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ON THE VISUALIZATION OF BOLYAI-LOBATCHEVSKY'S GEOMETRY

H. V. Fagundes*

Instituto de Física Teórica, Rua Pamplona, 145
Caixa Postal 5956
01405 - São Paulo
Brazil

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Instituto de Física Teórica - São Paulo - Brasil

Abstract

The plane geometry of Bolyai and Lobatchevsky has physical applications, e.g. in Friedmann's cosmological model. ~~Our~~ The approach to the problem is to use Reichenbach's general result (definition of congruence) and add a prescription on the observer's position as also essential for the visualization.

I. INTRODUCTION

If the Universe at large does indeed resemble something like Friedmann's open model[1], we may have to become more familiar with Bolyai-Lobatchevsky's (BL) geometry. Indeed, the latter is the geometry of a plane in Friedmann's model with negative curvature, and elsewhere we have shown how to take this into account in the process of triangulation for the determination of the great intergalatic distances[2].

Here we develop a few rules which may allow one to visualize plane BL geometry. In particular, we show how an observer is to make the necessary adjustments of perception so that BL geometry gradually becomes intuitive to him or her. That this is possible has been shown by H. Reichenbach in general[3]. His basic idea is that non-Euclidean geometries demand a new conception of the idea of congruence. We apply this idea to plane BL geometry.

In Section II we make a distinction between the primitive intuition of space (which is Euclidean) and the corrected intuition needed to visualize non-Euclidean space. In Sec. III we show that an appropriate image of the BL plane is obtained from polar coordinates where the pole is always at the point of observation. Using BL trigonometry, we determine the appearance of geodesics as seen from the pole,using the concepts of primitive and corrected intuition. We also study how such appearance changes as the pole is changed. Then (Sec.IV) we illustrate how some properties or theorems of BL geometry become visualizable by this method.

II. PRIMITIVE AND CORRECTED GEOMETRICAL INTUITION

If we make a polar plot of the plane (Fig.1), we have for the measure of a segment AB the expression

$$d\ell^2 = dr^2 + r^2\, d\varphi^2 \tag{1}$$

Equation (1) gives the Euclidean length of the segment AB. We say that such measure of AB corresponds to <u>primitive intuition</u>, because it is a matter of everyday experience that Eq.(1) gives the length of AB. But this is because our ordinary experience of space is quite limited to small distances.

The BL measure of length AB is[4]

$$d\ell^2 = dr^2 + (R \sinh \tfrac{r}{R})^2\, d\varphi^2 \tag{2}$$

where R is the space constant of the geometry, which in the application to cosmology is of the order of magnitude of the observable universe. It is convenient to take $R = 1$, so we rewrite eq.(2) as

$$d\ell^2 = dr^2 + \sinh^2 r\, d\varphi^2 \tag{3}$$

(In the application to Friedmann's open model one has to take into account that $R = R(t)$, See Ref.2).

Eq.(3) does not differ from Eq.(1) in any practical calculation with small distances $(r \ll 1)$. But for distances of the order $r \gtrsim 1$, Eq.(3) is different from Eq.(1): purely radial distances are the same, but transverse lines have different lengths. Thus Reichenbach[3] asks us to change the definition of congruence for transverse line elements located at different distances from the origin. In our case the appropriate definition is (cf. Fig.2):

$$AB = CD$$

if

$$(\sinh OA)d\varphi_1 = (\sinh OC)d\varphi_2 \, . \qquad (4)$$

For example, let OA = 1, and OC = 2 then AB = CD if $d\varphi_1/d\varphi_2 = \sinh 2/\sinh 1 = 3.09$. (In Euclidean geometry this ratio would be 2).

If we get accustomed to automatically think of Eq.(4) when we compare transverse segments, we get a corrected intuition, appropriate for the visualization of BL geometry.

III. POLAR COORDINATES WITH THE OBSERVER AT THE POLE

The polar coordinates used in Eq.(3) above are our point of departure for the process of visualization. This is because in this system of coordinates not only radial distances, but also central angles need no correction from the primitive intuition. Indeed, in Fig. 3, the distance OP is just r, and angle POQ is just $\varphi_2 - \varphi_1$.

However, a straight line (geodesic !) not passing through O, like PXR on Fig. 3, will look curved to the primitive intuition. The reason, of course[3], is that PXR is covered by less measuring rods than is the "apparently straighter" line PYR. Therefore, the corrected intuition will see PXR as the straight line through P and Q. Analogously, angle SPT on Fig. 3 is in general different from its primitive appearence. This is a consequence of the changed definition of congruence for transverse line elements (Eq.4).

As the next important step for the visualization, we specify that O, the coordinate's pole, be regarded as the point of observation of the plane, i.e., the point where the observer is located. To illustrate these ideas, let ABC be a triangle with side AC = b = 0,80, and angles A = 45°, and C = 90° given (Fig.4). The other elements are calculated by Bl. trigonometry[5]. Thus

$$\tanh b = \tanh c \cos A \qquad (5)$$

gives

$$c = 1.73 \qquad (6)$$

Analogously, from

$$\cosh c = \cosh a \cosh b \qquad (7)$$

we get

$$a = 1.42 \qquad (8)$$

and from

$$\tanh b = \sinh a \tan B \qquad (9)$$

$$B = 0.329 \text{ rad} = 18°50' \qquad (10)$$

Now let us look at this triangle according to our rules for visualization: i.e., using polar plots, with the observer O on the pole.We shall do this for three positions of O: on A , on the middle point between A and C, and on C.

1) O = A. We get Fig. 5, where two other radial distances are given, besides OC and OB. They also have been calculated by BL trigonometry.

2) O = Middle point between A and C. The appearance is that of Fig. 6.

3) O = C. The observer sees Fig. 7. Now CB looks immediately straight, while AB needs the corrected intuition.

Even in cases 1 and 2, where it is not a central angle, C has the common appearance of 90° angle. (This is not always so).

We speculate, like G. Gamow in a similar context, that if a human observer did actualy have these experiences by travelling between A and C- i.e., if the space constant R were of the human scale of dimensions - he or she would regard our corrected intuition as the natural one.

IV. VISUALIZATION OF SOME OTHER PROPERTIES OF THE BL PLANE

We now illustrate a few more facts of BL geometry, applying our visualization rules.

1) The parallel angle $\Pi(d)$ in a point P, with regard to a straight line ℓ which passes at a distance d from P, is an acute angle (Ref. 5, p. 58)(Fig. 8A, B)

The parallels to ℓ through P meet ℓ at infinity. As any straight line not through O looks convex to O (cf. Fig.3) it is obvious that

$\Pi(d) < 90°$.

2) Two non-intersecting straight lines a and b continuously diverge on each side of their common perpendicular (Ref. 4, p. 89)

3) There are no similar triangles (Ref. 5, p. 82)

We shall begin with Fig. 5 and add to it a perpendicular C'B' to AC, from a point C' distant AC' = 0.50 from A

In Euclidean geometry it would be

$$\frac{AB'}{AB} = \frac{C'B'}{CB} = \frac{AC'}{AC}$$

Here, however, a simple calculation gives

$$AB'/AB = 0.45$$

and

$$C'B'/CB = 0.41$$

while, by construction,

$$\frac{AC'}{AC} = \frac{5}{8}$$

If we consider radial distances (from O) in Fig. 10, it becomes visually clear that the triangles ABC and AB'C' are not similar .

Acknowledgement

I am grateful to Dr. Charles E. Bures, Professor of Philosophy at the California Institute of Technology, who some years ago called my attention to Reichenbach's work on space and time[3].

References

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Figure Captions

Fig. 1 Line element AB in polar coordinates.

Fig. 2 AB is defined as congruent to CD.

Fig. 3 The main advantage of the polar coordinates is that radial lines and central angles need no correction for the visualization. Non-radial lines, like PXR , and non-central angles, like TPS, do need correction.

Fig. 4 On this figure one cannot see the values given in Eqs. (6), (8) and (10).

Fig. 5 Triangle ABC as seen from A.

Fig. 6 Triangle ABC as seen from the middle of basis AC.

Fig. 7 Triangle ABC as seen from C.

Figs. 8A, B Illustrating that angle $\Pi(d)$ is acute.

Fig. 9A, B Illustrating the divergence of non-intersecting lines.

Fig. 10 Illustrating the non-similarity of triangles with two equal angles.

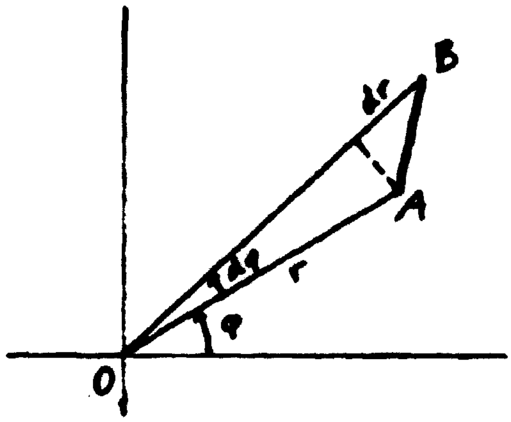

Fig. 1

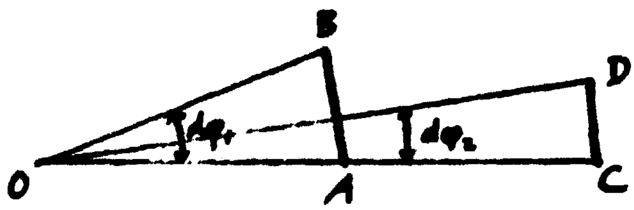

Fig. 2

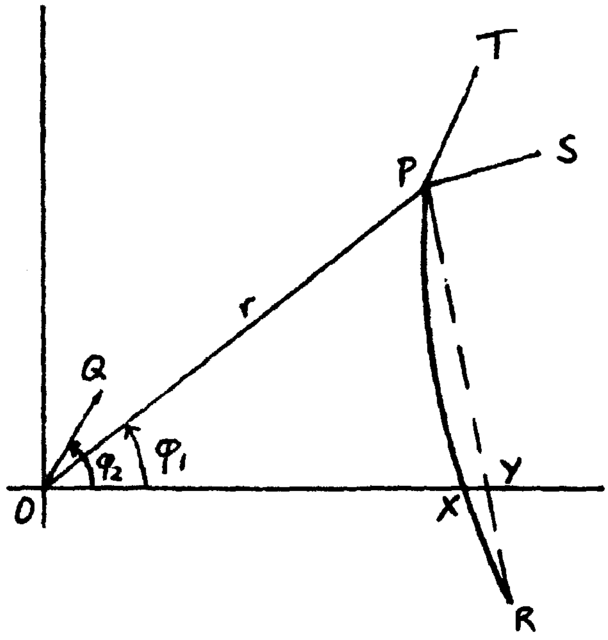

Fig. 3

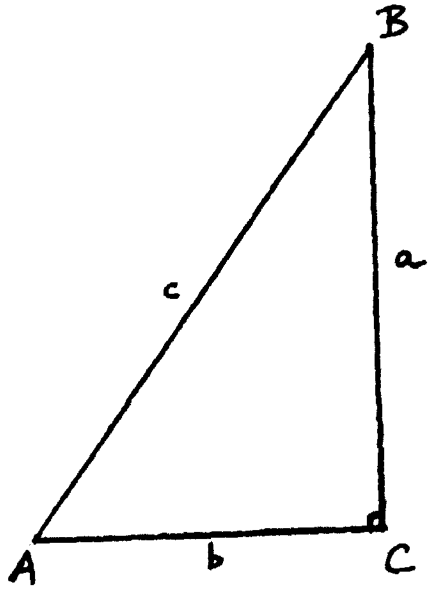

Fig. 4

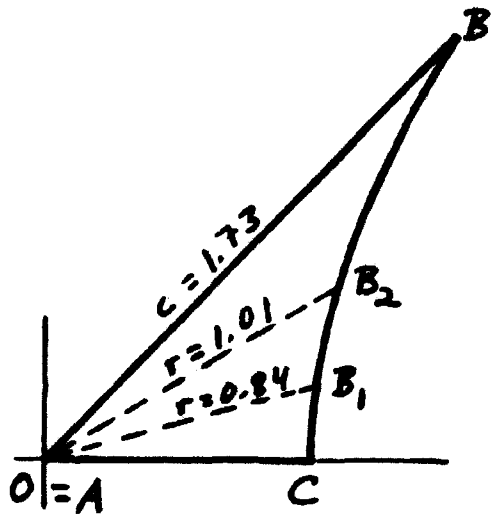

Fig. 5

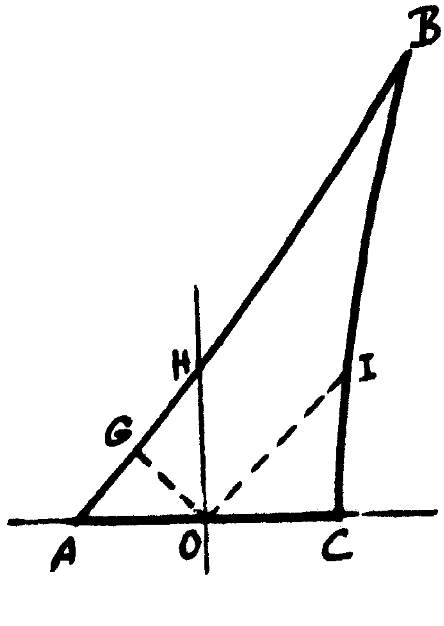

Fig. 6

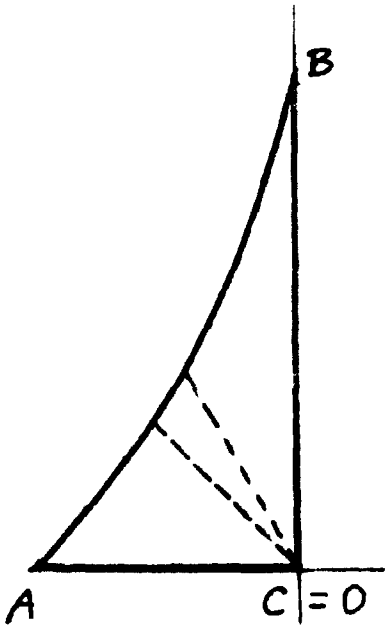

Fig. 7

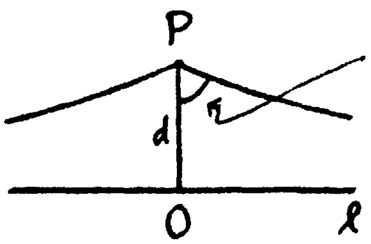

Fig. 8 A

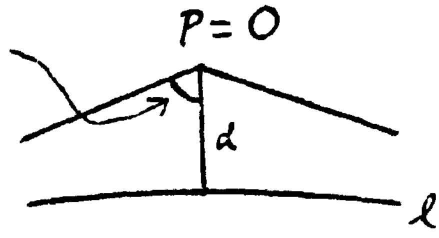

Fig. 8 B

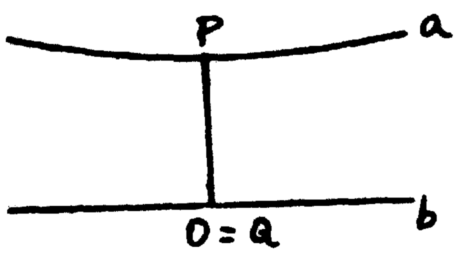

Fig. 9 A

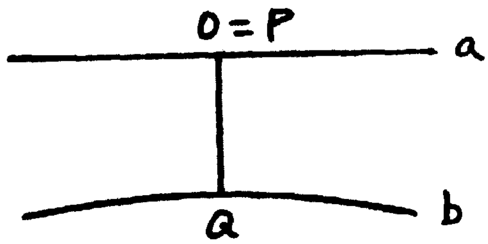

Fig. 9 B

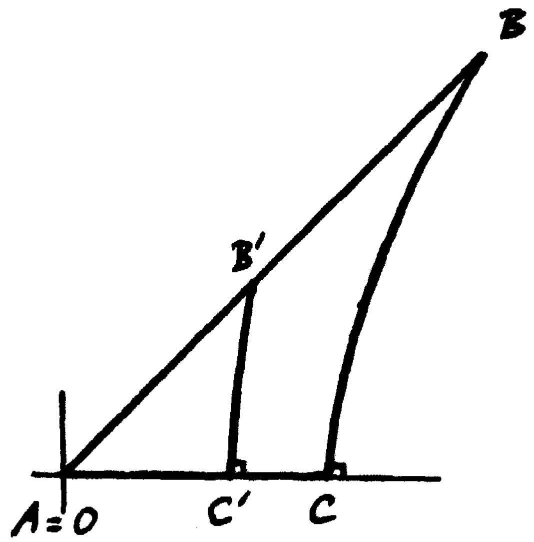

Fig. 10