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Correlations between r_π^2, $a_\mu(\pi^+\pi^-)$ and error functionals of spacelike data for the pion form factor *)

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*) Work performed under contract with the State Committee for Nuclear Energy.

CORRELATIONS BETWEEN r_π^2 , $a_\mu(\pi^+\pi^-)$, AND ERROR FUNCTIONALS OF SPACELIKE DATA FOR THE PION FORM FACTOR ≠

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Abstract

We formulate the information about the mean squared pion charge radius r_π^2 and the pionic contribution $a_\mu(\pi^+\pi^-)$ to the muon magnetic moment, contained in spacelike data for the pion form factor, in terms of a three-dimensional set $\Delta(d)$. This set expresses, for the class of all (real) analytic functions $f(z)$ in the unit disk $|z|<1$ and normalized to $f(o)=1$, to which the pion form factor belongs, the correlation between the values of r_π^2 , $a_\mu(\pi^+\pi^-)$ and their distances $d(f)$ to the data. The correlation is computed for two forms of the distance: an Euclidean distance $\chi(f)$ related to the usual χ^2 and a Chebyshev distance $\pi(f)$, which controls the deviations in the data points individually. It implies bounds for r_π^2 and $a_\mu(\pi^+\pi^-)$ and characterizes their stability properties and their dependence on the chosen form of $d(f)$.

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1. Introduction

The determination of physical quantities related to the pion form factor, like the mean squared pion charge radius r_π^2 or the pionic contribution $a_\mu(\pi^+\pi^-)$ to the muon magnetic moment, from spacelike data of the form factor includes two important elements:

a) the selection of a class of real analytic functions $f(z)$ (in the unit disk $|z|<1$) to which the pion form factor is believed to belong,

b) the choice of a distance function which measures the degree of compatibility between various functions of this class and the data.

The information contained in the data is then expressed as the set of values which these quantities take over the set of all functions out of the chosen class, situated within a given distance from the data.

If we consider measurements performed in the points $z=x_1, \ldots, x_n$ with the mean values $b_1, \ldots, b_n$ and standard deviations $\sigma_1, \ldots, \sigma_n$ there is an unlimited number of possibilities to define a distance $d(f)$ from a function to the data in terms of the values $f(x_i)$, b_i, and σ_i. Especially two of them are of common use and of intuitive appeal:

$$\chi(f) = \left(\frac{1}{n} \sum_{i=1}^{n} \left(\frac{f(x_i) - b_i}{\sigma_i} \right)^2 \right)^{\frac{1}{2}} \tag{1.1}$$

and

$$\pi(f) = \max_{1 \le i \le n} \left| \frac{f(x_i) - b_i}{\sigma_i} \right| \tag{1.2}$$

Mathematically, these functions have the properties of a norm in an n-dimensional space R^n of coordinates $\frac{f(x_i)-b_i}{\sigma_i}$. Physically, they imply different attitudes towards errors: whereas a given value of $\chi(f)$ only limits globally the deviations of $f(x_i)$ from b_i, without a control on their distribution between various points x_i, a value of $\bar{\pi}(f)$ controls each deviation individually. The quantitative relation between $\chi(f)$ and $\pi(f)$ is expressed by the inequalities

$$\chi(f) \leq \pi(f) \leq n^{\frac{1}{2}} \chi(f) \tag{1.3}$$

which indicate how much stronger $\pi(f)$ can be than $\chi(f)$. Although $\pi(f)$ is stronger than $\chi(f)$, it turns out that sometimes its use is computationally inconvenient because of bad differentiability properties.

The set of values of r_π^2 and $a_\mu(\pi^+\pi^-)$ allowed by a chosen class of functions is determined by

$$r_\pi^2 = 0.375\, ƛ^2 f'(0) \tag{1.4}$$

($ƛ$ = Compton wave length of the pion) and

$$a_\mu(\pi^+\pi^-) = \frac{\alpha^2}{96\pi} \frac{m_\mu^2}{16 m_\pi^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(z)|^2 d\theta \quad , \quad z = e^{i\theta} , \tag{1.5}$$

($\frac{\alpha^2}{96\pi} \frac{m_\mu^2}{16 m_\pi^2} = 0.63 \times 10^{-8}$), where $h(z) = g(z) f(z)$ and $g(z)$ is a known real function /1/ without zeros in $|z|<1$ (and of values $g(0)=0.5033$, $g'(0)=-0.8030$), when $f(z)$ goes through this class. If the class is chosen very narrow, e.g. as polynomials of a certain degree or linear combinations of a given number of other functions, as it is often done, then the set may be made very small. If one, further, accepts only the functions of distance smaller than a given value, it becomes even smaller or just empty.

The weakness of a narrow choice of functions lies in the risk to eliminate just the (true) form factor ; this implies a high degree of uncertainty for apparently rather precise results.

In this paper we describe a procedure which permits to determine the information contained in spacelike data for the pion form factor for both r_π^2 and $a_\mu(\pi^+\pi^-)$, without an artificial restriction of the class of functions f(z). Namely, we take all real functions f(z) which are normalized to f(o)=1 and for which the integral (1.5) is finite, i.e. a hyperplane in a Hilbert space. If we choose a distance d(f) in R^4 , then the information is represented by the set Δ(d) of points of coordinates (r_π^2 ,$a_\mu(\pi^+\pi^-)$, d(f)) one obtains when f(z) goes through the whole hyperplane. It certainly depends on the chosen distance. Since the procedure by which we determine the set does not use specific properties of d(f), we keep the arguments general. The numerical results are given, however for $\chi(f)$ and $\pi(f)$.

The set Δ(d) expresses the correlation between r_π^2, $a_\mu(\pi^+\pi^-)$ and d(f), particularly all conditional bounds of these magnitudes. The bounds for r_π^2, given by the condition that $a_\mu(\pi^+\pi^-)$ and d(f) do not exceed certain values, give a typical example where a natural selection of a narrow class of functions makes its appearance.

In the next section we reduce the determination of the set Δ(d) to a problem of n-dimensional convex optimization. Then, in sect.3, we determine general (mainly) convexity properties of Δ(d), and in sect.4 we derive from Δ(d) other correlations. These properties are true for any d(f). In sect.5 we mention a few specific properties of $\Delta(\pi)$ and $\Delta(\chi)$. The numerical results, for $\Delta(\pi)$ and $\Delta(\chi)$, are presented in sect.6, together with a discussion of some of their qualitative and quantitative features. Several general comments are added in the last section of the paper.

2. Description of the correlation

We start with a convenient representation /2/ of real functions h(z) in the Hilbert space H^2 with squared norm

$$\|h\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(z)|^2 d\theta \quad , \tag{2.1}$$

given by

$$\begin{aligned} h(z) &= h(o) + h'(o)z + \sum_{i=1}^{n} c_i\, \tau_i(z) + B_n(z)\, h_{n+1}(z) \quad , \\ \|h\|^2 &= h^2(o) + h'^2(o) + \sum_{i=1}^{n} c_i^2 + \|h_{n+1}\|^2 \end{aligned} \tag{2.2}$$

where

$$\tau_i(z) = z^2\, \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_{i-1}}{1-x_{i-1}z}\, \frac{(1-x_i^2)^{\frac{1}{2}}}{1-x_i z} \quad ,$$

$$B_n(z) = z^2\, \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_n}{1-x_n z}$$

and $h_{n+1}(z)$ belongs to H^2 . A very elementary derivation for general expansions of this type has been given in refs./3-5/. The functions we are interested in have a fixed value of h(o), i.e. h(o)=g(o). They represent a hyperplane $\mathcal{H}$ in H^2 .

Because of the simple relation between f(z) and h(z) we shall present all arguments for h(z) and the quantities $\|h\|^2$, h'(o), and d(h), where d(h) is the norm in R^n of the Introduction, considered for coordinates $\frac{h(x_i)-a_i}{\delta_i}$ ($a_i=g(x_i)b_i$, $\delta_i=g(x_i)\sigma_i$), e.g.

$$\lambda(h) = \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{h(x_i)-a_i}{\delta_i}\right)^2\right)^{\frac{1}{2}} \quad , \tag{2.3}$$

$$\pi(h) = \max_{1\le i\le n}\left|\frac{h(x_i)-a_i}{\delta_i}\right| \quad . \tag{2.4}$$

The parameters c_i of (2.2) are related linearly and nonsingularly to the values $h(x_k)$,

$$h(x_k) = h(0) + h'(0)\,x_k + \sum_{i=1}^{k} c_i\, t_i(x_k) \quad . \tag{2.5}$$

Mathematically, the correlation problem described in the Introduction may be formulated in terms of a mapping from H^2 into the space R^3 , performed by the system of functionals $(h'(o), \|h\|^2, d(h))$: the set $\Delta(d)$ is the image in R^3, through this mapping, of the hyperplane $\mathcal{H}$.

We start its solution by the observation that for fixed h'(o) the value of $\|h\|^2$ is bounded by

$$\|h\|^2 \geq h^2(0) + h'^2(0) \quad . \tag{2.6}$$

The inequality (2.6) defines, when h'(o) takes all real values, a 2-dimensional convex and closed set, which we denote by $\mathcal{D}$:

$$\mathcal{D} : \begin{array}{l} -\infty < h'(0) < \infty \quad , \\ \|h\|^2 \geq h^2(0) + h'^2(0) \quad . \end{array} \tag{2.7}$$

To each point $(h'(o), \|h\|^2 = h^2(o) + (h'(o))^2\,)$ of its boundary (a parabola) there corresponds a single function, $h(z)=h(o)+h'(o)z$. With $(h'(o),\|h\|^2)$ fixed in $\mathcal{D}$, the parameters c_i are allowed to take the values given by

$$\sum_{i=1}^{n} c_i^2 \leq \|h\|^2 - h^2(o) - h'^2(o) \quad , \tag{2.8}$$

i.e. in a sphere $S_n(h'(o),\|h\|^2)$ of radius $(\|h\|^2-h^2(o)-(h'(o))^2)^{1/2}$; when $(h'(o),\|h\|^2)$ is outside $\mathcal{D}$ there are no values allowed for c_i . A sphere is a bounded, closed and strict convex set (in R^n) and, since a linear transformation preserves these properties, the set (2.3) expressed in terms of $h(x_k)$ is also bounded, closed and strict convex in R^n (in fact an ellipsoid). We denote this set by $\mathcal{D}_n(h'(o), \|h\|^2)$; it is not empty only if $(h'(o),\|h\|^2) \in \mathcal{D}$. On the set $\mathcal{D}_n(h'(o), \|h\|^2)$ the norm $d(h)$ takes the values between the (global) minimum

$$d_m(h'(o),\|h\|^2) = \min_{(h(x_1),\ldots,h(x_n)) \in \mathcal{D}_n(h'(o),\|h\|^2)} d(h) \tag{2.9}$$

and the (highest local) maximum

$$d_M(h'(o),\|h\|^2) = \max_{(h(x_1),\ldots,h(x_n)) \in \mathcal{D}_n(h'(o),\|h\|^2)} d(h) \tag{2.10}$$

$$d_m(h'(o),\|h\|^2) \leq d(h) \leq d_M(h'(o),\|h\|^2) \quad . \tag{2.11}$$

These extremal values are thus defined only for $(h'(o),\|h\|^2) \in \mathcal{D}$; on the boundary $\partial\mathcal{D}$ they coincide since there $\mathcal{D}_n(h'(o),\|h\|^2)$ consists of the single point $(h(o)+h'(o)x_1 ,\ldots,h(o)+h'(o)x_n)$.

The set $\Delta(d)$ is thus described by the following inequalities :

$$\Delta(d):\quad \begin{aligned} &-\infty < h'(o) < \infty \quad , \\ &\|h\|^2 \geq h^2(o) + h'^2(o) \quad , \\ &d_m(h'(o),\|h\|^2) \leq d(h) < d_M(h'(o),\|h\|^2) . \end{aligned} \qquad (2.12)$$

3. Properties of the correlation set $\Delta(d)$

Since the sphere $S_n(h'(o),\|h\|_o^2)$ is strictly included (i.e. with no common boundary points) in the sphere $S_n(h'(o),\|h\|_1^2)$ when $\|h\|_o^2 < \|h\|_1^2$ and this property is preserved by the linear transformation which leads to the sets $\mathcal{D}_n(h'(o),\|h\|^2)$, there is also a strict inclusion of $\mathcal{D}_n(h'(o),\|h\|_o^2)$ in $\mathcal{D}_n(h'(o),\|h\|_1^2)$ when $\|h\|_o^2 < \|h\|_1^2$. This fact implies the strict increase (decrease) of $d_M(h'(o),\|h\|^2)$ ($d_m(h'(o),\|h\|^2)$, as long as it is not zero) with $\|h\|^2$, for fixed $h'(o)$. The curve $\mathcal{C}(d)$ of contact between the surfaces $d = d_m(h'(o),\|h\|^2)$ and $d = d_M(h'(o),\|h\|^2)$, $(h'(o),\|h\|^2) \in \mathcal{D}$, given by $d = d(h)$ for $h(z)=h(o)+h'(o)z$, and by $\|h\|^2 = h^2(o) + (h'(o))^2$, therefore may serve as a first and simple quantitative information on the geometry of the set $\Delta(d)$.

Further simple information on $\Delta(d)$ is gained from its projection on the plane $R^2(\|h\|^2,d(h))$: The projection of $\mathcal{D}$ is given by $\|h\|^2 \geq h^2(o)$. For a given $\|h\|^2$ obeying this inequality we define

$$d'_m(\|h\|^2) = \inf_{h'^2(o) \leq \|h\|^2 - h^2(o)} d_m(h'(o),\|h\|^2) \qquad (3.1)$$

and

$$d_m(\|h\|^2) = \sup_{h'^2(o) \leq \|h\|^2 - h^2(o)} d_m(h'(o), \|h\|^2) \tag{3.2}$$

Then the projection is given by the inequalities

$$\|h\|^2 \geq h^2(o) \quad ,$$

$$d_m(\|h\|^2) \leq d(h) \leq d_M(\|h\|^2) \quad . \tag{3.3}$$

We shall construct this projection in an independent way and use it for the determination of the values $h'_m(\|h\|^2,d)$ and $h'_M(\|h\|^2,d)$ of $h'(o)$ which realize the infima and suprema in (3.1) and (3.2), respectively, for various values of $\|h\|^2$. This allows to determine in R^3 two curves $\mathcal{C}_m(d) = (h'_m(\|h\|^2,d), d_m(\|h\|^2))$ and $\mathcal{C}_M(d) = (h'_M(\|h\|^2,d), d_M(\|h\|^2))$, for $\|h\|^2 \geq h^2(o)$, around which the surfaces $d_m(h'(o),\|h\|^2)$ and $d_M(h'(o),\|h\|^2)$ are "centred".

The construction proceeds through the decomposition

$$h(z) = h(o) + \sum_{i=1}^{n} \bar{c}_i \, z \, \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_{i-1}}{1-x_{i-1}z} \, \frac{(1-x_i^2)^{\frac{1}{2}}}{1-x_i z}$$

$$+ \; z \, \frac{z-x_1}{1-x_1 z} \cdots \frac{z-x_n}{1-x_n z} \, h_{n+1}(z) \quad , \tag{3.4}$$

$$\|h\|^2 = h^2(o) + \sum_{i=1}^{n} \bar{c}_i^2 + \|h_{n+1}\|^2 \quad ,$$

analogous to (2.2). For $\|h\|^2$ fixed, $\|h\|^2 \geq h^2(o)$, the parameters $\bar{c}_i$ are confined to a sphere $S_n(\|h\|^2)$ of radius $(\|h\|^2 - h^2(o))^{1/2}$ and the values $h(x_k)$, again linearly and nonsingularly related to $\bar{c}_i$, are confined to a bounded, closed, and strict convex set which we denote by $\mathcal{D}_n(\|h\|^2)$. The functions defined by (3.1) and (3.2) are then given by

$$d_m(\|h\|^2) = \min_{(h(x_1),\ldots,h(x_n)) \in \mathcal{D}_n(\|h\|^2)} d(h) \quad , \tag{3.5}$$

$$d_M(\|h\|^2) = \max_{(h(x_1),\ldots,h(x_n)) \in \mathcal{D}_n(\|h\|^2)} d(h) \quad . \tag{3.6}$$

The strictness of the convexity of $\mathcal{D}_n(\|h\|^2)$ has as a consequence that any of these minima and maxima is attained by a single function $h(z)$ of the form (3.4) with $h_{n+1}(z) \equiv 0$. Therefore the values $h_m'(\|h\|^2, d)$ and $h_M'(\|h\|^2, d)$ are well defined.

Since $d(h)$ has been chosen as a norm in R^n , it is convex, i.e.

$$d(\alpha h_1 + (1-\alpha) h_0) \leq \alpha d(h_1) + (1-\alpha) d(h_0) \tag{3.7}$$

for any pair of functions $h_0(z), h_1(z) \in H^2$.

On the other hand, if $(h_0(x_1),\ldots,h_0(x_n)) \in \mathcal{D}_n(h_0'(0), \|h_0\|^2)$ and $(h_1(x_1),\ldots,h_1(x_n)) \in \mathcal{D}_n(h_1'(0), \|h_1\|^2)$ then $(\alpha h_1(x_1) + (1-\alpha) h_0(x_1)$ $\ldots, \alpha h_1(x_n) + (1-\alpha) h_0(x_n)) \in \mathcal{D}_n(\alpha h_1'(0) + (1-\alpha) h_0'(0), \alpha \|h_1\|^2$ $+ (1-\alpha)\|h_0\|^2)$, for any α in $0 \leq \alpha \leq 1$. This statement makes sense since if the points $(h_0'(0), \|h_0\|^2), (h_1'(0), \|h_1\|^2)$ belong to $\mathcal{D}$ and therefore $\mathcal{D}_n(h_0'(0), \|h_0\|^2)$, $\mathcal{D}_n(h_1'(0), \|h_1\|^2)$ are not empty, then because of convexity $(\alpha h_1'(0) + (1-\alpha) h_0'(0), \alpha \|h_1\|^2 + (1-\alpha)\|h_0\|^2)$ belongs to $\mathcal{D}$ and $\mathcal{D}_n(\alpha h_1'(0) + (1-\alpha) h_0'(0), \alpha \|h_1\|^2 + (1-\alpha)\|h_0\|^2)$ is not empty. The truth of the statement follows from the facts that if $(u_1,\dots,u_n)$ belongs to $S_n(h_1'(o),\|h_1\|^2)$ and $(v_1,\dots,v_n)$ to $S_n(h_o'(o),\|h_o\|^2)$, then $(\alpha u_1+(1-\alpha)v_1,\dots,\alpha u_n+(1-\alpha)v_n)$ belongs to $S_n(\alpha h_1'(o)+(1-\alpha)h_o'(o),\alpha\|h_1\|^2+(1-\alpha)\|h_o\|^2)$ and that this inclusion is preserved by a linear transformation. The statement is conveniently formulated as the inclusion of the convex combination $\alpha\,\hat{\vartheta}_n(h_1'(o),\|h_1\|^2)+(1-\alpha)\,\hat{\vartheta}_n(h_o'(o),\|h_o\|^2)$ of $\hat{\vartheta}_n(h_1'(o),\|h_1\|^2)$, $\hat{\vartheta}_n(h_o'(o),\|h_o\|^2)$ in the set $\hat{\vartheta}_n(\alpha h_1'(o)+(1-\alpha)h_o'(o),\alpha\|h_1\|^2+(1-\alpha)\|h_o\|^2)$.

The inclusion of sets just proved leads to the inequality

$$\min_{(h(x_1),\dots,h(x_n))\in\hat{\vartheta}_n(\alpha h_1'(o)+(1-\alpha)h_o'(o),\,\alpha\|h_1\|^2+(1-\alpha)\|h_o\|^2)} d(h) \tag{3.8}$$

$$\leq \min_{(h(x_1),\dots,h(x_n))\in\alpha\hat{\vartheta}_n(h_1'(o),\|h_1\|^2)+(1-\alpha)\hat{\vartheta}_n(h_o'(o),\|h_o\|^2)} d(h)$$

and (3.7) leads to

$$\min_{(h(x_1),\dots,h(x_n))\in\alpha\hat{\vartheta}_n(h_1'(o),\|h_1\|^2)+(1-\alpha)\hat{\vartheta}_n(h_o'(o),\|h_o\|^2)} d(h) \tag{3.9}$$

$$\leq \alpha \min_{(h(x_1),\dots,h(x_n))\in\hat{\vartheta}_n(h_1'(o),\|h_1\|^2)} d(h) + (1-\alpha)\min_{(h(x_1),\dots,h(x_n))\in\hat{\vartheta}_n(h_o'(o),\|h_o\|^2)} d(h)$$

Their combination then leads to the inequality

$$d_m(\alpha h_1'(o)+(1-\alpha)h_o'(o),\,\alpha\|h_1\|^2+(1-\alpha)\|h_o\|^2)$$

$$\leq \alpha\, d_m(h_1'(o),\|h_1\|^2) + (1-\alpha)\, d_m(h_o'(o),\|h_o\|^2)\,, \tag{3.10}$$

which expresses the convexity of the function $d_m(h'(o),\|h\|^2)$.

From inequality (3.1o) we may derive an additional information on the set $\Delta(d)$, namely the convexity of the subset $\hat{\mathcal{D}}_o$ of $\hat{\mathcal{D}}$, where $d_m(h'(o),\|h\|^2)=o$. This set is, like $\hat{\mathcal{D}}$, independent of the choice of the norm $d(h)$, since $\hat{\mathcal{D}}_n(h'(o),\|h\|^2)$ is independent of $d(h)$ and $d_m(h'(o),\|h\|^2)=o$ for any $d(h)$ only for those values $(h'(o),\|h\|^2)$ for which $(a_1,\dots,a_n)$ belongs to $\hat{\mathcal{D}}_n(h'(o),\|h\|^2)$. These values are readily determined : If $(a_1,\dots,a_n) \in \hat{\mathcal{D}}_n(h'(o),\|h\|^2)$, then the values $c_i(h'(o),a_1,\dots,a_n)$ computed from $h(x_k)=a_k$ according to (2.5) belong to $S_n(h'(o),\|h\|^2)$ and have, therefore to obey (2.8). The linear dependence of $c_i(h'(o),a_1,\dots,a_n)$ on $h'(o)$ leads to the fact that the inequality

$$\|h\|^2 \geq h^2(o) + h'^2(o) + \sum_{i=1}^{n} c_i^2(h'(o),a_1,\dots,a_n) \qquad (3.11)$$

which defines $\hat{\mathcal{D}}_o$, shows that it is bounded by a parabola.

4. Correlations derived from $\Delta(d)$

Of main interest to us is the intersection $\Delta(d,\kappa)$ of $\Delta(d)$ with the half-space $d \leq \kappa$ ($\kappa \geq o$) since it corresponds exactly to those functions $h(z)$ of the hyperplane $\mathcal{H}$, which are situated within a distance κ from the data. The values allowed for $h'(o),\|h\|^2$ by these functions are given by the projection $\mathcal{D}(d,\kappa)$ of $\Delta(d,\kappa)$ on the $(h'(o),\|h\|^2)$-plane. As long as κ does not exceed the minimal value of d on the curve $\mathcal{C}(d)$, this (convex) set is bounded by the curve $d_m(h'(o),\|h\|^2)=\kappa$. When κ exceeds this minimal value, the boundary of $\mathcal{D}(d,\kappa)$ consists of the curve $d_m(h'(o),\|h\|^2)=\kappa$, where it exists, and of the curve $\|h\|^2= h^2(o) + h'^2(o)$ in the rest. In fact $\mathcal{D}(d,\kappa)$ may be represented as the projection on the $(h'(o),\|h\|^2)$-plane of the intersection $\Delta_m(d,\kappa)$ of the convex set $\Delta_m(d)$, defined by

$$\Delta_m(d) : \quad \begin{aligned} & -\infty < h'(o) < \infty , \\ & \|h\|^2 \geqslant h^2(o) + h'^2(o) , \\ & d(h) \geqslant d_m(h'(o),\|h\|^2) , \end{aligned} \tag{4.1}$$

with $d \leqslant \kappa$.

The set $\mathcal{D}(d,\kappa)$ expresses the correlation between $h'(o)$ and $\|h\|^2$, under the condition $d \leqslant \kappa$. Another projection, of the intersection of $\Delta(d)$ (or $\Delta_m(d)$) with $\|h\|^2 \leqslant h^2$ ($h^2 \geqslant h^2(o)$) on the $(h'(o),d(h))$-plane, gives the set $\mathcal{D}(d,h^2)$ (or the convex set $\mathcal{D}_m(d,h^2)$) which puts the quantities $h'(o),d(h)$ in the foreground and attributes to $\|h\|^2$ the role of regulator of the correlation between $h'(o)$ and $d(h)$. Similarly, one may project on the $(\|h\|^2,d(h))$-plane the intersection of $\Delta(d)$ (or $\Delta_m(d)$) with $h'_m \leqslant h'(o) \leqslant h'_M$ if one is mainly interested in the quantities $\|h\|^2$ and $d(h)$.

5. Particular properties of $\Delta(\pi)$ and $\Delta(\chi)$

These properties refer to the curves $\mathcal{C}(\pi)$ and $\mathcal{C}(\chi)$ and have their origin in the simplicity of the functions $h(z)=h(o)+h'(o)z$. We investigate them through the three projections on the coordinate planes of the space $R^3(h'(o),\|h\|^2,d(h))$. The projections on the

$(h'(o),\|h\|^2)$-plane need no discussion : they are in both cases the parabola $\|h\|^2 = h^2(o) + h'^2(o)$. The other two projections differ and we discuss first those of $\mathcal{C}(\pi)$.

The quantities $\dfrac{h(x_i) - a_i}{\rho_i}$ are, for $h(x_i) = h(o) + h'(o)x_i$, linear functions of $h'(o)$. Therefore $\pi(h) = \max\limits_{1 \le i \le n} \left| \dfrac{h(o) - a_i + x_i h'(o)}{\rho_i} \right|$ is in the $(h'(o),d(h))$-plane a convex polygon with at most 2n sides. The projection of $\mathcal{C}(\pi)$ on the $(\|h\|^2,d(h))$-plane consists of pieces of parabolas : If for a certain interval of $h'(o)$ the $(h'(o),d(h))$-projection of $\mathcal{C}(\pi)$ is on $\pi(h) = \varepsilon_k \dfrac{h(o) - a_k + x_k h'(o)}{\rho_k}$ ($\varepsilon_k = \pm 1$), then the corresponding $(\|h\|^2,d(h))$-projection is on the parabola $\|h\|^2 = h^2(o)$ $+ \dfrac{\rho_k^2}{x_k^2}\left(\pi(h) - \varepsilon_k \dfrac{h(o)-a_k}{\rho_k}\right)^2$. These facts allow a complete qualitative description of $\mathcal{C}(\pi)$ in terms of properties of x_i , a_i , and ρ_i.

For $\mathcal{C}(\chi)$ only the $(h'(o),d(h))$-projection is simple : a hyperbola, situated below the projection of $\mathcal{C}(\pi)$. The $(\|h\|^2,d(h))$-projection is no more such an elementary curve, but certain points of it, which display its qualitative behaviour, may still easily be computed.

6. Computational results

We have used for our computations the seven data points of refs. /6-8/ (Table I), considered in ref./9/[+]. The aim of these computations, which we have performed for $d(h)=\pi(h)$ and $d(h)=\chi(h)$, is mainly qualitative : to illustrate

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[+] Table I of ref./9/ contains errors in the data and in the values of the extremal function.

1) the possible appearance of instabilities, i.e. of strong dependences of the bounds, given by a set $\Delta(d)$, of one quantity on the values assumed for the others, and

2) the comparison between $\Delta(\chi)$ and $\Delta(\pi)$.

The curves $\mathcal{C}(\pi)$ and $\mathcal{C}(\chi)$, represented together with their projections in fig.1 , show that for both distances the surfaces $d_M(h'(o), \|h\|^2)$ come down to about 4 or 5 units of d . The domains of interest to us are mainly $\Delta(d,\kappa)$ with $1 \leq \kappa \leq 3$, which include the functions $h(z)$ situated within a reasonable number of standard deviations (in the mean, for $\chi(h)$) from the data. Their projections $\mathcal{D}(d,\kappa)$ have, therefore, as boundaries the curves $d_m(h'(o), \|h\|^2) = \kappa$. We have computed these curves for $\kappa = 1, 2, 3$ in the regions suggested by the curves $\mathcal{C}_m(\pi)$, $\mathcal{C}_m(\chi)$, represented together with their projections in fig.2 . The results of these computations are shown in fig.3 .

The stability features of $\Delta(d)$ turn out to be independent of the form chosen for the distance : For both distances there are regions of instabilities towards increasing values of r_π^2 . For $d=\pi$ the region appears at $r_\pi^2 \gtrsim 0.5$ F^2 and for $d=\chi$ at $r_\pi^2 \approx 1.0$ F^2 . In order to exemplify its effect we first take the point of view that $a_\mu(\pi^+\pi^-)$ is known, e.g. given by the value $a_\mu(\pi^+\pi^-) = 4.2 \times 10^{-8}$ of ref./10/, and derive with it bounds on the pion charge radius. Then each of our two choices for the distance leads to upper bounds for r_π^2 which are strongly dependent on κ : 0.58 F^2, 1.26 F^2, 1.74 F^2 (for $d=\pi$) and 1.22 F^2, 1.97 F^2, 2.26 F^2 (for $d=\chi$) for the values $\kappa = 1, 2, 3$, respectively. The lower bounds are much more stable ; their low values reflect the weakness of the correlation obtained with only seven spacelike data points.

Table I Experimental data from refs./6-8/, in the form used in this paper.

$-t_i$ $(GeV)^2$	x_i	$F(t_i)$ (experim.)	$g(x_i)$	a_i	δ_i
0.176	-0.2870	0.786 ± 0.045	0.7269	0.5713	0.0327
0.294	-0.3720	0.606 ± 0.028	0.7832	0.4746	0.0219
0.396	-0.4230	0.550 ± 0.015	0.8127	0.4470	0.0122
0.620	-0.4991	0.453 ± 0.014	0.8486	0.3844	0.0119
0.795	-0.5399	0.380 ± 0.013	0.8629	0.3279	0.0112
1.216	-0.6059	0.292 ± 0.026	0.8764	0.2559	0.0228
1.712	-0.6547	0.246 ± 0.017	0.8770	0.2157	0.0149

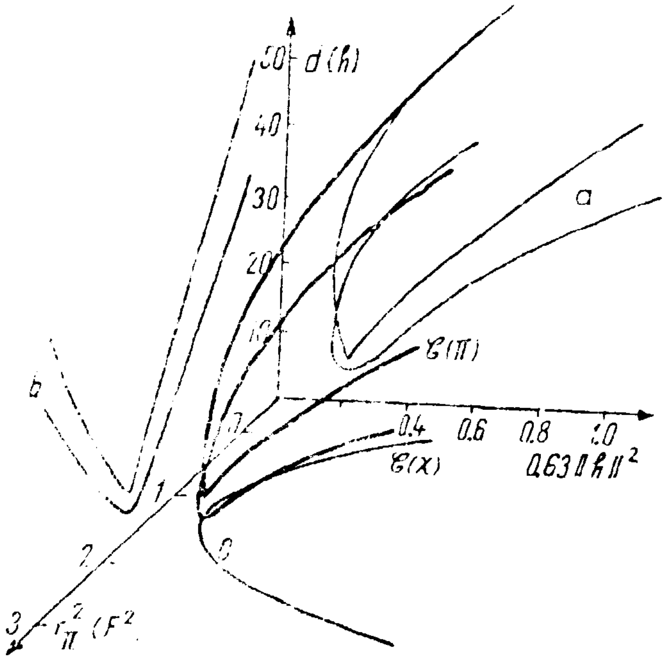

Fig. 1 : The curves $\mathcal{C}(\pi)$, $\mathcal{C}(\chi)$ and their projections on :

a) the plane $(\|h\|^2, d(h))$,

b) the plane $(h'(o), d(h))$,

c) the plane $(h'(o), \|h\|^2)$.

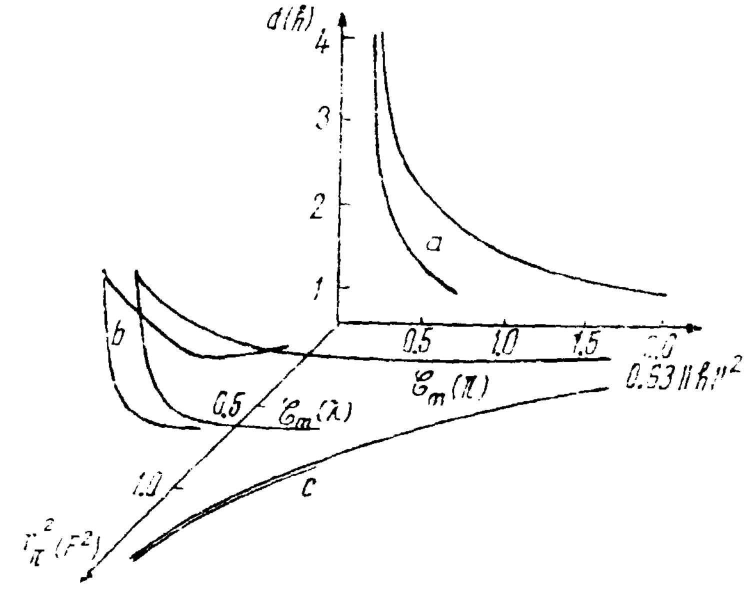

Fig. 2 : The curves $\mathscr{C}_m(\pi)$, $\mathscr{C}_m(\chi)$ and their projections on :

a) the plane $(\|h\|^2, d(h))$,

b) the plane $(h'(o), d(h))$,

c) the plane $(h'(o), \|h\|^2)$.

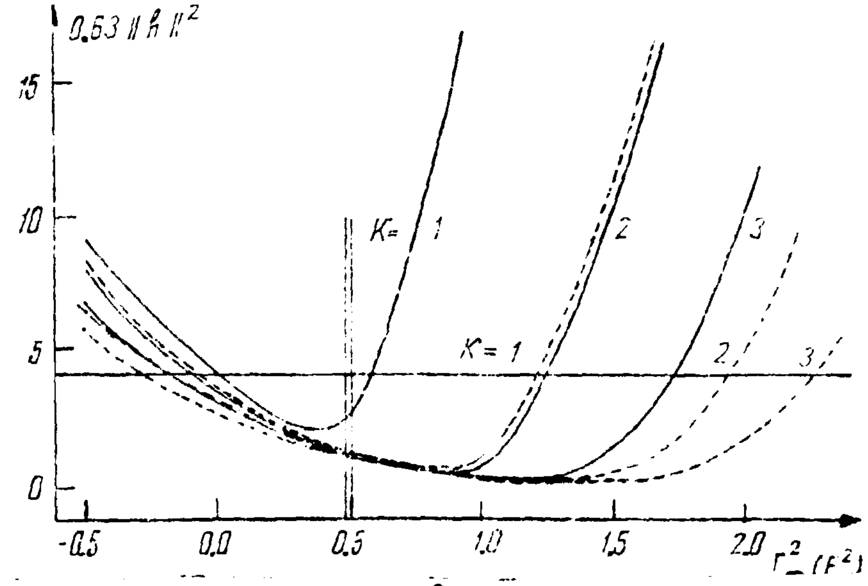

Fig. 3 : Curves $\pi_m(h'(o), \|h\|^2) = \kappa$ (full lines) and

$\chi_m(h'(o), \|h\|^2) = \kappa$ (dashed lines) for $\kappa = 1, 2, 3$.

The other point of view, the determination of bounds for $a_\mu(\pi^+\pi^-)$ when r_π^2 is fixed (at a value assumed to be known), is well illustrated by ref./9/. The magnitude computed in this paper corresponds to the minimal value of $\|h\|^2$ on the curve $\pi_m(h'(o),\|h\|^2)=1$ in the interval $o.697\ F \leq r_\pi \leq o.711\ F$; this is $a_\mu(\pi^+\pi^-) = 2.52\times1o^{-8}$ (attained at $r_\pi = o.697$ F). This (small) interval of r_π is situated in the region of transition from stability to instability, as shown by the(strong) decrease of the lower bound by roughly a factor 2, to $a_\mu(\pi^+\pi^-) = 1.22\times1o^{-8}$, when κ increases from 1 to 2, and by the weak decrease when κ further increases from 2 to 3 . For $d=\chi$ this interval of r_π is in the stability region. For large values of r_π^2 the instability of the lower bound of $a_\mu(\pi^+\pi^-)$ with respect to κ is seen to be really dramatic.

The comparison between $\Delta(\pi)$ and $\Delta(\chi)$ is very instructive, both on the curves $d_m(\|h\|^2)$ of fig.2 and on the curves $d_m(h'(o),\|h\|^2) = \kappa$ of fig.3 . Thus, from fig.2 we see that the lower bound for $a_\mu(\pi^+\pi^-)$ given at $\kappa=1$ by $\pi_m(\|h\|^2)$ ($2.16\times1o^{-8}$) is about three times stronger than that given by $\chi_m(\|h\|^2)$ ($o.72\times1o^{-8}$). It still remains two times stronger at $\kappa=2$ ($o.59\times1o^{-8}$ compared to $o.28\times1o^{-8}$). On fig.3 these values belong to the minima of the curves $\pi_m(h'(o),\|h\|^2)=\kappa$ and $\chi_m(h'(o), h^2)=\kappa$ (for $\kappa = 1, 2$). The curves of fig.3 also show e.g. that whereas for $\kappa=1$ one obtains with the constraints $o.697 \leq r_\pi \leq o.711$ F from our data for π a lower bound on $a_\mu(\pi^+\pi^-)$ of $2.52\times1o^{-8}$, for χ this bound is about two times smaller : $1.34\times1o^{-8}$. Also the bounds one obtains for r_π^2 with $\kappa=1$ and $a_\mu(\pi^+\pi^-)= 4.2\times1o^{-8}$ are $-o.o1 \leq r_\pi^2 \leq o.58\ F^2$ for π , by a factor of two stronger than those for γ: $-0.08 \leq r_\pi^2 \leq 1.22\ F^2$.

These numbers strongly suggest the use of $\pi(h)$ in correlations, if expectations in a strong result make it worthwhile facing possible computational difficulties. In our problem the computing times were comparable in the two situations, but this was due to the particular structure of this problem (i.e. of the set $\mathcal{H}$), which permits an interchange between constraints and functions to be minimized.

7. Comments

If one uses the set $\Delta(d)$ in order to derive through the projection $\hat{\mathcal{L}}(d,\kappa)$ lower bounds for $\|h\|^2$ ($a_\mu(\pi^+\pi^-)$) from known values of $h'(o)$ (r_π^2) one has to be careful in order to avoid logical inconsistencies in the sense that the value of r_π^2 should not come from data which are included in the norm d(h). This is difficult to achieve and also ref./9/ suffers from this shortcoming.

Namely one may derive, in terms of the expansion (3.4), directly from spacelike data the correlation (3.3) between d(h) and $\|h\|^2$ and gain by imposing the condition $d(h) \leq \kappa$ a lower bound on $a_\mu(\pi^+\pi^-)$, given implicitly by the smallest value of $\|h\|^2$ obeying $d_m(\|h\|^2) \leq \kappa$ (fig.2). This bound expresses the information contained in spacelike data on $a_\mu(\pi^+\pi^-)$, formulated in terms of the norm d(h).

But if one wishes to derive from these data (with any norm d(h)) information on r_π^2 , then one faces the problem, discussed in the Introduction, of a proper choice of a class of functions which one accepts as candidates for the pion form factor. The hyperplane $\mathcal{H}$ in H^2 , which is physically large enough, since it only requires that

$a_{\pi}(\pi^+\pi^-)$ be finite, clearly gives the trivial answer : it allows for the functions obeying $d(h) \leq \kappa$ with any $\kappa \geq 0$ all positive and negative values of $h'(o)$. This is so since any set $\mathcal{D}(d,\kappa)$ includes $\mathcal{D}_0 = \mathcal{D}(d,o)$ and therefore its projection on the $h'(o)$-axis includes that of $\mathcal{D}_0$, which is infinite (the projection of a parabola). Thus the whole information one derives from spacelike data alone for r_π^2 in fact stems from the selection of the class of functions. From the point of view of logical consistency any subset Ω (of $\mathcal{H}$) in H^2 is acceptable ; it may be selected in such a way that the functions of it with $d(h) \leq \kappa$ give a finite range for $h'(o)$. But any such selection needs further physical motivation. In ref./11/ e.g. it was done by using data from $e^+e^- \to \pi^+\pi^-$ and by assuming a certain threshold and high energy extrapolation of these data. The selection of a family of resonant form factors with correct analytic properties and asymptotic behaviour also satisfies the requirements of consistency, only risks to be too narrow.

Let us assume that we have chosen such a subset Ω of $\mathcal{H}$ that the intersection of its image through the mapping $h(z) \to (h'(o),d(h))$ with any $d(h) \leq \kappa$ has a bounded projection

$$\inf_{\substack{h(z)\in\Omega \\ d(h)\leq\kappa}} h'(o) \leq h'(o) \leq \sup_{\substack{h(z)\in\Omega \\ d(h)\leq\kappa}} h'(o) \tag{7.1}$$

on the $h'(o)$-axis. This means that we get with this Ω and any κ finite upper and lower bounds for r_π^2 . From the image of Ω through another mapping, $h(z) \to (\|h\|^2,d(h))$, we derive lower (and perhaps also finite upper) bounds

$$\inf_{\substack{h(z)\in\Omega \\ d(h)\leq\kappa}} \|h\|^2 \leq \|h\|^2 \leq \sup_{\substack{h(z)\in\Omega \\ d(h)\leq\kappa}} \|h\|^2 \tag{7.2}$$

on $\|h\|^2$ by projecting its intersection with $d(h)\leq\kappa$ on the $\|h\|^2$-axis.

If we now denote by $\Omega(\kappa)$ the set of all functions of H^2, which obey (7.1), and compute the lower (and upper) bounds

$$\inf_{\substack{h(z)\in\Omega\cap\Omega(\kappa) \\ d(h)\leq\kappa}} \|h\|^2 \leq \|h\|^2 \left(\leq \sup_{\substack{h(z)\in\Omega\cap\Omega(\kappa) \\ d(h)\leq\kappa}} \|h\|^2\right), \tag{7.3}$$

we end up exactly with the values (7.2), i.e.

$$\begin{aligned} \inf_{\substack{h(z)\in\Omega\cap\Omega(\kappa) \\ d(h)\leq\kappa}} \|h\|^2 &= \inf_{\substack{h(z)\in\Omega \\ d(h)\leq\kappa}} \|h\|^2 , \\ \sup_{\substack{h(z)\in\Omega\cap\Omega(\kappa) \\ d(h)\leq\kappa}} \|h\|^2 &= \sup_{\substack{h(z)\in\Omega \\ d(h)\leq\kappa}} \|h\|^2 , \end{aligned} \tag{7.4}$$

since $\Omega(\kappa)$ includes all functions of Ω, which have $d(h)\leq\kappa$. This means that a two step calculation of the bounds for $\|h\|^2$ from Ω and $d(h)\leq\kappa$, by first deriving bounds for $h'(o)$ and then using them as additional constraints, gives the same results as the direct calculation from Ω and $d(h)\leq\kappa$ alone and therefore should not be undertaken.

On the other hand, if one first computes bounds for $h'(o)$ from one set ($\Omega(h'(o))$), and then the bounds for $\|h\|^2$ from another ($\Omega(\|h'\|^2)$), then a two step calculation of $\|h\|^2$ is logically inconsistent, although the magnitudes in (7.3) still make sense. An inconsistency of this type appears in ref./9/. The inconsistency may become especially striking if $\Omega(h'(o))$ is not even situated in H^2 (as it happens e.g. if it is a family of simple or double pole functions, as frequently used).

Thus consistency requires that $\Delta(d)$ should not be used to derive bounds on $a_\mu(\pi^+\pi^-)$ in terms of values for r_π^2 , obtained (at least partially) from spacelike data of the pion form factor. It may, however, be used in order to determine consistently bounds for r_π^2 from spacelike data and from values of $a_\mu(\pi^+\pi^-)$, computed either directly from $e^+e^- \to \pi^+\pi^-$ /1o/ or obtained from the comparison between the experimental value of the muon magnetic moment and the value computed for it by quantum electrodynamics. Quantitatively, these bounds improve with the number of data points. The relative easiness of their computation apparently makes them the simplest candidates for bounds for r_π^2 derived from spacelike data without assumptions ad hoc.

Very strong bounds for r_π^2 are expected to follow from a correlation set $\Delta(d)$ constructed from the subset Ω of $\mathcal{H}$, which takes into account also data from $e^+e^- \to \pi^+\pi^-$. A procedure for the construction of this set has been developed in ref./12/, but it is expected that the computations are considerably longer than those of the present paper.

Finally, we expect from our present comparison between $\Delta(\pi)$ and $\Delta(\chi)$ that a calculation of lower bounds for $a_\mu(\pi^+\pi^-)$, of the type performed in ref./1/, but with $d(h) = \pi(h)$ instead of $\chi(h)$ and with the simplifications given in ref./12/, will lead to considerably stronger results.

On the formal side we observe that the procedure of this paper can be extended to data with asymmetric and correlated errors.

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