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SUMMATION OF SERIES OF QUASICLASSICAL PERTUBATION THEORY AND SPECULATIONS ABOUT THE STRUCTURE OF EXACT SOLUTIONS

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A b s t r a c t

Quasiclassical series of pertubation theory for ground energy state of anharmonic oscillator with O(2 n) symmetry and for Gell-Mann-Low function of scalar theory with strong nonlinearity with O (2 n) internal symmetry group are summed up. It is shown, that there exists the logarithmic singularity $e^{1/g}$ Ei $(-1/g)$ at $g = 0$. Possible modifications due to the corrections of quasiclassical quautization are discussed.

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In his pioneering papers [1, 2] Lipatov has proposed the quasiclassical method of calculating of the functional integral and has found the behavior of high order terms of the pertubation series for the Gell-Mann-Loy function (GLF) in renormalizable scalar field theory φ^{2N} $(N \geq 2)$. This method gives the possibility to perform the analogous calculation in the other quantum field theories and in quantum mechanics. In particular, it was applied by Brezin, Le Guillou, Zinn-Justen [3] to consideration of the ground energy state of the anharmonic oscillator with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i=1}^{n} x_i^2 + g \left(\sum_{i=1}^{n} x_i^2 \right)^N . \qquad (1)$$

They also have discussed the coefficients in the pertubation series for GLF in renormalizable scalar field theory with

$$\mathcal{L}_{int} = - \frac{g}{(2N)!} \int d^d x \left(\sum_{i=1}^{n} \varphi_i^2 \right)^N , \quad d = \frac{2N}{N-1} . \qquad (2)$$

In all cases the final results are represented by the following structure

$$B_g = c \sum_{k=r}^{\infty} (k(N-1))! \, a^k k^b g^k \left(1 + O\left(\frac{1}{k}\right) \right), \qquad (3)$$

where a, b, c are numbers related to numbers of degree of freedom n and degree of nonlinearity N. It is worth noting that the series obtained for GLF corresponding to the Lagrangian (2) coincides at n = 1 with Lipatov's result. For n = 1 Shirkov [4] has summed up the series and has obtained the answer in the closed form.

The purpose of this paper is the Borel summation of the series (3) for oscillator and for the GLF of the field theory (2). We consider only the case of even n (see Eqs. (1) - (2)). On the other hand, it is clear that the answers for odd mode oscillators and for the GLF Eq.(2) with odd numbers of fields have the structure analogous to that discovered by Shirkov [4] .

Let's consider the Eq.(3), Since $a < 0$ in Eq.(3) (see [1-3]) one may perform the Borel summation of the series (3) by Eiler transformation

$$f(x) = \int_0^\infty dy\, e^{-y} F(xy). \tag{4}$$

We introduce a multiplier $x^{k(N-1)}$ in the series (3) ($x = 1$ series thus obtained coincides with initial series (3)) and make the tranformation which is inverse that of Eq.(4). This yields the series $c\sum_{k=r}^{\infty} a^k k^b g^k y^{k(N-1)}$, which is convergent at $y < 1$. For convenience, let's make this transformation again with respect to variable $z = y^{N-1}$. So the final result has the form

$$\widetilde{B}(t) = c\sum_{k=r}^{\infty} \frac{(ag)^k\, k^b\, t^k}{k!}. \tag{5}$$

Eq.(5) is related to the initial series (3) by twofold Eiler transformation

$$B_b(x=1) = \iint_0^\infty dy\,dt\; e^{-(t+y)}\,\widetilde{B}(t\,y^{N-1}). \tag{6}$$

If $b = m$ is integer, one may use the method suggested in [5] for summation of (5). After summation B_m has the form $(A = -ag)$

$$B_m = \sum_{k=0}^{m} S(m,k)\,(-A)^k \cdot k! \int_0^\infty dy\, \frac{e^{-y}\, y^{k(N-1)}}{(1+Ay^{N-1})^{k+1}} - P_r(A) \tag{7}$$

where $S(m, k)$ are Stirling numbers of the second kind, $P_r(A)$ is polynomial of the power r, which appears, when the summation in Eq.(3) begins from $k = r$. For simplicity let's consider the particular case $N = 2$ (see (1)-(2)). In this case general expression (7) takes the more simple form

$$B_m(A) = \left(-\frac{1}{A}\right)\sum_{k=0}^{m} S(m,k)(-)^k k! \left\{ e^{\frac{1}{A}} E_i\left(-\frac{1}{A}\right) \sum_{\ell=0}^{k} \frac{\binom{k}{\ell}}{\ell! A^\ell} - \sum_{\ell=1}^{k} \frac{\binom{k}{\ell}(-)^\ell}{\ell!} \sum_{i=0}^{\ell-1} \frac{(-)^i (\ell-1-i)!}{A^i} \right\} - P_r(A) \tag{8}$$

where $Ei(-\frac{1}{A})$ is the integral exponential. Thus the correction to ground oscillator energy state* is equal to:

$$\Delta E = \frac{6^p}{3\pi g} B_{p-1}(3g) = e^{\frac{1}{3g}} \cdot Ei\left(-\frac{1}{3g}\right) P_p\left(\frac{1}{g}\right) + P'_{p-1}\left(\frac{1}{g}\right). \tag{9}$$

And GLF $\beta(G)$ $(G = g/16\pi^2)$ of Eq.(2) has the form

$$\beta(G) = \frac{c}{G} B_{p+3}(G) + cG = e^{\frac{1}{G}} Ei\left(-\frac{1}{G}\right) \widetilde{P}_{p+4}\left(\frac{1}{G}\right) + \widetilde{P}'_{p+3}\left(\frac{1}{G}\right) + cG, \tag{10}$$

where P_n, P'_n, $\widetilde{P}_n$, $\widetilde{P}'_n$ are polynomials of the n-th power without constant terms, but $P'_0 = 6/\pi$. It is worth noting that the asymptotical expansion of Eqs.(9)-(10) at $g = 0$ gives the initial series (3) and there exists the branch point at $g = 0$.

In order to investigate the structures (9)-(10) we write some particular cases**

$$\Delta E(n=2) = \frac{2}{\pi g} e^{\frac{1}{3g}} Ei\left(-\frac{1}{3g}\right) + \frac{6}{\pi} \cdot \tag{9.1}$$

$$\Delta E(n=4) = -\frac{12}{\pi g}\left[e^{\frac{1}{3g}} Ei\left(-\frac{1}{3g}\right)\left(1 + \frac{1}{3g}\right) + 1\right] \tag{9.2}$$

and GLF $\beta(G)$ at $n = 2$ has the form

$$-\frac{\beta(G)}{c} = \frac{e^{1/G} Ei(-1/G)}{G}\left(1 + \frac{15}{G} + \frac{25}{G^2} + \frac{10}{G^3} + \frac{1}{G^4}\right) + \frac{1}{G}\left(4 + \frac{17}{G} + \frac{9}{G^2} + \frac{1}{G^3}\right) - G \tag{10.1}$$

---

* If $n = 2p$ and $N = 2$, then $m = p-1$, $A = 3g$, $c = -6^p/\pi$ in Eq.(1) and $m = p+3$, $A = g/16\cdot\pi^2$, c is the number which is equal to 0,7676 for $n = 2$ for GLF of Eq.(2).

** Constant $6/\pi$ in Eq.(9.1) is related to nonzero coefficient at $k = 0$ in Eq.(3). This constant term appears only in two-mode oscillator.

Now consider the strong coupling limit $g \to \infty$. Then the Eqs.(9)-(10) change so that

$$\Delta E(n=2) \propto 6/\pi - \frac{2}{\pi}\frac{\ln g}{g} + O(1/g), \tag{11.1}$$

$$\Delta E(n=4) \propto d_p \frac{\ln g}{g} + O(1/g), \tag{11.2}$$

$$\beta_{n=2p}(G) \propto cG + \tilde{d}_p \frac{\ln g}{g} + O(1/G), \tag{12}$$

where d_p, $\tilde{d}_p$ are some calculatable numbers.

Now we consider the problem of corrections of order $O(\frac{1}{k})$ to Eq.(3). It is clear these corrections may be very important as for as Eq.(3) is divergent series. On the other hand, the result (11.2) seems to be strange from the physical point of view. This fact emphasizes the importance of correction. It is natural to try to estimate the correction terms in the framework of the most common suggests ous and to say something about exact answer.

It is known [1-4] that the comparison of the first few terms of Eq.(3) with results of exact pertubation calculations [6] for GLF of Eq.(2) when n = 1,2,3 shows that the relative error diminishes rapidly as k grows.* If this fact is not occasional, than we may suppose that the correction series is fast convergent. Let is suppose the exact coefficient in Eq.(3) has the form $A_k^{asymp.} \cdot f(\frac{1}{k})$, where at $k \to \infty$ $f(\frac{1}{k})$ is the regular function, has singularities at $|k| < 1$ and does not contain the multiplier $(-)^k$. Then $f(\frac{1}{k})$ has a representation in the form of Loran series

$$f(1/k) = \sum_{i=0}^{j} \alpha_i / k^i , \tag{13}$$

where α_i are some constants.

* For k = 4 relative error is smaller than 10 %.

Finally, we will consider some examples Eq.(13):

1) If $j = p - 1$ in Eq.(1) and $j = p+3$ in Eq.(3), then the structure Eqs. (9)-(10) remains unchanged, but some coefficients of polynomials P_n, P'_n, $\widetilde{P}_n$, $\widetilde{P}'_n$ varying. In particular, constant terms appear in P'_n and $\widetilde{P}'_n$, which change the asymptotics (11.2). and preasymptotical behavior (12)

$$\Delta E(n=2p) \propto Const + O\left(\frac{\ln g}{g}\right), \quad p \geq 1 \qquad (11.3)$$

$$\beta_p(G) \propto Const \cdot G + O(1). \qquad (12.1)$$

2) If $j = p$ in Eq.(1) and $j = p+4$ in Eq.(2), then the structure (9)-(10) are unchanged too and the only modification is the varying of polynomical coefficients. However, constant terms appear in P_n, $\widetilde{P}_n$. They yield the following asymptotics

$$\Delta E(n=2p) \propto Const \cdot \ln g, \quad p \geq 1 \qquad (11.4)$$

$$\beta(G) \propto Const \cdot G + O(\ln G) \qquad (12.2)$$

3) If $j > p$ in Eq.(1) and $j > p+4$ in Eq.(2) the situation is indefined. Beginning from $i = p+1$ in Eq.(1) and $i = p+5$ in Eq.(2) (see Eq.(13)) corrections change the structures (9)-(10) and terms of the form $Const \cdot (\ln g)^{l}$ arise in asymptotics, where l grows with each next correction, but the singularity character at $g = 0$ remains constant.

Let is notice the analogous situation arises in odd-mode oscillator with O (2n+1) space group Eq.(1) and in field theory with odd numbers of fields with O (2n+1) internal group Eq.(2). It is clear by means of GLF, discowered by Shirkov [4] for field theory (2) with one field in leading order on 1/k, that the structures for E of Eq.(1) and $\beta(G)$ of Eq.(2) have the forms

$$\Delta E = \frac{1}{\sqrt{ag}} e^{1/ag} Erfc\left(\frac{1}{\sqrt{ag}}\right) P\left(\frac{1}{ag}\right) + P'\left(\frac{1}{ag}\right) \qquad (14)$$

$$\beta(G) = \frac{1}{\sqrt{G}} e^{1/G} Erfc\left(\frac{1}{\sqrt{G}}\right) \widetilde{P}\left(\frac{1}{G}\right) + \widetilde{P}'\left(\frac{1}{G}\right) + Const\cdot G \qquad (15)$$

with root asymptotics at $g(G) \to \infty$.

In conclusion I would like to note an attractive opportunity. If quasiclassical quantization for Eqs.(1)-(2) coincides with exact one as in harmonic oscillator energy levels or Sine-Gordon equation mass spectrum [7]. It would give rises to answer on many above questions.

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