} = O(t^\circ) \left[\mathrm{tr}\, e^{-t \frac{1+\gamma_5}{2} \overset{\circ}{\not\nabla}{}^2} - \mathrm{tr}\, e^{-t \frac{1-\gamma_5}{2} \overset{\circ}{\not\nabla}{}^2} \right] \tag{39a}$$

$$= O(t^\circ) \left[\mathrm{tr}\, \gamma_5\, e^{t \overset{\circ}{\not\nabla}{}^2} \right. \tag{39b}$$

where the symbol $O(t^\circ)$ means : selecting the zeroth order term in the asymptotic expansion [28] of the indicated quantity for $t \to 0_+$, from which the negative powers of t involved in each term of (39a) drop out. This method of calculation turns out to be quite close to the physicist's version [9] based on the evaluation of Feynman graphs, to which it provides a firm foundation. The result can also be obtained by the purely algebraic methods involving the relevant representations of $SO2 \times SO2 \times \mathcal{G}$. The coefficient of n comes out a half of what it was in the Yang Mills case, but the coefficient of $\dim G$ changes significantly, in particular the term proportional to $\chi(S^4)$ is missing : in non flat space the relationship between the Dirac and Yang Mills problems is unclear, since here again positivity and the irreducibility of the connection take care of the absence of solutions of the adjoint system.

4. The manifold of connections minimizing the Euclidean action, global aspects.

Progress has recently been made [31] [32] towards a global study of self or antiself duality equations (Eq. 4)

$$F^{\alpha}_{\mu\nu} = \pm (*F)^{\alpha}_{\mu\nu}$$

So far, only $G = SU2$ has been dealt with. Since this is a quasilinear elliptic system, one expects solutions to possess analyticity properties. These analyticity properties have been found [31] and restrict the differential geometry framework to the algebraic geometry framework. Most of the geometry involved is related to general views put forward by R. Penrose [33], which ought to apply to a general class of conformal invariant euclidean field theories.

It may be of interest, for the purpose of orientation to review a simpler problem which bears some resemblance to the Yang Mills problem, namely the non linear σ model in two dimensions [34].

One looks for minima of

$$S = \int \partial_\mu \vec{\varphi}(x) . \partial_\mu \vec{\varphi}(x)\, d^2x \tag{40}$$

with the constraint

$$\vec{\varphi}^{\,2}(x) = \sum_1^3 \varphi_i^2(x) = 1 \tag{41}$$

and the boundary condition

$$\vec{\varphi}(x) \xrightarrow[|x| \to \infty]{} \vec{\varphi}_0 \qquad \vec{\varphi}_0^{\,2} = 1 \tag{42}$$

Thus

$$R^2 \cup \infty \ni x \longrightarrow \vec{\varphi}(x) \in S^2$$

defines a mapping from $R^2 \cup \infty$ to S^2 whose degree is given by

$$n = \int (\vec{\varphi}, \partial_\mu \vec{\varphi}, \tilde{\partial}_\mu \vec{\varphi})\, d^2x \tag{43}$$

$$\tilde{\partial}_\mu = \epsilon_{\mu\nu} \partial^\nu$$

Given n, S reaches an absolute minimum for

$$\partial_\mu \vec{\varphi} = \pm \vec{\varphi} \times \tilde{\partial}_\mu \vec{\varphi} \tag{44}$$

$$n = \pm |n|$$

as stems from the positivity condition

$$\int \sum_{\mu} \left(\partial_\mu \vec{\varphi} \pm \vec{\varphi} \times \tilde{\partial}_\mu \vec{\varphi} \right)^2 d^2x \geq 0 \tag{45}$$

$$S \geq \mp n$$

It is convenient to use the variables

$$\begin{aligned} z &= x_1 + i x_2 \\ \zeta &= \xi_1 + i \xi_2 \end{aligned} \tag{46}$$

where ξ_1, ξ_2 are the coordinates of the stereographic projection of $\vec{\varphi}$ on R^2. In terms of these variables

$$\begin{aligned} S &= \int \left(\left| \frac{\partial \zeta}{\partial z} \right|^2 + \left| \frac{\partial \zeta}{\partial \bar{z}} \right|^2 \right) \frac{dz\, d\bar{z}}{(1+|\zeta|^2)^2} \\ n &= \int \left(\left| \frac{\partial \zeta}{\partial z} \right|^2 - \left| \frac{\partial \zeta}{\partial \bar{z}} \right|^2 \right) \frac{dz\, d\bar{z}}{(1+|\zeta|^2)^2} \end{aligned} \tag{47}$$

and (44) reads

$$\frac{\partial \zeta}{\partial z} = 0 \qquad \text{or} \qquad \frac{\partial \zeta}{\partial \bar{z}} = 0 \tag{48}$$

according to the sign of n.

The general solution reads

$$\zeta(z) = \sum_{1}^{n} \frac{\lambda_p}{z - z_p} + \zeta_0 \tag{49}$$

in the holomorphic case. z should be replaced by $\bar{z}$ in the antiholomorphic case. λ_p, z_p, are arbitrary complex numbers, ζ_0, the stereographic projection of $\vec{\varphi}_0$.

Now, it has been often argued that there are similarities between the non linear σ model in two dimensions and the Yang Mills model in four dimensions. There is an obvious analogy here in the derivation of Eq. 48, to be compared with the self-duality condition Eq. 4, from the positivity condition (45) analogous to Eq. 7.

On the other hand, there is a substantial difference between S^2 and S^4: there is a unique complex structure [35] on S^2 (up to a sign), which makes it an analytic manifold, and is $SO2$ invariant. It is associated with the system of isotropic lines on the sphere. On S^4, there is *no global complex structure*. Locally, the analogue of the isotropic generators, of S^2 is provided by any isotropic 2-plane. Such two planes are parametrized by a point on $P^1(\mathbb{C})$, i.e. S^2, and so are, locally, the complex structures of S^4 : working with $P^1(H)$, for convenience, an isotropic two plane is parametrized by

$$T(P^1(H)) \ni \tilde{X} = u \otimes \lambda \quad (\text{resp } \lambda \otimes u \tag{50}$$

with u fixed spinor up to scaling, λ variable spinor :

$$(x,x) = \det \tilde{X} = 0 \tag{51}$$

Thus the direction of this two plane is parametrized by $P^1(\mathbb{C})$. Similarly, a complex structure compatible with $SO4$.

$$\tilde{X} \to \widetilde{JX} \qquad J^2 = -1 \qquad J J^T = 1 \tag{52}$$

can be parametrized by

$$\begin{gathered} \tilde{X} \to u(J)\, \tilde{X}\, v(J) \\ u(J),\ v(J) \in SU2 \end{gathered} \tag{53}$$

Now, all possible J's are of the form [35]

$$J = S J_0 S^{-1} \qquad S \in SO4$$

where J_0 is a special solution.
So, choosing J_0 :

$$\tilde{X} \to \widetilde{J_0 X} = \tilde{X}\, i\sigma_2$$

we can parametrize J by

$$\tilde{X} \to \tilde{X}\, v(J) \qquad , \quad v(J) \in SU2\,, \quad v^2(J) = -1$$

$V(J)$ is uniquely determined by its eigensubspace pertaining to the eigenvalue $+i$, so that the $SO4$ invariant complex structures are locally labelled by $P^1(\mathbb{C})$.

Now, $P^1(H)$ can be fibred by its complex structures into $P^3(\mathbb{C})$ which is a complex analytic manifold :

$$P^3(\mathbb{C}) \xrightarrow{P^1(\mathbb{C})} P^1(H) \tag{54}$$

The fibration can be described as follows. Let

$$\mathbb{C}^4 \ni Z = \begin{pmatrix} u \\ v \end{pmatrix} \qquad u, v \in \mathbb{C}^2 \tag{55}$$

$$P^3(\mathbb{C}) = \mathbb{C}^4 / \{ \tilde{Z} \sim \lambda Z \ , \ \mathbb{C} \ni \lambda \neq 0 \}$$

The involution σ on $\mathbb{C}^4$

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} i\sigma_2 u^* \\ i\sigma_2 v^* \end{pmatrix} \tag{56}$$

with square -1 induces an involution $[\sigma]$ with square +1 on $P^3(\mathbb{C})$. This involution has no fixed point, but fixed lines images of

$$\lambda \begin{pmatrix} u \\ v \end{pmatrix} + \mu \begin{pmatrix} i\sigma_2 u^* \\ i\sigma_2 v^* \end{pmatrix} \qquad \lambda, \mu \in \mathbb{C} \tag{57}$$

$$= \begin{pmatrix} q(u)\omega \\ q(v)\omega \end{pmatrix} \quad ; \quad \omega = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \in P^1(\mathbb{C}) \qquad \begin{matrix} q(u) = (u, i\sigma_2 u^*) \\ q(v) = (v, i\sigma_2 v^*) \end{matrix}$$

These lines are labelled by $P^1(H)$ since the change

$$\begin{matrix} q(u) \to q(u)\,q \\ q(v) \to q(v)\,q \end{matrix} \qquad 0 \neq q \in H \tag{58}$$

only corresponds to a different parametrization.

The subgroup of $SL(4,\mathbb{C})$, which acts on $P^3(\mathbb{C})$, and commutes with $[\sigma]$ is just $SL(2,H)$ which therefore acts on the real lines images of

$$Z_x = \begin{pmatrix} x v \\ v \end{pmatrix} \qquad v \in P^1(\mathbb{C}) \tag{59}$$

according to

$$\left\{ SL(2,H) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \ x \in H \right\} \longrightarrow Ax + B \, \frac{1}{Cx + D} \quad \in H \tag{60}$$

as expected.

Another useful description of these real lines is through the introduction of their Plücker coordinates. Let

$$F = Z_x \wedge \sigma Z_x \tag{61}$$

Then it is easy to check that, up to an overall scale which depends on v alone, F is determined by x and that

$$\operatorname{tr} F * F = 0 \tag{62}$$

The components of F are called the Plücker coordinates of X [36], [37] and Eq. 62 is the equation of the Plücker quadric Q_4 in P^5, which is left invariant by $SO(5,1)$.

This concludes a brief summary of the euclidean version of the geometrical framework introduced by R. Penrose.

The relevance of isotropic two planes is the following [32], [38], [39] : if one assumes that the solutions to our problem can be analytically continued in some neighbourhood of X, the self duality condition Eq. 4 means that the curvature vanishes on all isotropic two-planes. One thus infers that the connection, restricted to such a two plane is a gauge. If u is the projective spinor defining the direction of an isotropic two-plane, and $\tilde{a}$ the quaternion associated to a,

$$u\,\tilde{a} = H^{-1}(x,u)\, u \frac{\partial}{\partial \tilde{x}} H(x,u) \tag{63}$$

The fundamental theorem of the theory, due to Atiyah and Ward [31] actually shows how the gauge function $H(x,u)$ can be found, in principle :

To each principal $SU2$ bundle, and connection with (anti) self dual curvature, there corresponds in a unique way a holomorphic (algebraic) $\mathbb{C}^2$ bundle over $P^3(\mathbb{C})$. The complex structure is defined uniquely by the connection.

The construction is summarized by the following diagram [40]

$$\begin{tikzcd}
 & & \mathcal{E}(SL2,C) \arrow[dll, "\pi^* a^c"'] \\
P^3(\mathbb{C}) \arrow[d, "\pi"', "P^1(\mathbb{C})"] & & \mathcal{E}(SU2) \arrow[ll, "\pi^* a"'] \\
 & & \mathcal{B}(SL2,C) \arrow[dl, "a^c"', "SL(2,C)"] \\
P^1(H) & & \mathcal{B}(SU2) \arrow[ll, "a"', "SU2"]
\end{tikzcd} \tag{64}$$

where :

- $\mathcal{B}(SU2)$ is the principal $SU2$ bundle over $P^1(H)$, a the connection defining horizontal subspaces.
- $\mathcal{B}(SL2,C)$ is the unique SL2,C extension of $\mathcal{B}(SU2)$, a^c the extension of

- $\mathcal{B}(SU2)$, $\mathcal{B}(SL2,C)$ are the inverse images of $\mathcal{B}(SU2)$, $\mathcal{B}(SL2,C)$ by the projection $\pi : P^3(\mathbb{C}) \xrightarrow{\pi} P^1(H)$ previously described ; $\pi^* a$, $\pi^* a^c$ the inverse images of a, a^c respectively.

An almost complex structure J is defined on $\mathcal{B}(SL2,C)$ by the complex structure of $SL(2,C)$, along the fibres, and by lifting the complex structure of $P^3(\mathbb{C})$ to the horizontal subspaces defined by $\pi^* a^c$.
The integrability condition [35]

$$J\left([X,Y] - [JX,JY]\right) = [JX,Y] + [X,JY] \tag{65}$$

where X, Y are arbitrary vector fields, which insures that the Lie bracket of two holomorphic vector fields $X + iJX$, $Y + iJY$ is again holomorphic, i.e. of the form $Z + iJZ$ is known to be sufficient for J to define a complex structure. It can be checked here by choosing for X, and Y either horizontal or vertical fields. The self-duality condition is part of the integrability condition (63), for X, Y horizontal, $[X,Y]$ vertical, the other parts being trivially checked. This shows that $\mathcal{B}(SL2,C)$ is holomorphic. $\mathcal{B}(SL2,C)$ has two further properties : a reality property related to its construction from $\mathcal{B}(SU2)$ of which it is a kind of complexification, and a triviality property : the restriction of $\mathcal{B}(SL2,C)$ to real lines of $P^1(H)$ is trivial. This leads to the construction of the gauge function $H(x,u)$ in Eq. 63 as indicated by Ward [32] and also used by other authors [38], [39], as follows. Cover $P^1(\mathbb{C})$ by the two open sets Ω_0, Ω_∞:

$$\Omega_{\substack{0\\ \infty}} = \left\{ u = \begin{pmatrix} \tilde{z} \\ 1 \end{pmatrix}, \quad \tilde{z} \neq \begin{matrix}\infty \\ 0\end{matrix} \right\} \tag{66}$$

The restriction of $\mathcal{B}(SL2C)$ to a real line x is defined by a transition function $F(x,u)$ holomorphic in $\Omega_0 \cap \Omega_\infty$, with value in $SL(2,\mathbb{C})$. On the other hand $F(x,u)$ is not arbitrary :

$$F(x,u) = G(v,u)\Big|_{v = xu} \tag{67}$$

where G is the transition function of $\mathcal{B}(SL2C)$ in $\Omega_0^\varepsilon \cap \Omega_\infty^\varepsilon$, where $\Omega^\varepsilon_{\substack{0\\ \infty}}$ denotes a neighbourhood of $\Omega_{\substack{0\\ \infty}}$ in $P^3(\mathbb{C})$. Thus, F is invariant under the translation $x \to x + \lambda \otimes u\sigma_2$ and thus fulfills the differential equation

$$D_u F = 0 \tag{68}$$

where

$$D_u = u \frac{\partial}{\partial \tilde{x}} \tag{69}$$

is the operator appearing in Eq. 63. Of course G is homogeneous of degree 0 in u, v, and so F is homogeneous of degree 0 in u. Triviality means that one may split F according to

$$F = F_0 \, F_\infty^{-1} \tag{70}$$

F_{∞}^{0} in $SL(2,C)$, holomorphic in Ω_{∞}^{0}

It follows that

$$F_0^{-1} D_u F = F_\infty^{-1} D_v F \tag{71}$$

both sides continue each other in $\Omega_0 \cup \Omega_\infty$ and because they are homogeneous of degree 1 in u, they are linear in u, so that one may write

$$u\,\tilde{a} = F_0^{-1} D_u F = F_\infty^{-1} D_u F_\infty \tag{72}$$

where $\tilde{a}$ depends only on x and is identified with the connection; furthermore the identity

$$D_u\,(F_0^{-1} D_v F) - D_v\,(F_0^{-1} D_u F) - [F_0^{-1} D_u F, F_0^{-1} D_v F] = 0 \tag{73}$$

yields the self duality condition on the curvature. This essentially summarizes Ward's argument [32].

Now the structure of $\mathcal{E}(SL2,C)$ contains lots of information into which the author cannot go by mere lack of competence. For instance, the lines x on which triviality breaks down are represented by intersecting the Plücker quadric with an algebraic surface of degree n

$$P_n(F) = 0 \qquad P_n \text{ homogeneous polynomial of degree } n \tag{74}$$

Such lines are called jumping lines, and will appear as singularities of a, since (72) will break down.

In particular, the n=1 instanton is associated with a hyperplane section of Q_4. The set of n=1 instantons is parametrized by such hyperplanes (5 parameters) and transformed into one another by $SO5,1\backslash SO5$, $SO5$ being the stabilizer of one real hyperplane. The corresponding model for the associated $\mathbb{C}^2$ bundle, the so-called null correlation bundle [36], [41] is explicit enough so that one may reconstruct the known solution [18], for which now uniqueness is proved.

Whereas the information on the manifold of singularities of a is far from complete, nothing is known about their nature. On the other hand, the problem can be linearized in principle through a sequence of nested Ansatz [31], [42] which generalize the 't Hooft Ansatz through the introduction of massless free fields with higher spins. However the gauge freedom

$$\begin{aligned} F_0 &\to G_0 F_0 \\ F_\infty &\to G_\infty F_\infty \end{aligned} \quad \text{i.e.} \quad F \to G_0 F G_\infty^{-1} \tag{75}$$

where $G_{\substack{0\\ \infty}}$ belongs to $SL(2,C)$, is holomorphic in $\Omega_{\substack{0\\ \infty}}$ and fulfills

$$D_U G_{\substack{0\\ \infty}} = 0 \tag{76}$$

does not seem to have lead so far to very tractable formulae. The occurrence of massless fields is a consequence of Eq. 68 together with detailed properties of $\mathcal{G}(SL2,C)$ according to which F can be gauged into a triangular form characterized by its non diagonal element.

This concludes a rather imperfect description of the subject. Many omissions, in particular in this last part, are due to the author's poor understanding of the already published results for which written proofs are yet to come but it is hoped that the bibliography will help the puzzled reader to obtain information at the right source.

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