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BUCHAREST - ROMANIA

INFLUENCE OF ERRORS ON THE CORRELATION BETWEEN r_π^2 AND $a_\mu(\pi^+\pi^-)$ THROUGH SPACELIKE DATA FOR THE PION FORM FACTOR (*)

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Abstract

We construct the correlation between the mean squared pion charge radius r_π^2 and the pionic contribution $a_\mu(\pi^+\pi^-)$ to the muon magnetic moment by means of spacelike data for the pion form factor and investigate the dependence of the correlation on errors of these data. We put into evidence, in the correlation picture, regions of (sometimes very high) instability of the conditional bounds of these magnitudes with respect to errors (and also with respect to each other). The existence of such instabilities in the physically interesting part of the correlation picture asks for extreme care in the interpretation of numerical values for the (conditional) bounds, considered without reference to this picture.

1. Introduction

In this paper we investigate the implications of the spacelike information for the pion form factor on the (squared) pion charge radius r_π^2 and the pionic contribution $a_\mu(\pi^+\pi^-)$ to the muon magnetic moment.

It is a well known fact that spacelike data alone can not predict limits on the values allowed for the pion charge radius, if one does not impose (explicitly or implicitly) severe restrictions on those analytic functions which one is prepared to accept as candidates for the pion form factor. These restrictions may take the form of hypotheses as e.g. that the pion form factor belongs to a family of functions (polynomials of a given degree, linear combinations of simple rational functions, etc.). They may, however, also be related to other quantities of physical interest, as the pionic contribution to the muon magnetic moment.

For this last quantity one gets nontrivial limitations (lower bounds) from spacelike data on the form factor, but they may depend, as was already noticed in Ref./1/, extremely streng on these data, particularly on their errors.

If one accepts as candidates for the pion form factor (the class H^2 of) all those analytic functions, for which the integral expressing $a_\mu(\pi^+\pi^-)$ is finite, then from those of these functions, for which $a_\mu(\pi^+\pi^-)$ takes a given value and which are (in a prescribed way) close to the spacelike data, one already gets nontrivial bounds on r_π^2. Instead of looking at bounds on r_π^2, given by fixed values of $a_\mu(\pi^+\pi^-)$, and of investigating the dependence of the bounds on these values, one may take the other point of view. This is to derive lower bounds on $a_\mu(\pi^+\pi^-)$ in the family of those functions of H^2, for which r_π^2 takes a given value and which are close (in the same sense as before) to the spacelike data, and to investigate the dependence of these bounds on the values assumed for r_π^2. Both points of view are but two ways of expressing the correlation between $a_\mu(\pi^+\pi^-)$ and r_π^2, established by the requirement of a certain degree of compatibility of the analytic functions of H^2 with the spacelike data. The information contained in these data on r_π^2 and $a_\mu(\pi^+\pi^-)$ may then be expressed through the dependence of their correlation on the degree of compatibility one requires.

In the next Section of the paper we bring the quantitative form of our considerations to the point suitable for numerical computation and in the last Section we discuss the results of these computations and their implications. In an Appendix we give a few mathematical details.

2. Quantitative Considerations

In Ref./1/ the quantity $a_\mu(\pi^+\pi^-)$ was brought to the form

(1) $$a_\mu(\pi^+\pi^-) = \frac{\alpha^2}{96\pi}\,\frac{m_\mu^2}{16\,m_\pi^2}\,\|\ell\|^2 ,$$

where

(2) $$\|\ell\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\ell(z)|^2\,d\theta , \qquad z = e^{i\theta}$$

and $\ell(z)$ is a (real) analytic function in the unit disk $|z|<1$, related to the pion form factor $F(z)$ by $\ell(z) = g(z)\,F(z)$. There also the explicit form of the known function $g(z)$ is given. Here we note that $h(0) = g(0)$ $(= 0.5033)$, due to normalization $f(0) = 1$, and $h'(0) = g'(0) + \frac{8}{3\lambda^2} g(0)\, r^2$ $(r^2 = r_\pi^2$, λ = the Compton wave length of the pion, $g'(0) = -0.8080$).

The translation of $a_n(\pi^+\pi^-)$ and r^2 into the framework of the analytic functions $h(z)$ leads to those functions $h(z)$, for which $\|h\|^2 < \infty$ (the Hardy class H^2) and for which the values $h(0)$, $h'(0)$ are prescribed. Further, the translation to $h(z)$ of measurements of $f(z)$ in the points $z = x_i$ ($i = 1, \dots, n$) leads, if we have mean values a_i and standard deviations σ_i, to the problem of choosing a way to state precisely the meaning of the distance from a function $h(z)$ to these data. Our present choice extends that of Ref./1/ : we take a (positive) number $\varkappa$ and the functions $h(z)$ for which the values $h(x_i)$ lie within $\varkappa$ standard deviations from a_i :

$$(3) \qquad a_i - \varkappa\sigma_i \le h(x_i) \le a_i + \varkappa\sigma_i \quad , \quad i = 1, \dots, n \quad ;$$

to these functions we refer as situated within a distance $\varkappa$ from the data. This choice is very natural, although not the only possible one /2/. It allows, in addition, such a quantitative formulation of this particular problem, which makes its solution very simple : the determination of the values $a_n(\pi^+\pi^-)$ and r^2, allowed for functions $h(z)$ lying within a distance $\varkappa$ from the data, in their dependence on the values of $\varkappa$.

The solution of the problem may be given by means of a convenient explicit parametrization of all functions $h(z)$ from H^2 through the quantities $h(0)$, $h'(0)$, $h(x_i)$, on which we then impose constraints.

Such a parametrization is /3/

$$h(z) = h(0) + h'(0)z + \sum_{k=1}^{n} c_k r_k(z) + \tag{4}$$

$$+ z^2 \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_n}{1-x_n z} h_{n+1}(z)$$

with

$$r_k(z) = z^2 \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_{k-1}}{1-x_{k-1} z} \frac{(1-x_k^2)^{\frac{1}{2}}}{1-x_k z} \tag{5}$$

where $h_{n+1}(z)$ is an arbitrary function belonging to H^2 and the coefficients c_k are linear combinations of $h(0)$, $h'(0)$, $h(x_i)$. For completeness we sketch in the Appendix, following Ref./4/, the essential part of a very elementary derivation of general expansions of the type (4). This expansion has the virtue of orthogonality, i.e.

$$\|h\|^2 = h^2(0) + h'^2(0) + \sum_{k=1}^{n} c_k^2 + \|h_{n+1}\|^2 \tag{6}$$

and, because our data have no impact on $h_{n+1}(z)$, as seen from the translation of (3) from $h(x_i)$ to c_k :

(7) $P_n(\varkappa, h'(0))$

$$u_i - h(0) - h'(0)x_i - \varkappa \, \leq \sum_{k=1}^{i} c_k r_k(x_i) \leq$$

$$\leq a_i - h(0) - h'(0)x_i + \varkappa \, ,$$

it leads for functions with fixed values $h(0)$, $h'(0)$, $h(x_i)$ to the inequality

$$\|h\|^2 \geq h^2(0) + h'^2(0) + \sum_{k=1}^{n} c_k^2 \, . \tag{8}$$

If we relax the condition that the $h(x_i)$ are fixed, then the lowest value allowed for $\|h\|^2$ will be determined by the minimum of

$$(9) \qquad \|h\|_n^2 \equiv h^2(1) + h'^2(0) + \sum_{k=1}^{n} c_k^2$$

over the values of c_k, allowed (for fixed $h'(0)$ and $\varkappa$) by the $2n$ inequalities (7). The determination of this minimum is geometrically the determination of the point of a parallelotope $(P(\varkappa, h'(0)))$ of smallest distance to the origin, and algebraically the minimization of a positive quadratic expression under linear constraints (quadratic optimization). We denote the minimum by $h^2(\varkappa, h'(0))$. The dependence of $h^2(\varkappa, h'(0))$ on $h'(0)$ (for $\varkappa$ fixed) then expresses the correlation of $\|h\|^2$ and $h'(0)$, with the degree of compatibility with the spacelike data measured in terms of the number $\varkappa$ of standard deviations of the functions from them. The whole picture of correlation of $\|h\|^2$ and $h'(0)$, in its dependence of $\varkappa$, is then given by the surface $h^2(\varkappa, h'(0))$. The object of our computation is, therefore, (the most significant part of) this surface.

3. Results and Comments

We have chosen for numerical computations those seven data points from Ref.s /5-7/ with the momentum transfers given in Ref. /3/ (Table I). The results of computation are represented in Figs. 1 ...

Before we comment on these results we shall present a few qualitative theoretical considerations which can be helpful in interpretation and in checking computations : One can easily show (Appendix) that $h^2(\alpha, t^2)$ $(h^2(\varkappa, h'(0))$, considered for convenience as a function of $\varkappa$ and τ^2 $= \ell^2(\varkappa, g'(0) + \frac{8}{3\lambda^2} g(0)\tau^2$) is a convex surface, i.e.

$$(10) \quad \alpha \ell^2(\varkappa_1, \tau_1^2) + (1-\alpha)\ell^2(\varkappa_0, \tau_0^2) \geq \ell^2(\alpha\varkappa_1 + (1-\alpha)\varkappa_0, \alpha\tau_1^2 + (1-\alpha)\tau_0^2)$$

for any number $0 \leq \alpha \leq 1$. Therefore also the curves describing the dependence of $\ell^2(\varkappa, \tau^2)$ on τ^2 (for $\varkappa$ fixed) are convex :

$$(11) \quad \alpha \ell^2(\varkappa, \tau_1^2) + (1-\alpha)\ell^2(\varkappa, \tau_0^2) \geq \ell^2(\varkappa, \alpha\tau_1^2 + (1-\alpha)\tau_0^2)$$

The minima ℓ^2_m of these curves and the values τ^2_m of τ^2, corresponding to them, depend on $\varkappa$: $\tau^2_m = \tau^2_m(\varkappa)$, $\ell^2_m = \ell^2(\varkappa, \tau^2_m(\varkappa)) \equiv \equiv \ell^2_m(\varkappa)$. The values $\ell^2_m(\varkappa)$ are exactly the minima of $\|\ell\|^2$, which one obtains if one imposes no constraints on $\ell'(0)$ (τ^2) /1/ and allows for the functions $\ell(z)$ a distance $\varkappa$ from the data. From (10) it also follows that $\ell^2_m(\varkappa)$ is a convex curve :

$$(12) \quad \alpha \ell^2_m(\varkappa_1) + (1-\alpha)\ell^2_m(\varkappa_0) \geq \ell^2_m(\alpha\varkappa_1 + (1-\alpha)\varkappa_0) \quad .$$

Now we describe the results by means of the quantities $\ell^2(\varkappa, \tau^2)$ and $\ell^2_m(\varkappa)$, $\tau^2_m(\varkappa)$. In Fig.1 we have represented the space curve $\tau^2 = \tau^2_m(\varkappa)$, $\|\ell\|^2 = \ell^2(\varkappa, \tau^2_m(\varkappa))$, the "edge" of the surface $\ell^2(\varkappa, \tau^2)$, together with its projections $\|\ell\|^2 = \ell^2_m(\varkappa)$, $\tau^2 = \tau^2_m(\varkappa)$, $\|\ell\|^2 = \ell^2_m(\tau^2)$ on the coordinate planes $(\|\ell\|^2, \varkappa)$, $(\tau^2, \varkappa)$, $(\tau^2, \|\ell\|^2)$, respectively. This curve was obtained by performing on our data the computations of Ref./1/ for various standard deviations $\varkappa$. Its usefullness is given by the fact that it shows the position and shape of the physically most interesting part of $\ell^2(\varkappa, \tau^2)$. For this part we have represented in Fig.2 the level curves $\ell^2_{m}(\tau^2$, [illegible] τ^2 [illegible] $\tau^2)$ in its dependence on τ^2, for various fixed values of [illegible]. The [illegible] part of this figure exhibits instability (high sensitivity) of the bound $\bar{\kappa}^2(\kappa,\tau^2)$ with respect to the variation of κ. Further, the steep parts of the level curves reflect instability regions of $\bar{\kappa}^2$ [illegible] with respect to the variation of τ^2.

Statistics suggests that for reliable results one should consider errors up to three standard deviations, i.e. [illegible]. If we then look for information from Fig. 2 on bounds for r_λ^2, determined by given values of [illegible], then it turns out that the upper bound depends very strongly on the chosen confidence level. On the other [illegible], if we fix r_λ^2 (or admit small variations [illegible]) in the neighbourhood of [illegible] and V^2, where it is believed to lie, [illegible] we already are in the region of instability [illegible] of about two or more by going from [illegible].

Because of the various kinds of [illegible] brought [illegible] by the correlation surface [illegible] to suggest preferred numbers [illegible] consider [illegible] illustration of a framework [illegible] make his own choice according to those arguments and [illegible] he considers to be strong enough to justify it.

The correlation of r_λ^2 and $\mathcal{Q}_\lambda(\kappa^2$ [illegible], of course, [illegible] if one computes it with more data (experimental and also theoretical) than we have done here. The volume of data used here, however, allows to gain again, without becoming involved in highly [illegible] numerical computations, an insight into the qualitative behaviour one expects for this correlation also in more complicated situations.

Appendix

1. Orthogonal Expansion in Rational Functions

If for a real function $h(z)$ in H^2 we know $h(0)$, then the natural (orthogonal) expansion is the Taylor series $h(z) = h(0) + \sum_{k=1}^{\infty} c_k z^k$ or

$$h(z) = h(0) + z\, h_1(z) \tag{A.1}$$

where $h_1(z) = \sum_{k=1}^{\infty} c_k z^{k-1}$ contains the "rest" of the expansion and is an arbitrary function in H^2. From $\|h\|^2 = h^2_{(0)} + \sum_{k=1}^{\infty} c_k^2$ it is evident that

$$\|h\|^2 = h^2_{(0)} + \|h_1\|^2 \tag{A.2}$$

These facts allow the derivation of the analogue of (A.1) and (A.2) for those $h(z)$ in H^2, which have a fixed value of $h(x)$ at $x \neq 0$. By a change

$$w = \frac{z-x}{1-xz} \qquad \left(z = \frac{w+x}{1+xw} \right) \tag{A.3}$$

of variables, which transforms the unit disk $|z| \leq 1$ (conformally) onto the unit disk $|w| \leq 1$ one brings $z = x$ to the origin $w = 0$. In the variable w the expansion then is the Taylor series, but the function to be expanded is not $h(z(w))$ if we want to preserve orthogonality, since in the definition (2) of $\|f\|^2$ there appears the factor

$$\frac{d\theta}{d\varphi} = \frac{|1-xz|^2}{1-x^2} \tag{A.4}$$

of the transformation (A.3) from $z = e^{i\theta}$ to $w = e^{i\varphi}$. The Taylor expansion is thus for $\tilde{h}(w) = h(z)\frac{(1-x^2)^{1/2}}{(1-xz)}$ and gives $h(x)\frac{(1-x^2)}{(1-xz)}$ [illegible] $\frac{z-x}{1-xz}\tilde{h}_1(w)$ or

(A.5) $$\ell(z) = c(x)\frac{(1-x^2)^{\frac{1}{2}}}{1-xz} + \frac{z-x}{1-xz}\,\ell_1(z)$$

and

(A.6) $$\|\ell\|^2 = c^2(x) + \|\ell_1\|^2$$

where $c(x) = \ell(x)(1-x^2)^{\frac{1}{2}}$ and $\ell_1(z) = \frac{(1-x^2)^{\frac{1}{2}}}{1-xz}\tilde{\ell}_1(z)$, since $\int_{-\pi}^{\pi}|\tilde{\ell}_1(e^{i\theta})|^2 d\theta = \int_{-\pi}^{\pi}|\ell_1(e^{i\theta})|^2 d\theta$.

Expansion (4),(6), for instance, is but the iteration of (A.5), (A.6) according to the scheme

(A.7)
$$\begin{aligned}
\ell(z) &= \ell(0) + \ell'(0)z + z^2\ell_1(z) && , \|\ell\|^2 = \ell^2(0) + \ell'^2(0) + \|\ell_1\|^2 \\
\ell_1(z) &= c_1\frac{(1-x_1^2)^{\frac{1}{2}}}{1-x_1 z} + \frac{z-x_1}{1-x_1 z}\,\ell_2(z) && , \|\ell_1\|^2 = c_1^2 + \|\ell_2\|^2 , \\
&\;\;\vdots \\
\ell_n(z) &= c_n\frac{(1-x_n^2)^{\frac{1}{2}}}{1-x_n z} + \frac{z-x_n}{1-x_n z}\,\ell_{n+1}(z) && , \|\ell_n\|^2 = c_n^2 + \|\ell_{n+1}\|^2
\end{aligned}$$

2. Convexity of the Surface $h^2(x, r^2)$

We consider two pairs of values $(x_0, \ell'_0(0))$; $(x_1, \ell'_1(0))$ and $(x_2, \ell'_2(0))$ and denote by $\ell_0(z)$ and $\ell_1(z)$ the functions which realise the minimum of (9) over the parallelotopes $P(x_0, \ell'_0(0))$ and $P(x_1, \ell'_1(0))$ respectively :

(A.8)
$$\begin{aligned}
h^2(x_0, \ell'_0(0)) &= \|\ell_0\|^2_n \\
h^2(x_1, \ell'_1(0)) &= \|\ell_1\|^2_n
\end{aligned}$$

Since the function $\hat{h}_\alpha(z) = \alpha \hat{h}_1(z) + (1-\alpha)\hat{h}_0(z)$, $0 < \alpha < 1$, obeys $\hat{h}_\alpha(0) = \varrho(0)$, $\hat{h}'_\alpha(0) = \alpha \hat{h}'_1(0) + (1-\alpha)\hat{h}'_0(0)$ and

(A.9) $$a_i - (\alpha\varkappa_1 + (1-\alpha)\varkappa_0)\,r_i \le \hat{h}_\alpha(x_i) \le a_i + (\alpha\varkappa_1 + (1-\alpha)\varkappa_0)\,r_i, \qquad i = 1, \ldots, n$$

it belongs to the parallelotope $P_n(\alpha\varkappa_1 + (1-\alpha)\varkappa_0,\ \alpha\hat{h}'_1(0) + (1-\alpha)\hat{h}'_0(0))$. On the other side the inequality

(A.10) $$\alpha|\hat{h}_1(z)|^2 + (1-\alpha)|\hat{h}_0(z)|^2 \ge |\alpha\hat{h}_1(z) + (1-\alpha)\hat{h}_0(z)|^2 \equiv |\hat{h}_\alpha(z)|^2$$

leads, by integration according to (2) and by the observation that together with $\|\hat{h}_i\|^2 = \|\hat{h}_i\|_n^2$, $i = 0, 1$, also $\|\hat{h}_\alpha\|^2 = \|\hat{h}_\alpha\|_n^2$, to

(A.11) $$\alpha\|\hat{h}_1\|_n^2 + (1-\alpha)\|\hat{h}_0\|_n^2 \ge \|\hat{h}_\alpha\|_n^2 .$$

But the function $\hat{h}_\alpha(z)$, even when it belongs to $P_n(\alpha\varkappa_1 + (1-\alpha)\varkappa_0,\ \alpha\hat{h}'_1(0) + (1-\alpha)\hat{h}'_0(0))$, does not necessarily give the minimum of (9) over this parallelotope. Therefore

(A.12) $$\|\hat{h}_\alpha\|_n^2 \ge \hat{\lambda}^2(\alpha\varkappa_1 + (1-\alpha)\varkappa_0,\ \alpha\hat{h}'_1(0) + (1-\alpha)\hat{h}'_0(0))$$

and the composition of (A.8), (A.11) and (A.12) gives the inequality (10) of the text.

Footnotes and References

(+) Work performed under contract with the Romanian Nuclear Energy Committee.

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/2/ I.Raszillier : Paralipomena to the rigorous phenomenology of the pion electromagnetic form factor, Institute of Physics, Bucharest, Preprint (1976).

/3/ Expansion (3) was given in I.Raszillier : On rigorous lower bounds for the hadronic contribution to the muon g-factor, Institute of Physics, Bucharest, Preprint (1972). There, and also in /1/ it was mentioned that the derivation of a general expansion of this type is straightforward.

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Captions

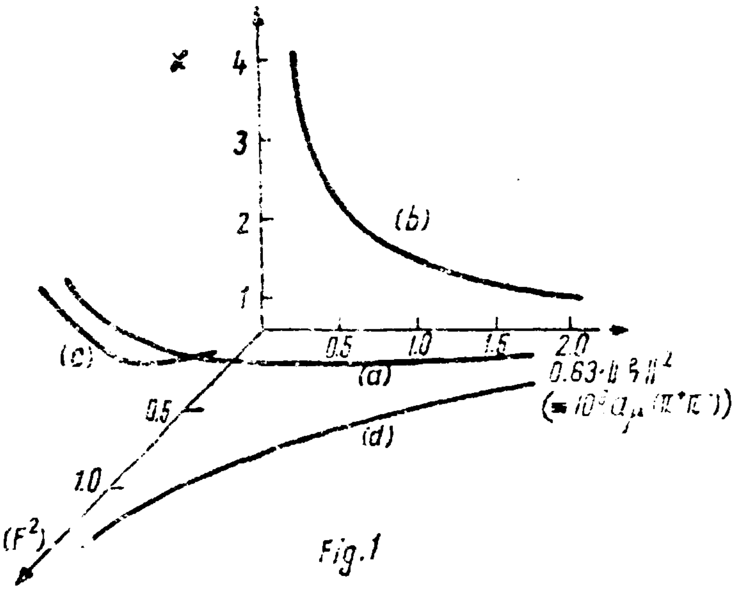
Table I : Experimental data, taken from Ref.s /5/-/7/, and their translation into the form used in this paper.

Fig.1 : The edge of the surface $k^2(x, r^2)$ (a), with its projections $\|k\|^2 = k_m^2(x)$ (b), $r^2 = r_m^2(x)$ (c), and $k_m^2(r_m^2)$ (d).

Fig.2 : Level curves of the surface $k^2(x, r^2)$ (the dashed line represents $k_m^2(r_m^2)$).

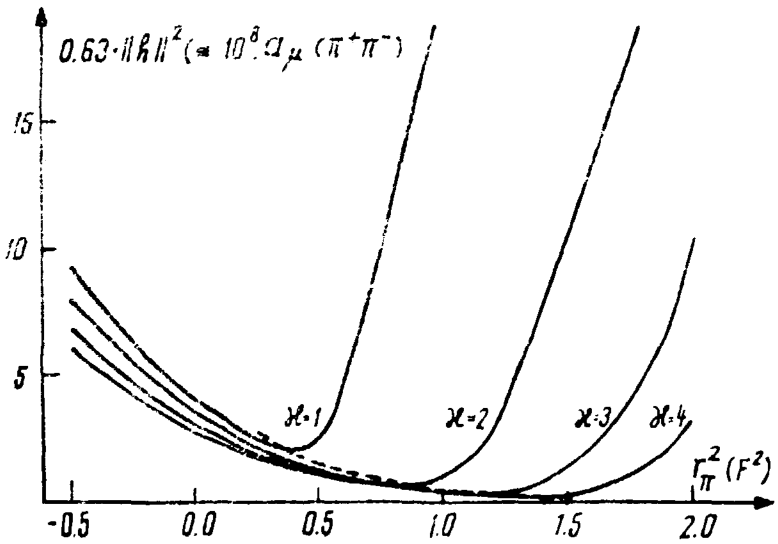
Table I

$-t_i$ $(GeV)^2$	x_i	$F_\pi(t_i)$ (exp)	$g(x_i)$	a_i	γ_i
0.176	-0.2870	0.786 ± 0.045	0.7269	0.5713	0.0327
0.294	-0.3720	0.606 ± 0.028	0.7832	0.4746	0.0219
0.396	-0.4230	0.550 ± 0.015	0.8127	0.4470	0.0122
0.620	-0.4991	0.453 ± 0.014	0.8486	0.3844	0.0119
0.795	-0.5399	0.380 ± 0.013	0.8629	0.3279	0.0112
1.216	-0.6059	0.292 ± 0.026	0.8764	0.2559	0.0228
1.712	-0.6547	0.246 ± 0.017	0.8770	0.2157	0.0149



(b)
(c)
(a)
(d)
$0.63 \cdot \|h\|^2$
$(\approx 10^8 a_\mu(\pi^+\pi^-))$
$r_\pi^2\ (F^2)$

Fig.1



$0.63 \cdot \|h\|^2 (\approx 10^8 \cdot a_\mu\ (\pi^+\pi^-)$
$\varkappa=1$
$\varkappa=2$
$\varkappa=3$
$\varkappa=4$
$r_\pi^2\ (F^2)$

Fig.2